REAL ANALYSIS: METHODS OF PROOF

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1. About

Mathematics is deductive science. The mathematicians spent a large amount of time performing calculations of special instances of phenomena they are studying in the hope of discovering unifying principles. The first formulate the conjectures and try to prove them using axioms of mathematics and already proven results (theorems).
Purpose
Main purpose of this chapter is to give insight of techniques of proof and understand the deductive process.

2. Statements

Statements are the meaningful sentences which are either true or false.
Examples:

(1) Sun moves round the earth.

(2) It is raining today.

(3) 5=2

(4) Sum of two even numbers is even.

Statements are denoted by small English alphabets, such as, p, q etc. Tautologies and falsities Some statements are always true, called tautologies.
Some statements are always false, called falsities.
Examples:

(i) p: 1+1=2

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(ii) q: Elephant eats meat.

Logically equivalent Two statements are said to be logically equivalent if one is true (or false) exactly when other is true (or false).

Forming new statements

3. Forming new statements

New statements can be formed from given ones by using logical connectives

Connectives If p and q are statements, then

- their **conjunction** is the statement p *and* q denoted by $p \wedge q$ which is true when both p and q are true, false otherwise. Clearly, $p \wedge q \equiv q \wedge p$.

- their **disjunction** is the statement p *or* q denoted by $p \vee q$ which is true if at least one of p and q is true, false when they are both false. Clearly, $p \vee q \equiv q \vee p$.

Negation If p is a statement, then its **negation** is the statement *not* p denoted by $\sim p$ which is true if p is false and false when p is true. Clearly, $p = \sim (\sim p)$.

Negation, conjunction and disjunction are related by *De Morgan's* laws as follows:

- $\sim (p \wedge q) \equiv (\sim p \vee \sim q)$

- $\sim (p \vee q) \equiv (\sim p \wedge \sim q)$

Implications Another way of forming new statements is the **implication** (or **conditional**) statement, denoted by $p \Rightarrow q$ or (if p then q) or (p implies q).

Here, p is called **hypothesis** and q is called **conclusion**.

if $p \Rightarrow q$ is logically equivalent to: $p \Rightarrow q \equiv (\sim p) \vee q \equiv \sim (p \wedge (\sim q))$

This statement is true unless p is true and q is false.

Contrapositive $p \Rightarrow q$ is logically equivalent to $\sim q \Rightarrow \sim p$ called **contrapositive** of the implication.

Example: If I am in Jaipur, I am in India. If I am not in India, I am not in Jaipur.

Converse-only if $q \Rightarrow p$ is converse of the implication $p \Rightarrow q$.

Example: If I am in India, I am in Jaipur.

Biconditional-if and only if Double implication $p \Rightarrow q$ and $q \Rightarrow p$ together called **biconditional** statement denoted by $p \Leftrightarrow q$.

We can write: $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

4. Quantifiers

Very often mathematical statements involves expressions such as 'for all', 'there exists', for every', 'for some', 'there are' and so on. Universal quantifier A statement 'for every (all)' can be made using universal quantifier denoted by $\forall$.

Existential quantifier A statement that involves 'there exists' make use of existential quantifier denoted by $\exists$.

Example: There exists integer x such that $x^2 = 1$.

The two quantifiers can be used in succession but their order of appearance is very important.

Example: $\forall x \exists y (x+y = 0)$ means for every integer x, there exists an integer y such that x+y=0.

$\exists y \forall x (x + y = 0)$ means there exists integer y such that for every integer x, x+y=0.

Negating the quantifiers

- $\sim (\forall x)P \equiv (\exists x) \sim P$
- $\sim (\exists y)P \equiv (\forall y) \sim P$.

Thus, to show that a property P is false for every element x in some set, it is enough to produce a **counter example** by showing the existence of an element x for which the property is false.

Example: Show that (0,1) is not complete.

To show that, it is false that there exists an element y in some set that satisfies a certain property P, we need to show that every element y in the set fails to have that property.

5. Methods of Proof

There are two major types of methods of proof

(1) Direct proofs

(2) Indirect proofs

Let the theorem be $p \Rightarrow q$ means whenever the 'hypothesis' p is true, q is true.

Definition 1 (Direct proofs)**.** Direct proof involves the construction of a string of statements $R_1, R_2, ..., R_n$ such that
$p \Rightarrow R_1$, $R_1 \Rightarrow R_2$,...,$R_n \Rightarrow q$
(Using the *law of syllogism*)

The direct proofs require insight, intuition and considerable effort. Example: Square of an odd integer is also an odd integer.

Remark 1. Q.E.D. stand for **quod erat demonstrandum** which is Latin for *which is to be demonstrated.*

Definition 2 (Indirect Proof)**.**

(1) Contrapositive proofs: To show $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$.

(2) Proof by contradiction: $p \Rightarrow q$ is established by showing that $p \wedge (\sim q)$ implies a contradiction.

Example 1: If n is an integer and n^2 is even, then n is even.
Example 2: No rational number satisfies $x^2 = 2$.

References

[1] Bartle R.G., Sherbert D. R., Introduction to real analysis, 4th Ed., Wiley publication, 2016.