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OPTIMAL DYNAMIC PORTFOLIO SELECTION: MULTIPERIOD MEAN-VARIANCE FORMULATION

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The mean-variance formulation by Markowitz in the 1950s paved a foundation for modern portfolio selection analysis in a single period. This paper considers an analytical optimal solution to the mean-variance formulation in multiperiod portfolio selection. Specifically, analytical optimal portfolio policy and analytical expression of the mean-variance efficient frontier are derived in this paper for the multiperiod mean-variance formulation. An efficient algorithm is also proposed for finding an optimal portfolio policy to maximize a utility function of the expected value and the variance of the terminal wealth.

KEY WORDS: multiperiod portfolio selection, multiperiod mean-variance formulation, utility function

1. INTRODUCTION

Portfolio selection is to seek the best allocation of wealth among a basket of securities. The mean-variance formulation by Markowitz (1959, 1989) provides a fundamental basis for portfolio selection in a single period. Analytical expression of the mean-variance efficient frontier in single-period portfolio selection was derived in Markowitz (1956) and Merton (1972). The problem of multiperiod portfolio selection has been studied by Smith (1967); Chen, Jen, and Zionts (1971), Mossin (1968), Merton (1969, 1990), Samuelson (1969), Fama (1970), Hakansson (1971a, 1971b), Elton and Gruber (1974a, 1974b, 1975), Winkler and Barry (1975), Francis (1976), Dumas and Luciano (1991), Östermark (1991), Grauer and Hakansson (1993) and Pliska (1997). Enormous difficulty was reported by Chen et al. (1971) in finding optimal solutions for a multiperiod mean-variance formulation. The literature in multiperiod portfolio selection has been dominated by the results of maximizing expected utility functions of the terminal wealth and/or multiperiod consumption. Specifically, investment situations where the utility functions are of power form, logarithm function, exponential function, or quadratic form have been extensively investigated in the literature.

To our knowledge, no analytical or efficient numerical method for finding the optimal portfolio policy for the multiperiod mean-variance formulation and determining the mean-variance efficient frontier has been reported in the literature. In this sense, the concept of the Markowitz's mean-variance formulation has not been fully utilized in multiperiod portfolio selection. This paper represents an extension of the existing literature

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to capture the spirit of risk management in dynamic portfolio selection. The analytical result (Markowitz 1956; Merton 1972) in single-period portfolio selection has been generalized in this paper to multiperiod portfolio selection. Analytical optimal portfolio policy is derived for the multiperiod mean-variance formulation along with the analytical expression of the mean-variance efficient frontier.

The organization of this paper is as follows. In Section 2, the mean-variance formulation for multiperiod portfolio selection is discussed. The analytical solution to the multiperiod mean-variance formulation is stated in Section 3. Detailed derivation of the analytical results is provided in Section 4. The mean-variance formulation for multiperiod portfolio selection is investigated in Section 5 for investment situations where there is a riskless asset. The multiperiod mean-variance formulation is then generalized in Section 6 to investment situations where a utility function of the expected terminal wealth and the risk is maximized. Three cases are studied in Section 7 and the paper concludes in Section 8 with a suggestion for further study.

2. MEAN-VARIANCE FORMULATION FOR MULTIPERIOD PORTFOLIO SELECTION

We consider a capital market with $(n+1)$ risky securities, with random rates of returns. An investor joins the market at time 0 with an initial wealth x_0. The investor can allocate his wealth among the $(n+1)$ assets. The wealth can be reallocated among the $(n+1)$ assets at the beginning of each of the following $(T-1)$ consecutive time periods. The rates of return of the risky securities at time period t within the planning horizon are denoted by a vector $\mathbf{e_t} = \left[e_t^0, e_t^1, \ldots, e_t^n\right]'$, where e_t^i is the random return for security i at time period t. It is assumed in this paper that vectors $\mathbf{e_t}$, $t = 0, 1, \ldots, T-1$, are statistically independent and return $\mathbf{e_t}$ has a known mean $\mathrm{E}(\mathbf{e_t}) = \left[\mathrm{E}\left(e_t^0\right), \mathrm{E}\left(e_t^1\right), \ldots, \mathrm{E}\left(e_t^n\right)\right]'$ and a known covariance

$$\mathrm{Cov}\,(\mathbf{e_t}) = \begin{bmatrix} \sigma_{t,00} & \cdots & \sigma_{t,0n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,0n} & \cdots & \sigma_{t,nn} \end{bmatrix}.$$

