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# Factor Analysis Regression

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Reinhold Kosfeld und Jorgen Lauridsen

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**Reinhold Kosfeld**[1] **and Jorgen Lauridsen**[2]

[1] University of Kassel, Department of Economics, Nora-Platiel-Str. 5, D-34117 Kassel, Germany, E-Mail: rkosfeld@wirtschaft.uni-kassel.de
[2] University of Southern Denmark, Department of Economics, Campusvej 55, DK-5230 Odense M, Denmark, E-Mail: jtl@sam.sdu.dk

**Abstract.** In presence of multicollinearity principal component regression (PCR) is sometimes suggested for the estimation of the regression coefficients of a multiple regression model. Due to ambiguities in the interpretation involved by the orthogonal transformation of the set of explanatory variables the method could not yet gain wide acceptance. Factor analysis regression (FAR) provides a model-based estimation method which is particular tailored to overcome multicollinearity in an errors in variables setting. In this paper we present a new FAR estimator that proves to be unbiased and consistent for the coefficient vector of a multiple regression model given the parameters of the measurement model. The behaviour of feasible FAR estimators in the general case of completely unknown model parameters is studied in comparison with the OLS estimator by means of Monte Carlo simulation.

JEL C13, C20, C51

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## 1. Introduction

In case of multicollinearity the identification of separate influences of highly collinear variables proves to be extremely difficult. Due to negative covariances of estimated regression coefficients an overrating of one regression coefficient is known to go along with an underrating of another when standard estimation methods are applied. Only the common influence of the regressors can be reliable estimated. This issue is highly relevant for testing hypotheses and evaluating policy measurements.

Several suggestions have been made to employ the methods of principal components in case of highly correlated explanatory variables in a multiple regression model in order to overcome - or at least mitigate - the problem of multicollinearity, see e.g. Amemiya (1985; pp. 57), Fomby et al. (1988, pp. 298). With principal components regression (PCR) one is willing to accept a biased estimation of the regression coefficients for the sake reduction of variability. However, due to ambiguities in the interpretation involved by the orthogonal transformation PCR could not gain wide acceptance, see Greene (2003, pp. 58).

Factor analysis regression (FAR) provides a model-based estimation method that is particular tailored to cope with multicollinearity in an errors in variables setting. Scott (1966, 1969) was the first to address this issue by deriving “factor analysis regression equations” from a factor model of both the dependent and the explanatory variables. The theoretical deficiencies of Scott's approach are criticized for the most part by King (1969).[1] He showed that Scott's FAR estimator is biased and that the bias still exists asymptotically. Scott's FAR approach has been reconsidered by Lawley and Maxwell (1973), Chan (1977) and Isogawa and Okamoto (1980). Chan's investigation focuses on how to overcome the inconsistency in predicting the dependent variable in this type of FAR model , see also Basilevsky (1994, pp. 694).

Basilevsky (1981) has developed an FAR estimator based on a factor analysis of only the explanatory variables. This approach of factor analysis regression gets particular attraction as it is the dependencies across the explanatory variables that are responsible for multicollinearity. Given the parameters of the multiple factor model Basilevsky's FAR estimator proves to be unbiased and consistent, see Basilevsky (1981) and Basilevsky (1994, pp. 672). The finite-sample sample properties are, however, completely unknown for any kind of FAR estimator.

The present paper aims at closing this gap. In section 2 the latter kind of the FAR approach is outlined. We distinguish two types of FAR estimators. The FAR estimator of first type is attached to the common factors, while the FAR estimator of second type refers to the "true" in the sense of flawless measured explanatory variables. An FAR estimator derived in this context differs from Basilevsky's proposal in respect to its entirely data-based design. In section 3.1 it is shown that new FAR estimator shares the properties of the Basilevsky estimator under the same set of assumptions. The finite-sample properties of two feasible FAR estimators are investigated by Monte Carlo simulation in section 3.2. Special features of FAR estimators disclosed by the simulation experiments are discussed. Section 4 concudes with some qualifications regarding the applicability of the FAR approach.

## 2. FAR model and estimation

Let **y** be an nx1 vector of the regressand y, $\Xi$ an nxp matrix of the stationary regressors $\xi_j$ measured as deviations from their means, $\beta$ a px1 vector of the regression coefficients $\beta_j$ and **v** an nx1 vector of disturbances v. Further assume that the structural equation of an econometric model is given by the multiple regression equation

[1] There still exist some additional problems with Scott's FAR approach. His so-called "factor analysis regression equations" e.g. do not result from a transformation of the factor matrix to a more simple and better interpretative structure in the sense of Thurstone's concept of simple structure (Thurstone, 1970). It is simply the result of a reduction of multiple factor model to a one factor model. A "rotation" of the factor matrix is unnecessary since the implied "regression coefficients" are invariant to orthogonal transformations.

(2.1) $\mathbf{y} = \Xi \cdot \beta + \mathbf{v}$.

The regressors $\xi_j$, however, are prone to measurement errors $u_j$ and thence are not directly observable. Only their flawed counterparts $x_j$, $\xi_j + u_j$, are accessible to observation. Hence, the nxp observation matrix **X** is composed of the matrix of "true" regressor values, $\Xi$, and the nxp matrix of measurement errors, **U**:

(2.2) $\mathbf{X} = \Xi + \mathbf{U}$.

By substituting $\Xi$ with **X-U** in Equation (2.1) one obtains the error in variables model

(2.3) $\mathbf{y} = \mathbf{X} \cdot \beta + \varepsilon$

with the error term

(2.4) $\varepsilon = \mathbf{v} - \mathbf{U} \cdot \beta$.

It is well-known that the properties of the OLS estimator $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ for the parameter vector $\beta$ depend on the kind of relationship among the regressors and the disturbances. If Equation (2.1) renders the "true" regression for which the standard assumptions

(2.5) (a) $E(\mathbf{v} \mid \Xi) = \mathbf{o}$ **and** (b) $\Sigma_v = E(\mathbf{v} \cdot \mathbf{v}' \mid \Xi) = \sigma_v^2 \cdot \mathbf{I}_n$

hold, it follows that the flawed regressors $x_j$ in model (2.3) are correlated with the disturbances $\varepsilon_j$. In this case the OLS estimator $\hat{\boldsymbol{\beta}}$ is neither unbiased nor consistent for $\beta$. Since the bias depends on the ratio $\sigma_v^2 / \sigma_\xi^2$ (Johnston and DiNardo, 1997, pp. 153), it may be negligible in situations when the disturbance variance $\sigma_v^2$ is small with regard to the variance $\sigma_\xi^2$ of the true regressors. Here we refer to the case of multicollinearity where the OLS estimator is bound up with high variability. Moreover, an overrating of one regression coefficient is known to come along with an underrating of another one. With increasing correlations the explonatory variables share larger common parts which makes a separation of their influences on the dependent variable more and more difficult.

From experience with principal components regression it can be expected that variability generally could be reduced by orthogonalising the collinear regressors. However, the increase of the bias with the orthogonalisation transformation has turned out to be dramatic.[2] The advantage of factor analysis regression consists in explicitly allowing for the model structure.

[2] This conclusion is drawn from an own simulation study which cannot be presented here due to space limitation.

Factor analysis serves as a measurement model for the common and specific parts of the regressors. After extracting common factors in the first step, the dependent variable y is regressed on the orthogonal factors $f_1, f_2, \ldots, f_m$ in a second step (FAR estimator of 1st type). In a third step a factor analysis regression estimator (FAR estimator of 2nd type) is derived from the estimated influences of the factors on the explanatory and dependent variable y.