Let x_t be the wealth of the investor at the beginning of the tth period, and let u_t^i, $i = 1, 2, \ldots, n$, be the amount invested in the ith risky asset at the beginning of the tth time period. The amount investigated in the 0th risky asset at the beginning of the tth time period is equal to $x_t - \sum_{i=1}^n u_t^i$. An investor is seeking a best investment strategy, $\mathbf{u}_t = \left[u_t^1, u_t^2, \ldots, u_t^n\right]'$ for $t = 0, 1, 2, \ldots, T-1$, such that (i) the expected value of the terminal wealth x_T, $\mathrm{E}(x_T)$, is maximized if the variance of the terminal wealth, $\mathrm{Var}(x_T)$, is not greater than a preselected risk level, or (ii) the variance of the terminal wealth, $\mathrm{Var}(x_T)$, is minimized if the expected terminal wealth, $\mathrm{E}(x_T)$, is not smaller than a preselected level. Mathematically, a mean-variance formulation for multiperiod portfolio selection can be posed as one of the following two forms when security 0 is taken as a reference:

$$(1) \qquad (P1(\sigma)): \qquad \begin{aligned} &\max\ \mathrm{E}(x_T) \\ &\text{s.t. } \mathrm{Var}(x_T) \le \sigma \\ &x_{t+1} = \sum_{i=1}^n e_t^i u_t^i + \left(x_t - \sum_{i=1}^n u_t^i\right) e_t^0 \\ &\quad = e_t^0 x_t + \mathbf{P_t}'\mathbf{u_t} \qquad t = 0, 1, 2, \ldots, T-1 \end{aligned}$$

and

$$
\begin{aligned}
(2) \qquad (P2(\epsilon)): \qquad & \min \ \mathrm{Var}(x_T) \\
& \text{s.t. } \mathrm{E}(x_T) \geq \epsilon \\
& x_{t+1} = \sum_{i=1}^{n} e_t^i u_t^i + \left(x_t - \sum_{i=1}^{n} u_t^i \right) e_t^0 \\
& \quad = e_t^0 x_t + \mathbf{P_t}' \mathbf{u}_t \qquad t = 0, 1, 2, \ldots, T-1,
\end{aligned}
$$

where

$$
(3) \qquad \mathbf{P_t} = \left[P_t^1, P_t^2, \ldots, P_t^n \right]' = \left[(e_t^1 - e_t^0), (e_t^2 - e_t^0), \ldots, (e_t^n - e_t^0) \right]'.
$$

Notice that $\mathrm{E}(\mathbf{e_t}(\mathbf{e_t})') = \mathrm{Cov}(\mathbf{e_t}) + \mathrm{E}(\mathbf{e_t})\mathrm{E}(\mathbf{e_t}')$. It is reasonable to assume in this paper that $\mathrm{E}(\mathbf{e_t}(\mathbf{e_t})')$ is positive definite for all time periods; that is,

$$
(4) \qquad \mathrm{E}(\mathbf{e_t}(\mathbf{e_t})') = \begin{bmatrix} \mathrm{E}((e_t^0)^2) & \mathrm{E}(e_t^0 e_t^1) & \ldots & \mathrm{E}(e_t^0 e_t^n) \\ \mathrm{E}(e_t^1 e_t^0) & \mathrm{E}((e_t^1)^2) & \ldots & \mathrm{E}(e_t^1 e_t^n) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{E}(e_t^n e_t^0) & \mathrm{E}(e_t^n e_t^1) & \ldots & \mathrm{E}((e_t^n)^2) \end{bmatrix} > 0 \qquad \forall t = 0, 1, \ldots, T-1.
$$

The following is true from equation (4):

$$
\begin{aligned}
& \begin{bmatrix} \mathrm{E}((e_t^0)^2) & \mathrm{E}(e_t^0 \mathbf{P_t'}) \\ \mathrm{E}(e_t^0 \mathbf{P_t}) & \mathrm{E}(\mathbf{P_t} \mathbf{P_t'}) \end{bmatrix} \\
(5) \qquad = & \begin{bmatrix} 1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & \ldots & 1 \end{bmatrix} \mathrm{E}(\mathbf{e_t}(\mathbf{e_t})') \begin{bmatrix} 1 & -1 & \ldots & -1 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix} > 0 \qquad \forall t = 0, 1, \ldots, T-1.
\end{aligned}
$$

Furthermore, we have the following from equation (5):

$$
(6) \qquad \mathrm{E}(\mathbf{P_t} \mathbf{P_t'}) > 0 \qquad \forall t = 0, 1, \ldots, T-1
$$

and

$$
(7) \qquad \mathrm{E}((e_t^0)^2) - \mathrm{E}(e_t^0 \mathbf{P_t'}) \mathrm{E}^{-1}(\mathbf{P_t} \mathbf{P_t'}) \mathrm{E}(e_t^0 \mathbf{P_t}) > 0 \qquad \forall t = 0, 1, \ldots, T-1.
$$