By using the multiple factor model (Dillon and Goldstein, 1984, pp. 53; Johnson and Wichern, 1992, pp. 396)

(2.6) $\mathbf{X} = \mathbf{F} \cdot \Lambda' + \mathbf{U}$

as a measurement model we assume a special data generating process for the explanatory variables. According to the factor model (2.6) the observed explanatory variables are generated by m common factors $f_k$, $m<p$, and m specific factors $u_j$. Each common factor is associated with more than one explanatory variable, whereas the specific factors are exactly assigned to a particular regressor. The specific factors match exactly with the measurement errors $u_j$ in Equation (2.2). With

(2.7) $\Xi = \mathbf{F} \cdot \Lambda'$

the "true" regressors $\xi_j$ are given by a linear combination of the common factors $f_1, f_2, \ldots, f_m$ which are arranged in the nxm factor score matrix **F**. The weights $\lambda_{jk}$ of the linear combination (2.7) are called factor loadings; the pxm matrix of the factor loadings, $\Lambda$, denotes the factor matrix.

Without loss of generality

(2.8) $E(\mathbf{F}) = E(\mathbf{U}) = \mathbf{0}$

and

(2.9) $\Sigma_f = E(\frac{1}{n}\mathbf{F}'\mathbf{F}) = \mathbf{I}_m$

can be assumed. The assumption (2.9) of uncorrelated common factors $f_k$ is always necessary for factor extraction. Although it can be altered in a later step it is retained in our case. The covariance matrix of the unique factors $u_j$ must be diagonal:

(2.10) $\Sigma_u = \text{diag}(\sigma^2_{u_1} \quad \sigma^2_{u_2} \quad \ldots \quad \sigma^2_{u_p})$.

Note that the distinction of two kind of factors requires the common factors $f_k$ to be uncorrelated with the specific factors $u_j$:

(2.11) $E(\mathbf{F'U}) = \mathbf{0}$.

Finally, we assume the common and unique factors to be independently normally distributed with zero expectation and covariance matrices given by (2.9) and (2.10), respectively.

By inserting the hypothesis (2.7) on the generation of "true" regressors into the multiple regression equation (2.1) one obtains

$$\mathbf{y} = \mathbf{F}\cdot\Lambda'\cdot\beta + \mathbf{v}$$

or, with

(2.12) $\beta^* = \Lambda'\cdot\beta$

(2.13) $\mathbf{y} = \mathbf{F}\cdot\beta^* + \mathbf{v}$.

Equation (2.13) can be interpreted as a factor regression where the endogenous variable of the original model, y, is explained by a set of common factors $f_1, f_2, \ldots, f_m$. While the explanatory variables $x_1, x_2, \ldots, x_p$ will be highly correlated in case of multicollinearity, the common factors $f_1, f_2, \ldots, f_m$ are uncorrelated. If the factor scores were assumed to be known, OLS applied on (2.13) would produce the factor analysis regression (FAR) estimator with respect to the common factors $f_1, f_2, \ldots, f_m$ (FAR estimator of 1st type):

(2.14) $\hat{\boldsymbol{\beta}}^*_{FAR} = (\mathbf{F'}\cdot\mathbf{F})^{-1}\cdot\mathbf{F'}\cdot\mathbf{y}$.

Of course, since the common factors are not directly observable, they have to be estimated in advance in order to measure the influences the common factors on the variable to be explained in the model.

At the outset, though, we our interest has been addressed to the issue of a stable estimation of the parameter vector β in case of multicollinearity. This type of factor analysis regression (FAR) estimator (FAR estimator of 2nd type) has to capture the influences of the "true" regressors $\xi_1, \xi_2, \ldots, \xi_p$ on the dependent variable y. To obtain an FAR estimator $\hat{\boldsymbol{\beta}}_{FAR}$ for β compatible to $\hat{\boldsymbol{\beta}}^*_{FAR}$, the relationship (2.12) between the parameter vectors β* and β has to be translated to both FAR estimators:

(2.15) $\hat{\boldsymbol{\beta}}^*_{FAR} = \Lambda'\cdot\hat{\boldsymbol{\beta}}_{FAR}$.

After substitution of $\hat{\boldsymbol{\beta}}^*_{FAR}$ in Equation (2.15) by (2.14) and premultiplication by the factor matrix Λ, the relation

(2.16) $\Lambda\cdot\Lambda'\cdot\hat{\boldsymbol{\beta}}_{\mathrm{FAR}} = \Lambda\cdot(\mathbf{F}'\cdot\mathbf{F})^{-1}\cdot\mathbf{F}'\cdot\mathbf{y}$

results. The FAR estimator with respect to the explanatory variables $\xi_1$, $\xi_2$, …, $\xi_p$ is then given by

(2.17) $\cdot\hat{\boldsymbol{\beta}}_{\mathrm{FAR}} = (\Lambda\cdot\Lambda')^{+}\cdot\Lambda\cdot(\mathbf{F}'\cdot\mathbf{F})^{-1}\cdot\mathbf{F}'\cdot\mathbf{y}$**,**

where $(\Lambda\cdot\Lambda')^{+}$ denotes the Moore-Penrose pseudo inverse of the product matrix $\Lambda\cdot\Lambda'$. By replacing **y** by (2.13) it is easily shown with (2.12) that the FAR estimator (2.17) is invariant to orthogonal transformations of the factor matrix. Hence, the rotation problem of factor analysis does not matter at all in factor analysis regression.

Note that in (2.17) both the factor matrix $\Lambda$ and the factor score matrix **F** are unknown. In order to determine the FAR estimator numerically, $\Lambda$ and **F** have to be estimated in advance. An orthogonal estimator of **F** is generally obtained, when $\Lambda$ is estimated by maximum likelihood or generalised least squares factor analysis, see Jöreskog (1977). In ML factor analysis the relation

(2.18) $\frac{1}{n}\hat{\mathbf{F}}'\cdot\hat{\mathbf{F}} = \mathbf{I}$

is met – in contrary to other extraction methods - not only approximately but exactly. Moreover, ML factor analysis is preferable as well in order to ensure the consistency of an estimator $\hat{\boldsymbol{\Lambda}}$ for the factor matrix $\Lambda$. To avoid a stodgy notation $\hat{\boldsymbol{\Lambda}}$ always denote a consistent estimator of $\Lambda$. Then a feasible FAR estimator of type (2.17) reads

(2.19) $\hat{\hat{\boldsymbol{\beta}}}_{\mathrm{FAR}} = (\hat{\boldsymbol{\Lambda}}\cdot\hat{\boldsymbol{\Lambda}}')^{+}\hat{\boldsymbol{\Lambda}}\cdot(\hat{\mathbf{F}}'\hat{\mathbf{F}})^{-1}\cdot\hat{\mathbf{F}}'\mathbf{y}$.

The factor scores can be estimated by the well-known “regression estimator” (Thomson estimator)

(2.20) $\hat{\mathbf{F}}_{\mathrm{T}}' = \boldsymbol{\Lambda}'(\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}' + \boldsymbol{\Sigma}_{\mathrm{u}})^{-1}\mathbf{X}' = (\mathbf{I} + \boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{\mathrm{u}}^{-1/2}\boldsymbol{\Lambda})^{-1}\boldsymbol{\Lambda}'\cdot\mathbf{X}'$

that is known to be a biased minimum variance estimator, see Brachinger and Ost (1996, pp. 691). Alternatively, the Bartlett estimator

(2.21) $\hat{\mathbf{F}}_{\mathrm{B}}' = (\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{\mathrm{u}}^{-1/2}\boldsymbol{\Lambda})^{-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{\mathrm{u}}^{-1/2}\mathbf{X}'$

can be employed. $\hat{\mathbf{F}}_B'$ can be interpreted as a weighted least-squares estimator. It proved to be the best linear unbiased estimator of **F** given the parameters Λ and $\Sigma_u$ of the multiple factor model (2.6), see McDonald and Burr (1967) and Brachinger and Ost (1996, pp. 690).