One of the advantages of adopting problem formulation $(P1(\sigma))$ or $(P2(\epsilon))$ in multi-period portfolio selection over the expected utility approach is that formulation $(P1(\sigma))$ or $(P2(\epsilon))$ enables an investor to specify a risk level he can afford when he is seeking to maximize his expected terminal wealth or specify an expected terminal wealth he would like to achieve when he is seeking to minimize the corresponding risk. It is easier and more direct for investors to provide this kind of subjective information than for them to construct a utility function in terms of terminal wealth.

A multiperiod portfolio policy is an investment sequence,

$$\pi = \{\mu_0, \mu_1, \mu_2, \ldots, \mu_{T-1}\} = \left\{ \begin{bmatrix} \mu_0^1 \\ \mu_0^2 \\ \vdots \\ \mu_0^n \end{bmatrix}, \begin{bmatrix} \mu_1^1 \\ \mu_1^2 \\ \vdots \\ \mu_1^n \end{bmatrix}, \begin{bmatrix} \mu_2^1 \\ \mu_2^2 \\ \vdots \\ \mu_2^n \end{bmatrix}, \ldots, \begin{bmatrix} \mu_{T-1}^1 \\ \mu_{T-1}^2 \\ \vdots \\ \mu_{T-1}^n \end{bmatrix} \right\}. \tag{8}$$

More specifically, π is a feedback policy and μ_t maps the wealth at the beginning of the tth period, x_t, into a portfolio decision in the tth period,

$$\begin{bmatrix} u_t^1 \\ u_t^2 \\ \vdots \\ u_t^n \end{bmatrix} = \begin{bmatrix} \mu_t^1(x_t) \\ \mu_t^2(x_t) \\ \vdots \\ \mu_t^n(x_t) \end{bmatrix}. \tag{9}$$

A multiperiod portfolio policy, π^*, is said to be efficient if there exists no other multiperiod portfolio policy, π, such that $\mathrm{E}(x_T)|_\pi \geq \mathrm{E}(x_T)|_{\pi^*}$ and $\mathrm{Var}(x_T)|_\pi \leq \mathrm{Var}(x_T)|_{\pi^*}$ with at least one equality strictly. By varying the value of σ in $(P1(\sigma))$ or the value of ϵ in $(P2(\epsilon))$, the set of efficient multiperiod portfolio policies can be generated.

An equivalent formulation to either $(P1(\sigma))$ or $(P2(\epsilon))$ in generating efficient multiperiod portfolio policies is

$$\begin{aligned} (\mathrm{E}(w)): \quad & \max \mathrm{E}(x_T) - w\mathrm{Var}(x_T) \\ & \text{s.t. } x_{t+1} = e_t^0 x_t + \mathbf{P_t}'\mathbf{u}_t \qquad t = 0, 1, 2, \ldots, T-1, \end{aligned} \tag{10}$$

where $w \in [0, \infty)$. It is well known that if π^* solves $(\mathrm{E}(w))$, then π^* solves $(P1(\sigma))$ with $\sigma = \mathrm{Var}(x_T)|_{\pi^*}$ and π^* solves $(P2(\epsilon))$ with $\epsilon = \mathrm{E}(x_T)|_{\pi^*}$. Note the relationship $w = \partial\mathrm{E}(x_T)/\partial\mathrm{Var}(x_T)$ at the optimal solution of $(\mathrm{E}(w))$. Problem formulation $(\mathrm{E}(w))$ is preferable to be adopted in investment situations where an investor is able to specify his desirable trade-off between the expected terminal wealth and the associated risk.

3. ANALYTICAL SOLUTION TO THE MULTIPERIOD MEAN-VARIANCE FORMULATION

Analytical solutions to all three problems $(P1(\sigma))$, $(P2(\epsilon))$, and $(\mathrm{E}(w))$ are derived in this paper. The major results of the analytical optimal multiperiod portfolio policy and the analytical expression of the mean-variance efficient frontier are stated in this section; the detailed derivation of these results is given in the next section.