## 3. Properties of the FAR estimators

### 3.1 Properties when Λ and $\Sigma_u$ are known

In factor analysis the properties of estimators $\hat{\mathbf{F}}$ of the factor score matrix **F** have been established for the case that the parameters of the factor analytics model Λ and $\Sigma_u$ are known, see Anderson and Rubin; McDonald and Burr, 1967; Lawley and Maxwell, 1971. Using the same set of assumptions here, the FAR estimator (2.17) proves to be unbiased for the parameter vector β for any conditional unbiased estimator $\hat{\mathbf{F}}$ for **F**.

**Theorem 3.1**

For any conditional unbiased estimator $\hat{\mathbf{F}}$ for **F**,

(3.1) $\quad \mathrm{E}(\hat{\mathbf{F}} \mid \mathbf{F}) = \mathbf{F}$

given Λ and $\Sigma_u$ the FAR estimator

(3.2) $\quad \hat{\boldsymbol{\beta}}_{\mathrm{FAR}} = (\boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}')^{+} \boldsymbol{\Lambda} \cdot (\hat{\mathbf{F}}' \cdot \hat{\mathbf{F}})^{-1} \cdot \hat{\mathbf{F}}' \cdot \mathbf{y}.$

is an unbiased estimator for the parameter vector β:

(3.3) $\quad \mathrm{E}(\hat{\boldsymbol{\beta}}_{\mathrm{FAR}}) = \boldsymbol{\beta}.$

*Proof*

Substituting **y** in (3.2) by Equation (2.13) and using (2.12) the FAR estimator takes the form

$$\hat{\boldsymbol{\beta}}_{\mathrm{FAR}} = (\boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}')^{+} \boldsymbol{\Lambda} \cdot (\hat{\mathbf{F}}' \cdot \hat{\mathbf{F}})^{-1} \cdot \hat{\mathbf{F}}' \cdot (\mathbf{F} \cdot \boldsymbol{\Lambda}' \cdot \boldsymbol{\beta} + \mathbf{v}).$$

With regard to (3.1) the expectation reads

$$\mathrm{E}(\hat{\boldsymbol{\beta}}_{\mathrm{FAR}}) = \mathrm{E}[(\boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}')^{+} \boldsymbol{\Lambda} \cdot (\mathbf{F}' \cdot \mathbf{F})^{-1} \cdot \mathbf{F}' \cdot \mathbf{F} \cdot \boldsymbol{\Lambda}' \cdot \boldsymbol{\beta} + (\boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}')^{+} \boldsymbol{\Lambda} \cdot (\mathbf{F}' \cdot \mathbf{F})^{-1} \cdot \mathbf{F}' \mathrm{E}(\mathbf{v} | \mathbf{F}]$$

from which (3.3) immediately follows on account of (2.5a). □

When the parameters Λ and $\Sigma_u$ are known unbiasedness of FAR estimation is ensured. A theoretical assessment of FAR estimator, however, has to establish its large-sample properties, too. In this connection the property of consistency becomes the focus of attention.

Since unbiasedness also holds for $n\to\infty$, it is sufficient for $\hat{\boldsymbol{\beta}}_{FAR}$ to be consistent for β to show that its variances and covariances vanish asymptotically, see Judge et al. (1988, pp. 83 and p.260). With regard to Theorem (3.1) consistency $\hat{\boldsymbol{\beta}}_{FAR}$ is ensured if the covariance matrix of $\hat{\boldsymbol{\beta}}_{FAR}$, Cov($\hat{\boldsymbol{\beta}}_{FAR}$), does approach a pxp zero matrix $\mathbf{0}_p$ as n goes to infinity.

**Theorem 3.2**

For an asymptotical conditional unbiased estimator $\hat{\mathbf{F}}$ for $\mathbf{F}$

(3.4) $\lim_{n\to\infty} E(\hat{\mathbf{F}} \mid \mathbf{F}) = \mathbf{F}$

the FAR estimator (3.2) is a consistent estimator for the parameter vector β:

(3.5) $\operatorname*{plim}_{n\to\infty} \hat{\boldsymbol{\beta}}_{FAR} = \boldsymbol{\beta}$.

given the parameters Λ and $\Sigma_u$

*Proof*

On account of Theorem (3.1) consistency of the FAR estimator $\hat{\boldsymbol{\beta}}_{FAR}$ is ensured if its covariance matrix Cov($\hat{\hat{\boldsymbol{\beta}}}_{FAR}$) is proved to approach the zero matrix $\mathbf{0}_p$ for $n\to\infty$.

By plugging (2.13) into (3.1) it is easily verified that the expectation $E(\hat{\boldsymbol{\beta}}_{FAR}\, \hat{\boldsymbol{\beta}}'_{FAR})$ is given by the expression

$$E(\hat{\boldsymbol{\beta}}_{FAR}\, \hat{\boldsymbol{\beta}}'_{FAR}) =$$

$$(\boldsymbol{\Lambda\Lambda}')^{+}\boldsymbol{\Lambda}\cdot E[(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{F}\boldsymbol{\Lambda}'\boldsymbol{\beta\beta}'\boldsymbol{\Lambda}\mathbf{F}'\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1} + (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'E(\mathbf{v}|\mathbf{F})\boldsymbol{\beta\Lambda}\mathbf{F}'\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}$$

$$+ (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{F}\boldsymbol{\Lambda}'\boldsymbol{\beta}E(\mathbf{v}'|\mathbf{F})\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1} + (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'E(\mathbf{vv}'|\mathbf{F})\,\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}]\cdot\boldsymbol{\Lambda}'(\boldsymbol{\Lambda\Lambda}')^{+}.$$

From this

$$E(\hat{\boldsymbol{\beta}}_{FAR}\, \hat{\boldsymbol{\beta}}'_{FAR}) = \boldsymbol{\beta\beta}' + \frac{1}{n}\sigma_v^2(\boldsymbol{\Lambda\Lambda}')^{+}$$

follows considering (2.5a), (2.5b), (2.9) and (2.18).

Due to (3.2) the covariance of $\hat{\boldsymbol{\beta}}_{FAR}$ reads

(3.6) $$\mathrm{Cov}(\hat{\boldsymbol{\beta}}_{FAR}) = = \frac{1}{n}\sigma_v^2(\boldsymbol{\Lambda\Lambda}')^{+},$$

which approaches the zero matrix $\mathbf{0}_p$ as n goes to infinity: □

Theorems (3.1) and (3.2) ensure unbiasedness and consistency of the FAR estimator $\hat{\boldsymbol{\beta}}_{FAR}$ given the parameters $\Lambda$ and $\Sigma_u$ of the multiple factor model (2.6). However, with the usual extraction methods employed in factor analysis unbiased estimation of $\Lambda$ and $\Sigma_u$ cannot be assured. Moreover, consistent estimation of $\Lambda$ and $\Sigma_u$ by the method of maximum likelihood (ML) or the generalised least-squares (GLS) method does not necessarily translate this property to the feasible FAR estimator (2.19), see Greene (2000, p. 469) and Schmidt (1976, p. 69).

### 3.2 Properties when $\Lambda$ and $\Sigma_u$ are unknown

Usually the parameters $\Lambda$ and $\Sigma_u$ of the multiple factor model are not known in advance and, hence, have be estimated from sample data. Under the usual regular conditions maximum likelihood estimators the $\hat{\boldsymbol{\Lambda}}$ and $\hat{\boldsymbol{\Sigma}}_u$ are shown to be consistent, asymptotically efficient and asymptotically normal estimators, see Lawley and Maxwell (1971). Moreover, on the basis of the ML estimators for $\Lambda$ and $\Sigma_u$ the validity of orthogonality condition (2.18) is ensured.