Define

$$B_t = \mathrm{E}\left(\mathbf{P_t}'\right) \mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}(\mathbf{P_t}) \qquad t = 0, 1, \ldots, T-1 \tag{11}$$

$$A_t^1 = \mathrm{E}\left(e_t^0\right) - \mathrm{E}\left(\mathbf{P_t}'\right) \mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}\left(e_t^0 \mathbf{P_t}\right) \qquad t = 0, 1, \ldots, T-1 \tag{12}$$

$$A_t^2 = \mathrm{E}\left((e_t^0)^2\right) - \mathrm{E}\left(e_t^0 \mathbf{P_t}'\right) \mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}\left(e_t^0 \mathbf{P_t}\right) \qquad t = 0, 1, \ldots, T-1 \tag{13}$$

$$B_t^1 = B_t \frac{\prod_{k=t+1}^{T-1} A_k^1}{2\prod_{k=t+1}^{T-1} A_k^2} \qquad t = 0, 1, \ldots, T-1 \tag{14}$$

$$B_t^2 = B_t \left(\frac{\prod_{k=t+1}^{T-1} A_k^1}{2\prod_{k=t+1}^{T-1} A_k^2}\right)^2 \qquad t = 0, 1, \ldots, T-1, \tag{15}$$

where in equations (14) and (15) $\prod_{k=T}^{T-1} A_k^i$, $i = 1, 2$, are defined to equal to one. Define further

$$\mu = \prod_{t=0}^{T-1} A_t^1 \tag{16}$$

$$\nu = \sum_{t=0}^{T-1} \left(\prod_{k=t+1}^{T-1} A_k^1\right) B_t^1 \tag{17}$$

$$\tau = \prod_{t=0}^{T-1} A_t^2 \tag{18}$$

$$a = \frac{\nu}{2} - \nu^2 \tag{19}$$

$$b = \frac{\mu\nu}{a} \tag{20}$$

$$c = \tau - \mu^2 - ab^2. \tag{21}$$

The optimal multiperiod portfolio policy for problem (E(w)) is specified by the following analytical form:

$$\begin{aligned} \mathbf{u}_t^* &= -\mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}\left(e_t^0 \mathbf{P_t}\right) x_t \\ &\quad + \frac{1}{2}\left(bx_0 + \frac{\nu}{2wa}\right) \left(\prod_{k=t+1}^{T-1} \frac{A_k^1}{A_k^2}\right) \mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}(\mathbf{P_t}) \\ &\qquad \forall t = 0, 1, \ldots, T-2, \end{aligned} \tag{22}$$

$$\begin{aligned} \mathbf{u}_{T-1}^* &= -\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right) \mathrm{E}\left(e_{T-1}^0 \mathbf{P}_{T-1}\right) x_{T-1} \\ &\quad + \frac{1}{2}\left(bx_0 + \frac{\nu}{2wa}\right) \mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right) \mathrm{E}(\mathbf{P}_{T-1}). \end{aligned} \tag{23}$$

The optimal multiperiod portfolio policy for problems $(P1(\sigma))$ and $(P2(\epsilon))$ is specified by the following analytical form:

$$\begin{aligned}\mathbf{u}_t^* &= -\mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right)\mathrm{E}\left(e_t^0\mathbf{P_t}\right)x_t \\ &\quad +\frac{1}{2}(bx_0+\frac{\nu}{2w^*a})\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)\mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right)\mathrm{E}(\mathbf{P_t}) \\ &\qquad \forall t=0,1,\ldots,T-2, \end{aligned} \tag{24}$$

$$\begin{aligned}\mathbf{u}_{T-1}^* &= -\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)\mathrm{E}\left(e_{T-1}^0\mathbf{P}_{T-1}\right)x_{T-1} \\ &\quad +\frac{1}{2}(bx_0+\frac{\nu}{2w^*a})\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)\mathrm{E}(\mathbf{P}_{T-1}), \end{aligned} \tag{25}$$

where

$$w^* = \begin{cases} \frac{\nu}{2\sqrt{a(\sigma-cx_0^2)}} & \text{when } (P1(\sigma)) \text{ is solved} \\ \frac{\nu^2}{2a[\epsilon-(\mu+b\nu)x_0]} & \text{when } (P2(\epsilon)) \text{ is solved.} \end{cases} \tag{26}$$

The mean-variance efficient frontier for problems $(P1(\sigma))$, $(P2(\epsilon))$, and $(\mathrm{E}(w))$ is specified by the following analytical form:

$$\begin{aligned}\mathrm{Var}(x_T) &= \frac{a}{\nu^2}[\mathrm{E}(x_T)-(\mu+b\nu)x_0]^2+cx_0^2 \\ &\qquad \text{for } \mathrm{E}(x_T)\geq(\mu+b\nu)x_0. \end{aligned} \tag{27}$$

With the analytical solution, the implementation of optimal multiperiod portfolio policy for problem $(P1(\sigma))$, $(P2(\epsilon))$, or $(\mathrm{E}(w))$ is straightforward. The optimal multiperiod portfolio policy consists of two terms and exhibits a decomposition property between the investor's risk attitude and his current wealth. The second term in u_t^* is dependent on the investor's risk attitude and is independent of his current wealth. It can be calculated off-line before the real investment process starts. The first term in u_t^* is dependent on the current wealth and is independent of the investor's risk attitude. It is calculated on-line at every time period when the current wealth is observed.