To study the finite-sample properties of the FAR estimator in comparison with those of the OLS estimator we assume the common and specific factors to be multivariate normally distributed:

$$\mathbf{f} \sim MN(\mathbf{o}, \mathbf{I}_m),\ k=1,2,...,m \quad \text{and} \quad \mathbf{u} \sim MN(\mathbf{o}, \Sigma_{\mathbf{u}})$$

with

$$\mathbf{f} = (f_1\ \ f_2\ \ \ldots\ \ f_m)' \quad \text{and} \quad \mathbf{u} = (u_1\ \ u_2\ \ \ldots\ \ u_m)'.$$

Collinear multivariate normally distributed regressors $x_1, x_2, \ldots, x_p$ are generated by the measurement model (2.6) for given alternative factor patterns $\Lambda$:

$$\mathbf{x} \sim MN(\mathbf{o}, \Sigma_{\mathbf{x}})$$

with

$$\mathbf{x} = (x_1\ \ x_2\ \ \ldots, x_p)'.$$

The structure covariance matrix of the regressors $x_1, x_2, \ldots, x_p$

$$(3.7) \quad \Sigma_{\mathbf{x}} = \Lambda\cdot\Lambda' + \Sigma_{\mathbf{u}},$$

renders the so-called fundamental theorem of factor analysis, see e.g. Dillon and Goldstein (1984) or Johnson and Wichern (1992). The next step consists of estimating the factor matrix Λ by employing maximum factor analysis. Since the FAR estimator is invariant to orthogonal transformations, a factor rotation is redundant. After that the factor score matrix **F** is estimated by applying the Thompson and Bartlett estimators $\hat{\mathbf{F}}_T$ and $\hat{\mathbf{F}}_B'$ according to Equations (2.20) and (2.21), respectively. Finally, the properties of the two variants of the feasible FAR estimator (2.19) and the OLS estimator are investigated by Monte Carlo methods.

Table 3.1: Experimental design

| Experimentation factor | | | |
|---|---|---|---|
| 1. Number of factors (m) | One-factor model | Two-factor models | |
| 2. Number of explanatory variables (p) | 3, 4, 5 | 5, 6, 7 | |
| 3. Degree of multicollinearity | Moderate | Strong | High |
| 4. Regression coefficients | Identical | Different | |

The Monte Carlo simulation is stratified fourfold (Table 3.1). The first experimentation factor refers to the number of common factors, m, used to generate the regressors of the multiple regression model (2.3). Specifically, we study one- and two-factor models with a varying number of manifest variables, p. In the one-factor model the number of variables varies from three to five, in the two-factor model from five to seven. The second experimentation factor p is bounded downwards by the condition of non-negative degrees of freedom condition in the goodness-of-fit test. The third experimentation factor captures the degree of multicollinearity. Three degrees are distinguished: moderate, strong and high multicollinearity. Moderate multicollinearity reflects the situation where the factor pattern of the common factors implies correlations between 0.7 and 0.9 for respective groups of variables. Correlations between 0.90 and 0.96 within groups of variables refer define a strong degree of multicollinearity. In case of high multicollinearity the correlations within groups of variables are larger than 0.96. Extreme situations of virtually perfect multicollinearity are not covered. Table 3.2 exhibits the special factor patterns studied by Monte Carlo methods.[3] The fourth experimentation factor refers to the choice of identical or different components of the coefficient vector β. While all

[3] The simulations are carried out with MATLAB Version 6.5.

regression coefficients are set equal to 1 in the former case, they are incremented by 0.5 or – 0.5 in the latter case.

Table 3.2: Factor patterns and simulation design

| | Strong multicollinearity | High multicollinearity | Very high Multicollinearity |
|---|---|---|---|
| One-factor model | $\begin{bmatrix} 0.9 \\ 0.8 \\ 0.9 \end{bmatrix}; \begin{bmatrix} 0.9 \\ 0.8 \\ 0.9 \\ 0.8 \end{bmatrix}; \begin{bmatrix} 0.9 \\ 0.8 \\ 0.9 \\ 0.8 \\ 0.9 \end{bmatrix}$ | $\begin{bmatrix} 0.98 \\ 0.95 \\ 0.96 \end{bmatrix}; \begin{bmatrix} 0.98 \\ 0.95 \\ 0.95 \\ 0.96 \end{bmatrix}; \begin{bmatrix} 0.98 \\ 0.95 \\ 0.95 \\ 0.96 \\ 0.97 \end{bmatrix}$ | $\begin{bmatrix} 0.998 \\ 0.995 \\ 0.996 \end{bmatrix}; \begin{bmatrix} 0.998 \\ 0.995 \\ 0.995 \\ 0.996 \end{bmatrix}; \begin{bmatrix} 0.998 \\ 0.995 \\ 0.995 \\ 0.996 \\ 0.997 \end{bmatrix}$ |
| Two-factor model | $\begin{bmatrix} 0.9 & 0 \\ 0.8 & 0 \\ 0.9 & 0 \\ 0 & 0.9 \\ 0 & 0.8 \end{bmatrix}; \begin{bmatrix} 0.9 & 0 \\ 0.8 & 0 \\ 0.9 & 0 \\ 0 & 0.9 \\ 0 & 0.85 \\ 0 & 0.85 \end{bmatrix}; \begin{bmatrix} 0.9 & 0 \\ 0.8 & 0 \\ 0.9 & 0 \\ 0.9 & 0 \\ 0 & 0.85 \\ 0 & 0.85 \\ 0 & 0.9 \end{bmatrix}$ | $\begin{bmatrix} 0.98 & 0 \\ 0.95 & 0 \\ 0.98 & 0 \\ 0 & 0.98 \\ 0 & 0.96 \end{bmatrix}; \begin{bmatrix} 0.98 & 0 \\ 0.95 & 0 \\ 0.98 & 0 \\ 0 & 0.98 \\ 0 & 0.96 \\ 0 & 0.97 \end{bmatrix}; \begin{bmatrix} 0.98 & 0 \\ 0.95 & 0 \\ 0.98 & 0 \\ 0 & 0.98 \\ 0 & 0.96 \\ 0 & 0.97 \\ 0 & 0.95 \end{bmatrix}$ | $\begin{bmatrix} 0.998 & 0 \\ 0.995 & 0 \\ 0.998 & 0 \\ 0 & 0.998 \\ 0 & 0.996 \end{bmatrix}; \begin{bmatrix} 0.998 & 0 \\ 0.995 & 0 \\ 0.998 & 0 \\ 0 & 0.998 \\ 0 & 0.996 \\ 0 & 0.997 \end{bmatrix}; \begin{bmatrix} 0.998 & 0 \\ 0.995 & 0 \\ 0.998 & 0 \\ 0 & 0.998 \\ 0 & 0.996 \\ 0 & 0.997 \\ 0 & 0.995 \end{bmatrix}$ |

The Monte Carlo statistics are based on 10,000 repetitions and a sample size of 100. Table A3.1 exhibits the performance of the estimation methods for the one-factor models with identical regression coefficients. While the FAR estimators are generally slightly more biased than the OLS estimator for the lower degrees of multicollinearity, the overall bias of the OLS estimator exceeds that of the FAR1 estimator when multicollinearity is highly marked. The FAR1 estimator tends to be slightly downward biased, whereas a small upward bias is attached to the FAR2 estimator. The variance of the OLS estimator increases substantially with the degree of multicollinearity. For the highest degree of multicollinearity the variance of the OLS estimator exceeds those of the FAR1 and FAR2 estimators on the average by factors of about 50 and 19, respectively. In this case the OLS estimator becomes totally unreliable, whereas both FAR estimators scarcely lose their precision. According to the mean square error (MSE) criterion the OLS estimator is outperformed by both FAR estimators in the cases of strong and high multicollinearity for all one-factor models with identical coefficients. The FAR1 estimator proves to be slightly preferable to the FAR2 estimator.