4. DERIVATION OF THE ANALYTICAL SOLUTION

All three problems $(P1(\sigma))$, $(P2(\epsilon))$, and $(\mathrm{E}(w))$ are difficult to solve directly due to their nonseparability in the sense of dynamic programming. Variance minimization has been a notorious problem in stochastic control. Let I^t be an information set available at time t and $I^{t-1}\subset I^t$, $\forall t$. A key observation is that although the expectation operator satisfies the smoothing property: $\mathrm{E}[\mathrm{E}(\cdot \mid I^j) \mid I^k] = \mathrm{E}(\cdot \mid I^k)$, $\forall j > k$, the variance operation does not: $\mathrm{Var}[\mathrm{Var}(\cdot \mid I^j) \mid I^k] \neq \mathrm{Var}(\cdot \mid I^k)$, $\forall j > k$.

The optimal multiperiod portfolio policy for problem $(\mathrm{E}(w))$ will first be derived. The solutions to problems $(P1(\sigma))$ and $(P2(\epsilon))$ will then be obtained based on the relationships between $(P1(\sigma))$, $(P2(\epsilon))$, and $(\mathrm{E}(w))$.

A solution scheme adopted in this paper is to embed problem $(\mathrm{E}(w))$ into a tractable auxiliary problem that is separable, investigate the relationship between the solution sets

of problem (E(w)) and the auxiliary problem, and search for the solution to the auxiliary problem that attains the optimum point of problem (E(w)).

Define $\prod_{\mathrm{E}}(w)$ to be the set of optimal solutions of problem (E(w)) with given w; that is,

$$\Pi_{\mathrm{E}}(w) = \{\pi \,|\, \pi \text{ is a maximizer of } (\mathrm{E}(w))\}. \tag{28}$$

Define

$$\begin{aligned} &\widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right) \\ &= \mathrm{E}(x_T) - w\mathrm{Var}(x_T) \\ &= -w\mathrm{E}\left(x_T^2\right) + \left[w\mathrm{E}^2(x_T) + \mathrm{E}(x_T)\right]. \end{aligned} \tag{29}$$

It is obvious that $\widetilde{U}$ is a convex function of $\mathrm{E}\left(x_T^2\right)$ and $\mathrm{E}(x_T)$. The following auxiliary problem is now constructed for (E(w)),

$$\begin{aligned} (A(\lambda, w)): \quad & \max \mathrm{E}\left\{-wx_T^2 + \lambda x_T\right\} \\ & \text{s.t. } x_{t+1} = e_t^0 x_t + \mathbf{P_t}'\mathbf{u}_t \qquad t = 0, 1, 2, \ldots, T-1. \end{aligned} \tag{30}$$

Prominent features of problem $(A(\lambda, w))$ are that $(A(\lambda, w))$ is of a separable structure in the sense of dynamic programming and the objective function of $(A(\lambda, w))$ is of a quadratic form while the system dynamic is of a linear form. Define $\Pi_A(\lambda, w)$ to be the set of the optimal solutions of problem $(A(\lambda, w))$ for given λ and w; that is,

$$\Pi_A(\lambda, w) = \{\pi \,|\, \pi \text{ is a maximizer of } (A(\lambda, w))\}. \tag{31}$$

Denote

$$\begin{aligned} d(\pi, w) &= \frac{\partial \widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)}{\partial \mathrm{E}(x_T)}\Big|_{\pi} \\ &= 1 + 2w\mathrm{E}(x_T)\,|_{\pi}. \end{aligned} \tag{32}$$

THEOREM 1. *For any* $\pi^* \in \Pi_E(w)$, $\pi^* \in \Pi_A(d(\pi^*, w), w)$.

Proof. By contradiction, assume that $\pi^* \notin \Pi_A(d(\pi^*, w), w)$. Then, there exists a π such that

$$\left[-w, d\left(\pi^*, w\right)\right] \begin{bmatrix} \mathrm{E}\left(x_T^2\right) \\ \mathrm{E}(x_T) \end{bmatrix}\Bigg|_{\pi} > \left[-w, d\left(\pi^*, w\right)\right] \begin{bmatrix} \mathrm{E}\left(x_T^2\right) \\ \mathrm{E}(x_T) \end{bmatrix}\Bigg|_{\pi^*}. \tag{33}$$

Notice equation (32) and

$$\frac{\partial \widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)}{\partial \mathrm{E}\left(x_T^2\right)} = -w. \tag{34}$$

Since $\widetilde{U}$ is a convex function of $\mathrm{E}\left(x_T^2\right)$ and $\mathrm{E}(x_T)$, the following property is satisfied,

$$\begin{aligned} &\widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)|_{\pi} \geq \widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)|_{\pi^*} \\ &+ \left[-w, d\left(\pi^*, w\right)\right]\left\{\begin{bmatrix} \mathrm{E}\left(x_T^2\right) \\ \mathrm{E}(x_T) \end{bmatrix}\Bigg|_{\pi} - \begin{bmatrix} \mathrm{E}\left(x_T^2\right) \\ \mathrm{E}(x_T) \end{bmatrix}\Bigg|_{\pi^*}\right\}. \end{aligned} \tag{35}$$

Combining equations (33) and (35) yields

$$\widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)|_{\pi} > \widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)|_{\pi^*}, \tag{36}$$

which contradicts the assumption of $\pi^* \in \Pi_{\mathrm{E}}(w)$. □

The implication of Theorem 1 is that the solution set for problem $(\mathrm{E}(w))$ is a subset of the solution set for problem $(A(\lambda, w))$. We can embed the nontractable primal problem $(\mathrm{E}(w))$ into a tractable auxiliary problem $(A(\lambda, w))$ with a quadratic utility function. The following theorem provides a necessary condition under which a solution of $(A(\lambda, w))$ constitutes an optimal multiperiod portfolio policy of $(\mathrm{E}(w))$.

THEOREM 2. *Assume* $\pi^* \in \Pi_A(\lambda^*, w)$. *A necessary condition for* $\pi^* \in \Pi_{\mathrm{E}}(w)$ *is* $\lambda^* = 1 + 2w\mathrm{E}(x_T)|_{\pi^*}$.

Proof. For a given w, the solution set of $(A(\lambda, w))$ can be parameterized by λ. In other words, each point in $\cup_\lambda \Pi_A(\lambda, w)$ can be expressed in terms of λ as $\{\mathrm{E}(x_T^2(\lambda, w)), \mathrm{E}(x_T(\lambda, w))\}$. Since $\Pi_{\mathrm{E}}(w) \subseteq \cup_\lambda \Pi_A(\lambda, w)$, problem $(\mathrm{E}(w))$ can be reduced in abstract to the following equivalent form:

$$\begin{aligned} &\max_\lambda \widetilde{U}\left(\mathrm{E}(x_T^2(\lambda, w)), \mathrm{E}(x_T(\lambda, w))\right) \\ &= \max_\lambda -w\mathrm{E}\left(x_T^2(\lambda, w)\right) + \left[w\mathrm{E}^2(x_T(\lambda, w)) + \mathrm{E}(x_T(\lambda, w))\right]. \end{aligned} \tag{37}$$

A first-order necessary optimality condition for optimal λ^* is

$$-w\frac{\partial \mathrm{E}\left(x_T^2(\lambda^*, w)\right)}{\partial \lambda} + [1 + 2w\mathrm{E}(x_T)|_{\pi^*}]\frac{\partial \mathrm{E}(x_T(\lambda^*, w))}{\partial \lambda} = 0. \tag{38}$$

On the other side, when $\pi^* \in \Pi_A(\lambda^*, w)$, we have the following from Reid and Citron (1971),

$$-w\frac{\partial \mathrm{E}\left(x_T^2(\lambda^*, w)\right)}{\partial \lambda} + \lambda^*\frac{\partial \mathrm{E}(x_T(\lambda^*, w))}{\partial \lambda} = 0. \tag{39}$$

Hence, the vector $[-w, (1 + 2w\mathrm{E}(x_T)|_{\pi^*})]$ is proportional to $\left[-w, \lambda^*\right]$. We must have $\lambda^* = 1 + 2w\mathrm{E}(x_T)|_{\pi^*}$. □