The performance of FAR experiences a distinct alteration in the case of non-identical regression coefficients. Table A3.2 shows that the biases of the FAR estimators increase considerably for all one-factor models. A clear tendency of a downward or upward can no more be stated. However, a tendency of FAR estimation to equalize the effects of the explanatory variables on the dependent variable becomes obvious. With it the high precision of the FAR estimators is still retained. The average variance inflation factors (46 and 18) coming along with OLS estimation do not alter noticeable compared with the case of identical coefficients. Although OLS always ranks first in both lower degrees multicollinearity, its performance deteriorates in the factor analysis regression models which are subjugated to high multicollinearity. Only in the regression model with four explanatory variables the bias of the FAR estimators turns out to be more severe than the loss of precision by OLS estimation. In most of the cases the Bartlett-type FAR1 estimator outperforms the Thompson-type FAR2 estimator.

In the two-factor models with identical regression coefficients (Table A3.3) FAR estimation proved to be preferable to OLS estimation for both higher degrees of multicollinearity. Only in case of the lowest degree of multicollinearity the high precision of the FAR estimators cannot fully compensate their larger downward and upward biases. Again the variance inflation attached to the OLS estimator is much higher in relation to the FAR1 estimator as with the FAR2 estimator. In the highest degree of multicollinearity its variance is inflated on the average by the factors 54 and 14, respectively. Again the Bartlett-type FAR1 estimator proves to be superior to the Thompson-type FAR2 estimator due to its smaller bias and higher precision.

As with the one-factors models Table A3.4 exhibits a tendency of FAR estimation to equalization in the two-factor models with non-identical regression. More specifically estimated regression coefficients of the groups of variables generated by a common factor seem to differ only randomly from one another. For the lowest degree of multicollinearity the OLS estimators outperforms the FAR estimators in respect to both bias and precision. In case of strong multicollinearity the gain in precision of both estimators cannot level out the adverse equalization tendency. When multicollinearity is highly marked, however, the variance inflation of the OLS estimator again turns out to be considerable. Compared with the FAR1 estimator the variance inflation factor on the average takes a value of about 25. This illustrates once more that the FAR1 estimator clearly outperforms the FAR2 estimator with respect to precision. Although OLS can maintain its position against the Thompson-type FAR2 estimator for the highest degree of multicollinearity it falls short against the Bartlett-type FAR1 estimator.

On the whole the simulation study shows how the trade-off between bias and precision manifests with OLS and FAR estimation. On the one hand the bias attached with OLS estimation turns out to be negligible for typically low-dimensional factor models. Due to a tendency of averaging of the effects of the variables related to the same common factor, feasible FAR estimators can become considerably biased. However, they retain their high precision under all experimental conditions, while the variance of the OLS estimator is inflated substantially with the degree of multicollinearity. When multicollinearity is highly marked the OLS estimator becomes totally unstable. Although specifically the Bartlett-type feassible FAR estimator can outperform OLS under these circumstances, it only performs satisfactorily when the influences the explanatory variables within a factor group exert on the regressand do not differ significantly.

Thus, in case of high multicollinearity, generally neither the OLS estimator nor the FAR estimator of 2nd type, i.e. the FAR estimator with respect to the "true" regressors $\xi_1$, $\xi_2$, …, $\xi_p$ will be an adequate choice. When the explanatory variables are prone to measurement errors and multicollinearity is highly marked, the FAR estimator of 1st type, i.e. the FAR estimator with respect to the common factors $f_1$, $f_2$, …, $f_m$, gains attraction for it is not adversely affected by both data problems. A sensible application of this kind of FAR estimator in empirical research, however, crucially depends on the interpretability of the dimensions underlying the explanatory variables.

## 4. Conclusions

When multicollinearity is highly marked OLS goes along with highly inflated variances that can entirely invalidate statistical inference in an econometric model. Employing principal components regression in this situation could not yet gain broad acceptance, as its interpretation runs into difficulties (Greene, 1997, p. 427). Principal components regression basically stands for a pure transformation method and not for an explicit modelling approach. When the researcher intends to explicitly account for errors in variables a multiple factor model can be used as a measurement model for the explanatory variables. Factor analysis regression is a model-based approach to coping with multicollinearity when variables are measured with errors.

Although factor analysis regression has been treated in several papers by different modelling approaches, finite-sample properties of FAR estimators in the case of unknown parameters of the factor model have not yet been established. In this paper this issue has been addressed by means of Monte Carlo simulation. It turns out that unbiasedness and consistency given the parameters of the factor model are only of limited importance when they are actually unknown. Monte Carlo simulations uncover the particular behaviour of two feasible variants of a theoretically attractive FAR estimator in comparison with the OLS estimator.

The simulation study reveals that the OLS estimator becomes totally unstable when multicollinearity is highly marked. While the bias of the OLS estimator remains negligible, its variance is substantially inflated. In contrary, both FAR estimators are expelled by a high precision, whereas their biases cannot be ignored in all stratifications. Although more biased the Bartlett-type feasible FAR estimator outperforms the OLS estimator in the highest grade of multicollinearity.

Although an FAR estimator may be favourable compared with the OLS estimator when multicollinearity is highly marked, a distinct adverse feature is unmasked from the experimental study. FAR estimation tend to equalize the effects of the explanatory variables on the dependent variable within a factor group. Only when the regressors within a factor group exert identically influences on the regressand, the feasible FAR estimators solve the problem of multicollinearity very efficiently. In case of different influences FAR estimation is suitable to identify only the average effect of the explanatory variables loading on the same common factor. Single effects are mixed by balancing low and high effects.

As a result both OLS and FAR do not provide satisfactory procedures to cope with high multicollinearity. However, the problem of the division of a common factor effect on the dependent variable in FAR does only refer to feasible estimators with respect to the explanatory variables (FAR estimator of 2nd type). The effects of the common factors on the dependent variable are not mixed. On this ground the employment of the FAR estimator of 1st type can be of advantage provided that the dimensions of the set of explanatory variables are accessible to a sensible economic interpretation.

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**Appendix: Simulation results**

Table A3.1: Simulation results for one-factor models (identical regression coefficients)

| | Moderate multicollinearity | | | Strong multicollinearity | | | High multicollinearity | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 |
| One-Factor Model (p=3) | | | | | | | | | |
| Bias | 0.0010 | 0.0319 | 0.1246 | 0.0019 | 0.0142 | 0.0353 | 0.0055 | -0.0016 | 0.0136 |
| (p=3) | -0.0013 | -0.0839 | -0.0015 | -0.0027 | -0.0169 | 0.0035 | -0.0086 | -0.0040 | 0.0113 |
| | 0.0007 | 0.0314 | 0.1240 | 0.0011 | -0.0066 | 0.0140 | 0.0034 | -0.0030 | 0.0123 |
| Var | 0.0067 | 0.0048 | 0.0044 | 0.0249 | 0.0048 | 0.0046 | 0.2464 | 0.0053 | 0.0104 |
| (p=3) | 0.0049 | 0.0057 | 0.0059 | 0.0173 | 0.0051 | 0.0049 | 0.1668 | 0.0053 | 0.0105 |
| | 0.0068 | 0.0049 | 0.0045 | 0.0198 | 0.0050 | 0.0048 | 0.1921 | 0.0053 | 0.0105 |
| MSE | 0.0067 | 0.0059 | 0.0200 | 0.0249 | 0.0050 | 0.0058 | 0.2464 | 0.0053 | 0.0106 |
| (p=3) | 0.0049 | 0.0128 | 0.0059 | 0.0173 | 0.0054 | 0.0049 | 0.1669 | 0.0053 | 0.0106 |
| | 0.0068 | 0.0059 | 0.0199 | 0.0198 | 0.0050 | 0.0050 | 0.1922 | 0.0053 | 0.0106 |
| One-Factor Model (p=4) | | | | | | | | | |
| | -0.0000 | 0.0519 | 0.1350 | -0.0000 | 0.0180 | 0.0359 | 0.0009 | -0.0021 | 0.0983 |
| Bias | 0.0007 | -0.0660 | 0.0078 | 0.0014 | -0.0133 | 0.0041 | 0.0050 | -0.0034 | 0.0969 |
| (p=4) | -0.0007 | 0.0517 | 0.1348 | -0.0009 | -0.0132 | 0.0041 | -0.0041 | -0.0021 | 0.0983 |
| | -0.0003 | -0.0657 | 0.0081 | -0.0007 | -0.0028 | 0.0147 | -0.0021 | -0.0028 | 0.0976 |
| | 0.0076 | 0.0041 | 0.0035 | 0.0305 | 0.0045 | 0.0044 | 0.3668 | 0.0052 | 0.0165 |
| Var | 0.0049 | 0.0054 | 0.0052 | 0.0175 | 0.0049 | 0.0047 | 0.2157 | 0.0052 | 0.0166 |
| (p=4) | 0.0077 | 0.0041 | 0.0035 | 0.0186 | 0.0049 | 0.0047 | 0.3737 | 0.0052 | 0.0165 |
| | 0.0051 | 0.0053 | 0.0051 | 0.0211 | 0.0048 | 0.0046 | 0.2775 | 0.0052 | 0.0165 |
| | 0.0076 | 0.0068 | 0.0218 | 0.0305 | 0.0049 | 0.0057 | 0.3668 | 0.0052 | 0.0261 |
| MSE | 0.0049 | 0.0097 | 0.0052 | 0.0175 | 0.0050 | 0.0047 | 0.2158 | 0.0053 | 0.0259 |
| (p=4) | 0.0077 | 0.0067 | 0.0217 | 0.0186 | 0.0050 | 0.0047 | 0.3737 | 0.0052 | 0.0261 |
| | 0.0051 | 0.0096 | 0.0052 | 0.0211 | 0.0048 | 0.0048 | 0.2775 | 0.0052 | 0.0260 |
| One-Factor Model (p=5) | | | | | | | | | |
| | 0.0010 | 0.0404 | 0.1019 | 0.0021 | 0.0166 | 0.0306 | 0.0064 | -0.0005 | 0.0511 |
| Bias | 0.0000 | -0.0755 | -0.0209 | 0.0000 | -0.0146 | -0.0010 | -0.0000 | -0.0029 | 0.0487 |
| (p=5) | -0.0004 | 0.0405 | 0.1020 | -0.0009 | -0.0145 | -0.0010 | -0.0029 | -0.0029 | 0.0487 |
| | -0.0008 | -0.0750 | -0.0205 | -0.0017 | -0.0041 | 0.0096 | -0.0056 | -0.0020 | 0.0495 |
| | 0.0006 | 0.0402 | 0.1017 | 0.0009 | 0.0062 | 0.0200 | 0.0025 | -0.0012 | 0.0504 |
| | 0.0086 | 0.0037 | 0.0032 | 0.0360 | 0.0043 | 0.0042 | 0.3572 | 0.0049 | 0.0131 |
| Var | 0.0053 | 0.0050 | 0.0048 | 0.0191 | 0.0046 | 0.0045 | 0.1857 | 0.0050 | 0.0132 |
| (p=5) | 0.0085 | 0.0037 | 0.0033 | 0.0193 | 0.0047 | 0.0045 | 0.1871 | 0.0050 | 0.0132 |
| | 0.0054 | 0.0051 | 0.0048 | 0.0231 | 0.0045 | 0.0044 | 0.2263 | 0.0050 | 0.0131 |
| | 0.0084 | 0.0037 | 0.0032 | 0.0281 | 0.0044 | 0.0043 | 0.2768 | 0.0049 | 0.0131 |
| | 0.0086 | 0.0053 | 0.0136 | 0.0360 | 0.0046 | 0.0051 | 0.3572 | 0.0049 | 0.0157 |
| MSE | 0.0053 | 0.0107 | 0.0052 | 0.0191 | 0.0049 | 0.0045 | 0.1857 | 0.0050 | 0.0155 |
| (p=5) | 0.0085 | 0.0054 | 0.0137 | 0.0193 | 0.0049 | 0.0045 | 0.1871 | 0.0050 | 0.0155 |
| | 0.0054 | 0.0107 | 0.0052 | 0.0231 | 0.0046 | 0.0045 | 0.2263 | 0.0050 | 0.0156 |
| | 0.0084 | 0.0053 | 0.0136 | 0.0281 | 0.0045 | 0.0047 | 0.2768 | 0.0049 | 0.0156 |

Table A3.2: Simulation results for the one-factor models (non-identical regression coefficients)

| | Moderate multicollinearity | | | Strong multicollinearity | | | High multicollinearity | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 |
| One-Factor Model (p=3) | | | | | | | | | |
| Bias | 0.0010 | 0.5314 | 0.6240 | 0.0019 | 0.5106 | 0.5316 | 0.0055 | 0.4981 | 0.5134 |
| (p=3) | -0.0013 | -0.0841 | -0.0017 | -0.0027 | -0.0204 | -0.0001 | -0.0086 | -0.0042 | 0.0111 |
| | 0.0007 | -0.4685 | -0.3758 | 0.0011 | -0.5101 | -0.4896 | 0.0034 | -0.5032 | -0.4879 |
| Var | 0.0067 | 0.0044 | 0.0039 | 0.0249 | 0.0048 | 0.0046 | 0.2464 | 0.0053 | 0.0104 |
| (p=3) | 0.0049 | 0.0059 | 0.0061 | 0.0173 | 0.0051 | 0.0050 | 0.1668 | 0.0053 | 0.0105 |
| | 0.0068 | 0.0059 | 0.0056 | 0.0198 | 0.0051 | 0.0049 | 0.1921 | 0.0053 | 0.0105 |
| MSE | 0.0067 | 0.2867 | 0.3934 | 0.0249 | 0.2655 | 0.2872 | 0.2464 | 0.2534 | 0.2740 |
| (p=3) | 0.0049 | 0.0130 | 0.0062 | 0.0173 | 0.0055 | 0.0050 | 0.1669 | 0.0054 | 0.0106 |
| | 0.0068 | 0.2253 | 0.1468 | 0.0198 | 0.2653 | 0.2446 | 0.1922 | 0.2585 | 0.2486 |
| One-Factor Model (p=4) | | | | | | | | | |
| | -0.0000 | 0.7992 | 0.9018 | -0.0000 | 0.7686 | 0.7909 | 0.0009 | 0.7473 | 0.8729 |
| Bias | 0.0007 | 0.1536 | 0.2447 | 0.0014 | 0.2295 | 0.2512 | 0.0050 | 0.2457 | 0.3711 |
| (p=4) | -0.0007 | -0.2008 | -0.0981 | -0.0009 | -0.2704 | -0.2487 | -0.0041 | -0.2527 | -0.1271 |
| | -0.0003 | -0.8454 | -0.7542 | -0.0007 | -0.7574 | -0.7355 | -0.0021 | -0.7534 | -0.6280 |
| | 0.0076 | 0.0059 | 0.0051 | 0.0305 | 0.0070 | 0.0068 | 0.3668 | 0.0080 | 0.0256 |
| Var | 0.0049 | 0.0081 | 0.0078 | 0.0175 | 0.0075 | 0.0073 | 0.2157 | 0.0081 | 0.0258 |
| (p=4) | 0.0077 | 0.0068 | 0.0061 | 0.0186 | 0.0076 | 0.0073 | 0.3737 | 0.0080 | 0.0256 |
| | 0.0051 | 0.0095 | 0.0095 | 0.0211 | 0.0074 | 0.0072 | 0.2775 | 0.0081 | 0.0257 |
| | 0.0076 | 0.6446 | 0.8184 | 0.0305 | 0.5978 | 0.6323 | 0.3668 | 0.5665 | 0.7875 |
| MSE | 0.0049 | 0.0317 | 0.0677 | 0.0175 | 0.0602 | 0.0704 | 0.2158 | 0.0685 | 0.1635 |
| (p=4) | 0.0077 | 0.0471 | 0.0157 | 0.0186 | 0.0807 | 0.0692 | 0.3737 | 0.0719 | 0.0418 |
| | 0.0051 | 0.7241 | 0.5783 | 0.0211 | 0.5811 | 0.5482 | 0.2775 | 0.5758 | 0.4200 |
| One-Factor Model (p=5) | | | | | | | | | |
| | 0.0010 | 0.4317 | 0.4868 | 0.0021 | 0.4111 | 0.4236 | 0.0064 | 0.3993 | 0.4458 |
| Bias | 0.0000 | -0.1720 | -0.1231 | 0.0000 | -0.1169 | -0.1047 | -0.0000 | -0.1028 | -0.0565 |
| (p=5) | -0.0004 | -0.5680 | -0.5130 | -0.0009 | -0.6168 | -0.6047 | -0.0029 | -0.6028 | -0.5565 |
| | -0.0008 | -0.1716 | -0.1227 | -0.0017 | -0.1074 | -0.0952 | -0.0056 | -0.1021 | -0.0557 |
| | 0.0006 | 0.4315 | 0.4866 | 0.0009 | 0.4017 | 0.4141 | 0.0025 | 0.3987 | 0.4451 |
| | 0.0086 | 0.0030 | 0.0026 | 0.0360 | 0.0036 | 0.0034 | 0.3572 | 0.0040 | 0.0106 |
| Var | 0.0053 | 0.0042 | 0.0041 | 0.0191 | 0.0038 | 0.0037 | 0.1857 | 0.0041 | 0.0107 |
| (p=5) | 0.0085 | 0.0033 | 0.0030 | 0.0193 | 0.0038 | 0.0037 | 0.1871 | 0.0041 | 0.0107 |
| | 0.0054 | 0.0042 | 0.0041 | 0.0231 | 0.0037 | 0.0036 | 0.2263 | 0.0041 | 0.0107 |
| | 0.0084 | 0.0029 | 0.0026 | 0.0281 | 0.0036 | 0.0035 | 0.2768 | 0.0040 | 0.0106 |
| | 0.0086 | 0.1894 | 0.2396 | 0.0360 | 0.1726 | 0.1829 | 0.3572 | 0.1635 | 0.2093 |
| MSE | 0.0053 | 0.0338 | 0.0192 | 0.0191 | 0.0175 | 0.0147 | 0.1857 | 0.0146 | 0.0139 |
| (p=5) | 0.0085 | 0.3260 | 0.2661 | 0.0193 | 0.3843 | 0.3694 | 0.1871 | 0.3675 | 0.3203 |
| | 0.0054 | 0.0337 | 0.0191 | 0.0231 | 0.0153 | 0.0127 | 0.2263 | 0.0145 | 0.0138 |
| | 0.0084 | 0.1892 | 0.2393 | 0.0281 | 0.1650 | 0.1750 | 0.2768 | 0.1630 | 0.2087 |