The optimal solution of the auxiliary problem $(A(\lambda, w))$ can be derived analytically using dynamic programming (Li, Chan and Ng (1998)). The optimal portfolio policy for auxiliary problem $(A(\lambda, w))$ at each time period t is of the following form,

$$\mathbf{u}_t^*(x_t; \gamma) = -\mathbf{K}_t x_t + \mathbf{v}_t(\gamma) \qquad t = 0, 1, \ldots, T-1, \tag{40}$$

where

$$\gamma = \frac{\lambda}{w} \tag{41}$$

$$\mathbf{K}_t = \mathrm{E}^{-1}\left(\mathbf{P_t}'\mathbf{P}_t\right)\mathrm{E}\left(e_t^0\mathbf{P_t}\right) \tag{42}$$

$$\mathbf{v}_t(\gamma) = \frac{\gamma}{2}\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)\mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P}_t'\right)\mathrm{E}(\mathbf{P_t})$$

$$t = 0, 1, 2, \ldots, T-2, \tag{43}$$

with the following boundary condition,

$$\mathbf{v}_{T-1}(\gamma) = \frac{\gamma}{2}\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)\mathrm{E}(\mathbf{P}_{T-1}), \tag{44}$$

where A_t^1 and A_t^2 are defined in equations (12) and (13), respectively. Substituting (40) into the equation of wealth dynamics yields the dynamics of the wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$x_{t+1}(\gamma) = \left(e_t^0 - \mathbf{P_t}'\mathbf{K}_t\right)x_t(\gamma) + \mathbf{P_t}'\mathbf{v}_t(\gamma). \tag{45}$$

Taking expectation on both sides of (45) and noticing the statistical independence between $(e_t^0, \mathbf{P_t})$ and x_t, we have the following recursive expression for the expected wealth between successive time periods under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$\mathrm{E}(x_{t+1}(\gamma)) = A_t^1\mathrm{E}(x_t(\gamma)) + \frac{\gamma}{2}\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)B_t, \tag{46}$$

where B_t is defined in equation (11). Squaring both sides of (45) yields

$$\begin{aligned} x_{t+1}^2(\gamma) = &\left[\left(e_t^0\right)^2 - 2e_t^0\mathbf{P_t}'\mathbf{K}_t + \mathbf{K}_t'\mathbf{P_t}\mathbf{P}_t'\mathbf{K}_t\right]x_t^2(\gamma) \\ &+2\left(e_t^0 - \mathbf{P_t}'\mathbf{K}_t\right)x_t(\gamma)\mathbf{P_t}'\mathbf{v}_t(\gamma) + \mathbf{v}_t(\gamma)'\mathbf{P_t}\mathbf{P_t}'\mathbf{v}_t(\gamma) \\ &t = 0, 1, \ldots, T-1. \end{aligned} \tag{47}$$

Taking expectation on both sides of the above equation and simplifying the resulting expression leads to the following recursive expression for the expected value of the squared wealth between successive time periods under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$\mathrm{E}\left(x_{t+1}^2(\gamma)\right) = A_t^2\mathrm{E}\left(x_t^2(\gamma)\right) + \frac{\gamma^2}{4}\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)^2 B_t. \tag{48}$$

Solving the two recursive equations (46) and (48) yields explicit expressions for the expected values of the terminal wealth and the square of the terminal wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$\mathrm{E}(x_T(\gamma)) = \mu x_0 + \nu\gamma \tag{49}$$

$$\mathrm{E}\left(x_T^2(\gamma)\right) = \tau x_0^2 + \frac{\nu}{2}\gamma^2, \tag{50}$$

where μ, ν, and τ are defined in equations (16), (17) and (18), respectively.

The variance of the terminal wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$ can be expressed in terms of γ using (49) and (50),

$$\begin{aligned} \mathrm{Var}(x_T(\gamma)) &= \mathrm{E}\left(x_T^2(\gamma)\right) - \mathrm{E}^2(x_T(\gamma)) \\ &= a(\gamma - bx_0)^2 + cx_0^2, \end{aligned} \tag{51}$$

where a, b, and c are defined in equations (19), (20), and (21), respectively.

It can be seen that the expected terminal wealth $\mathrm{E}(x_T(\gamma))$ is an increasing linear function of γ whereas the variance of the terminal wealth, $\mathrm{Var}(x_T(\gamma))$, is a quadratic function of γ. From (49) and (51), we can express $\widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right)$ as a function of γ,

$$\begin{aligned} &\widetilde{U}\left(\mathrm{E}\left(x_T^2\right), \mathrm{E}(x_T)\right) \\ &= \mu x_0 + \nu\gamma - w[a(\gamma - bx_0)^2 + cx_0^2]. \end{aligned} \tag{52}$$

Clearly, $\widetilde{U}$ is a concave function of γ. Differentiating (52) with respect to γ yields

$$\frac{d\widetilde{U}}{d\gamma} = \nu - 2wa(\gamma - bx_0). \tag{53}$$

The optimal γ must satisfy the optimality condition of $d\widetilde{U}/d\gamma = 0$; that is,

$$\gamma^* = bx_0 + \frac{\nu}{2wa}. \tag{54}$$

Substituting the optimal γ^* in (54) into equation (40) yields the optimal multiperiod portfolio policy for $(\mathrm{E}(w))$ specified in (22) and (23).