Table A3.3: Simulation results for the two-factor models (identical regression coefficients)

| | Moderate multicollinearity | | | Strong multicollinearity | | | High multicollinearity | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 |
| | | | | Two-Factor Models (p=5) | | | | | |
| | 0.0014 | 0.0307 | 0.1227 | 0.0026 | 0.0073 | 0.0236 | 0.0075 | -0.0021 | 0.1096 |
| Bias | 0.0010 | -0.0864 | -0.0047 | 0.0019 | -0.0243 | -0.0085 | 0.0057 | -0.0040 | 0.1074 |
| (p=5) | -0.0017 | 0.0294 | 0.1214 | -0.0039 | 0.0069 | 0.0232 | -0.0127 | -0.0021 | 0.1095 |
| | -0.0006 | -0.0285 | 0.0072 | -0.0015 | -0.0031 | 0.0035 | -0.0047 | -0.0014 | 0.1774 |
| | 0.0006 | -0.0273 | 0.0086 | 0.0014 | -0.0027 | 0.0040 | 0.0046 | -0.0015 | 0.1773 |
| | 0.0071 | 0.0086 | 0.0228 | 0.0319 | 0.0054 | 0.0058 | 0.3140 | 0.0053 | 0.0384 |
| MSE | 0.0050 | 0.0175 | 0.0108 | 0.0185 | 0.0064 | 0.0058 | 0.1788 | 0.0054 | 0.0382 |
| (p=5) | 0.0070 | 0.0083 | 0.0221 | 0.0315 | 0.0053 | 0.0057 | 0.3108 | 0.0053 | 0.0384 |
| | 0.0044 | 0.0291 | 0.0336 | 0.0186 | 0.0065 | 0.0066 | 0.1798 | 0.0056 | 0.0691 |
| | 0.0045 | 0.0288 | 0.0341 | 0.0188 | 0.0064 | 0.0064 | 0.1803 | 0.0055 | 0.0690 |
| | | | | Two-Factor Models (p=6) | | | | | |
| | 0.0005 | 0.0288 | 0.1210 | 0.0013 | 0.0061 | 0.0224 | 0.0046 | -0.0032 | 0.1175 |
| | -0.0008 | -0.0867 | -0.0048 | -0.0014 | -0.0252 | -0.0094 | -0.0042 | -0.0051 | 0.1155 |
| Bias | -0.0002 | 0.0271 | 0.1191 | -0.0004 | 0.0054 | 0.0217 | -0.0009 | -0.0034 | 0.1173 |
| (p=6) | 0.0010 | 0.0318 | 0.1335 | 0.0022 | 0.0068 | 0.0251 | 0.0069 | -0.0026 | 0.1169 |
| | -0.0013 | -0.0275 | 0.0688 | -0.0025 | -0.0142 | 0.0037 | -0.0080 | -0.0038 | 0.1156 |
| | 0.0002 | -0.0258 | 0.0706 | 0.0004 | -0.0032 | 0.0150 | 0.0012 | -0.0029 | 0.1166 |
| | 0.0071 | 0.0077 | 0.0218 | 0.0320 | 0.0052 | 0.0055 | 0.3153 | 0.0052 | 0.0376 |
| | 0.0050 | 0.0167 | 0.0102 | 0.0185 | 0.0063 | 0.0057 | 0.1787 | 0.0053 | 0.0374 |
| MSE | 0.0070 | 0.0076 | 0.0213 | 0.0317 | 0.0051 | 0.0054 | 0.3116 | 0.0052 | 0.0375 |
| (p=6) | 0.0067 | 0.0080 | 0.0245 | 0.0293 | 0.0053 | 0.0057 | 0.2897 | 0.0052 | 0.0348 |
| | 0.0059 | 0.0086 | 0.0133 | 0.0216 | 0.0058 | 0.0054 | 0.2094 | 0.0053 | 0.0346 |
| | 0.0058 | 0.0086 | 0.0136 | 0.0253 | 0.0054 | 0.0055 | 0.2473 | 0.0053 | 0.0348 |
| | | | | Two-Factor Models (p=7) | | | | | |
| | -0.0006 | 0.0273 | 0.1194 | -0.0011 | 0.0063 | 0.0227 | -0.0033 | -0.0022 | 0.1352 |
| | -0.0002 | -0.0871 | -0.0052 | -0.0001 | -0.0245 | -0.0086 | -0.0001 | -0.0038 | 0.1335 |
| Bias | 0.0003 | 0.0275 | 0.1197 | 0.0008 | 0.0064 | 0.0227 | 0.0029 | -0.0022 | 0.1352 |
| (p=7) | -0.0006 | 0.0231 | 0.0946 | -0.0012 | 0.0122 | 0.0281 | -0.0035 | -0.0021 | 0.0964 |
| | 0.0001 | -0.0340 | 0.0335 | 0.0001 | -0.0084 | 0.0072 | 0.0003 | -0.0033 | 0.0951 |
| | 0.0001 | -0.0349 | 0.0326 | 0.0003 | 0.0018 | 0.0175 | 0.0009 | -0.0026 | 0.0957 |
| | 0.0003 | 0.0214 | 0.0928 | 0.0006 | -0.0195 | -0.0041 | 0.0021 | -0.0042 | 0.0941 |
| | 0.0071 | 0.0076 | 0.0211 | 0.0320 | 0.0052 | 0.0056 | 0.3156 | 0.0052 | 0.0397 |
| | 0.0052 | 0.0166 | 0.0098 | 0.0192 | 0.0063 | 0.0057 | 0.1854 | 0.0053 | 0.0395 |
| MSE | 0.0071 | 0.0076 | 0.0212 | 0.0322 | 0.0052 | 0.0056 | 0.3174 | 0.0052 | 0.0397 |
| (p=7) | 0.0081 | 0.0069 | 0.0152 | 0.0334 | 0.0051 | 0.0056 | 0.3307 | 0.0051 | 0.0238 |
| | 0.0064 | 0.0089 | 0.0091 | 0.0223 | 0.0055 | 0.0053 | 0.2174 | 0.0052 | 0.0237 |
| | 0.0067 | 0.0092 | 0.0093 | 0.0284 | 0.0053 | 0.0054 | 0.2795 | 0.0051 | 0.0238 |
| | 0.0081 | 0.0068 | 0.0149 | 0.0189 | 0.0060 | 0.0055 | 0.1829 | 0.0052 | 0.0235 |

Table A3.4: Simulation results for the two-factor models (non-identical regression coefficients)

| | Moderate multicollinearity | | | Strong multicollinearity | | | High multicollinearity | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 | OLS | FAR1 | FAR2 |
| | Two-Factor Models (p=5) | | | | | | | | |
| | 0.0014 | 0.5301 | 0.6222 | 0.0026 | 0.5075 | 0.5238 | 0.0075 | 0.4980 | 0.6096 |
| Bias | 0.0010 | -0.0872 | -0.0054 | 0.0019 | -0.0246 | -0.0088 | 0.0057 | -0.0041 | 0.1073 |
| (p=5) | -0.0017 | -0.4697 | -0.3776 | -0.0039 | -0.4930 | -0.4768 | -0.0127 | -0.5021 | -0.3905 |
| | -0.0006 | 0.1796 | 0.2598 | -0.0015 | 0.2426 | 0.2575 | -0.0047 | 0.2468 | 0.6491 |
| | 0.0006 | -0.3051 | -0.2241 | 0.0014 | -0.2556 | -0.2407 | 0.0046 | -0.2533 | 0.1490 |
| | 0.0071 | 0.2992 | 0.4068 | 0.0319 | 0.2646 | 0.2813 | 0.3140 | 0.2535 | 0.3999 |
| MSE | 0.0050 | 0.0354 | 0.0310 | 0.0185 | 0.0093 | 0.0087 | 0.1788 | 0.0056 | 0.0400 |
| (p=5) | 0.0070 | 0.2398 | 0.1633 | 0.0315 | 0.2501 | 0.2342 | 0.3108 | 0.2576 | 0.1808 |
| | 0.0044 | 0.1414 | 0.2012 | 0.0186 | 0.0871 | 0.0948 | 0.1798 | 0.0868 | 0.6036 |
| | 0.0045 | 0.2468 | 0.2316 | 0.0188 | 0.0950 | 0.0879 | 0.1803 | 0.0900 | 0.2043 |
| | Two-Factor Models (p=6) | | | | | | | | |
| | 0.0005 | 0.5292 | 0.6214 | 0.0013 | 0.5066 | 0.5229 | 0.0046 | 0.4970 | 0.6179 |
| | -0.0008 | -0.0863 | -0.0044 | -0.0014 | -0.0252 | -0.0094 | -0.0042 | -0.0051 | 0.1156 |
| Bias | -0.0002 | -0.4732 | -0.3813 | -0.0004 | -0.4949 | -0.4786 | -0.0009 | -0.5035 | -0.3827 |
| (p=6) | 0.0010 | 0.5683 | 0.8217 | 0.0022 | 0.5151 | 0.5609 | 0.0069 | 0.4933 | 0.7919 |
| | -0.0013 | -0.0772 | 0.1627 | -0.0025 | -0.0369 | 0.0080 | -0.0080 | -0.0094 | 0.2890 |
| | 0.0002 | -0.5744 | -0.3344 | 0.0004 | -0.5103 | -0.4649 | 0.0012 | -0.5076 | -0.2092 |
| | 0.0071 | 0.2979 | 0.4073 | 0.0320 | 0.2638 | 0.2805 | 0.3153 | 0.2523 | 0.4080 |
| | 0.0050 | 0.0351 | 0.0330 | 0.0185 | 0.0097 | 0.0091 | 0.1787 | 0.0056 | 0.0399 |
| MSE | 0.0070 | 0.2429 | 0.1676 | 0.0317 | 0.2519 | 0.2360 | 0.3116 | 0.2588 | 0.1725 |
| (p=6) | 0.0067 | 0.3525 | 0.7002 | 0.0293 | 0.2947 | 0.3429 | 0.2897 | 0.2748 | 0.7553 |
| | 0.0059 | 0.0404 | 0.0621 | 0.0216 | 0.0323 | 0.0300 | 0.2094 | 0.0318 | 0.2124 |
| | 0.0058 | 0.3664 | 0.1497 | 0.0253 | 0.2908 | 0.2455 | 0.2473 | 0.2893 | 0.1723 |
| | Two-Factor Models (p=7) | | | | | | | | |
| | -0.0006 | 0.5273 | 0.6194 | -0.0011 | 0.5065 | 0.5229 | -0.0033 | 0.4978 | 0.6353 |
| | -0.0002 | -0.0867 | -0.0047 | -0.0001 | -0.0243 | -0.0085 | -0.0001 | -0.0038 | 0.1336 |
| Bias | 0.0003 | -0.4713 | -0.3790 | 0.0008 | -0.4933 | -0.4770 | 0.0029 | -0.5022 | -0.3647 |
| (p=7) | -0.0006 | -0.7200 | -0.6306 | -0.0012 | -0.7292 | -0.7093 | -0.0035 | -0.7522 | -0.6292 |
| | 0.0001 | -0.2921 | -0.2077 | 0.0001 | -0.2552 | -0.2357 | 0.0003 | -0.2537 | -0.1308 |
| | 0.0001 | 0.2062 | 0.2904 | 0.0003 | 0.2574 | 0.2772 | 0.0009 | 0.2471 | 0.3700 |
| | 0.0003 | 0.7763 | 0.8655 | 0.0006 | 0.7306 | 0.7500 | 0.0021 | 0.7451 | 0.8680 |
| | 0.0071 | 0.2860 | 0.3919 | 0.0320 | 0.2621 | 0.2788 | 0.3156 | 0.2531 | 0.4253 |
| | 0.0052 | 0.0189 | 0.0126 | 0.0192 | 0.0068 | 0.0062 | 0.1854 | 0.0054 | 0.0398 |
| MSE | 0.0071 | 0.2314 | 0.1534 | 0.0322 | 0.2489 | 0.2330 | 0.3174 | 0.2574 | 0.1547 |
| (p=7) | 0.0081 | 0.5278 | 0.4068 | 0.0334 | 0.5393 | 0.5103 | 0.3307 | 0.5737 | 0.4183 |
| | 0.0064 | 0.0961 | 0.0540 | 0.0223 | 0.0732 | 0.0634 | 0.2174 | 0.0723 | 0.0397 |
| | 0.0067 | 0.0532 | 0.0951 | 0.0284 | 0.0742 | 0.0845 | 0.2795 | 0.0690 | 0.1595 |
| | 0.0081 | 0.6111 | 0.7571 | 0.0189 | 0.5421 | 0.5705 | 0.1829 | 0.5632 | 0.7760 |