Substituting (54) into (49) and (51) yields the expression for the expected value and the variance of the terminal wealth on the efficient frontier in terms of w,

$$\mathrm{E}(x_T(w)) = (\mu + b\nu)x_0 + \frac{\nu^2}{2wa} \tag{55}$$

$$\mathrm{Var}(x_T(w)) = \frac{\nu^2}{4aw^2} + cx_0^2. \tag{56}$$

Given a problem $(P1(\sigma))$ or $(P2(\epsilon))$, we can first calculate the associated w in terms of σ or ϵ using (55) or (56) and then compute the corresponding optimal γ^* using (54). Substituting the optimal γ^* into (40) yields the optimal multiperiod portfolio policy for $(P1(\sigma))$ or $(P2(\epsilon))$ specified in (24), (25), and (26).

The mean-variance efficient frontier given in equation (27) can be obtained by eliminating the parameter w in (55) and (56).

5. INVESTMENT SITUATIONS WITH ONE RISKLESS ASSET

Investment situations where there exists a riskless asset can be regarded as a special case in the general multiperiod mean-variance formulation discussed above. Let the 0th security be riskless. In other words, we consider now a capital market with n risky assets and a riskless asset offering a sure rate of return. In this case e_t^0 equals to a constant s_t

and $\operatorname{cov}\left(e_t^0, e_t^i\right) = 0$, $i = 0, 1, \ldots, n$, $\forall t = 0, 1, \ldots, T-1$. The parameters defined in equations (11)–(15) now take the following forms:

$$B_t = \mathrm{E}\left(\mathbf{P}_\mathbf{t}'\right) \mathrm{E}^{-1}\left(\mathbf{P}_\mathbf{t}\mathbf{P}_\mathbf{t}'\right) \mathrm{E}(\mathbf{P}_\mathbf{t}) \qquad t = 0, 1, \ldots, T-1 \tag{57}$$

$$A_t^1 = s_t\,(1 - B_t) \qquad t = 0, 1, \ldots, T-1 \tag{58}$$

$$A_t^2 = s_t^2\,(1 - B_t) \qquad t = 0, 1, \ldots, T-1 \tag{59}$$

$$B_t^1 = \frac{B_t}{2\prod_{k=t+1}^{T-1} s_k} \qquad t = 0, 1, \ldots, T-1 \tag{60}$$

$$B_t^2 = \frac{B_t}{4(\prod_{k=t+1}^{T-1} s_k)^2} \qquad t = 0, 1, \ldots, T-1, \tag{61}$$

where in (60) and (61) $\prod_{k=T}^{T-1} s_k$ is defined as equal to one. The expressions for μ, ν, τ, a, b, and c in (16)–(21) can be then simplified to the following using (57) to (61):

$$\mu = \prod_{t=0}^{T-1} s_t\,(1 - B_t) \tag{62}$$

$$\nu = \frac{1}{2}[1 - \prod_{t=0}^{T-1}(1 - B_t)] \tag{63}$$

$$\tau = \prod_{t=0}^{T-1} s_t^2\,(1 - B_t) \tag{64}$$

$$a = \frac{1}{4}\prod_{t=0}^{T-1}(1 - B_t)\,[1 - \prod_{t=0}^{T-1}(1 - B_t)] \tag{65}$$

$$b = 2\prod_{t=0}^{T-1} s_t \tag{66}$$

$$c = 0. \tag{67}$$

Notice that the relationship of $\prod_{t=0}^{T-1}(1 - B_t) = 1 - \sum_{t=0}^{T-1}\prod_{k=t+1}^{T-1}(1 - B_k)B_t$ is used in the above derivation.

The optimal parameter γ^* for problem (E(w)) in the investment situations with a riskless asset can be found using (54), (63), (65), and (66):

$$\gamma^* = 2\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{w\left(\prod_{t=0}^{T-1}(1 - B_t)\right)}. \tag{68}$$

The optimal portfolio policy for problem (E(w)) in the investment situations with a riskless asset is given as follows from (54), (22), (23), and (68),

$$\mathbf{u}_t^* = -s_t \mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}(\mathbf{P_t})\, x_t + \left[\prod_{k=0}^{T-1} s_k x_0 + \frac{1}{2w\left(\prod_{k=0}^{T-1}(1-B_k)\right)}\right]\left(\prod_{k=t+1}^{T-1}\frac{1}{s_k}\right)\mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P}_t'\right)\mathrm{E}(\mathbf{P_t})$$

$$t = 0, 1, \ldots, T-2; \tag{69}$$

$$\mathbf{u}_{T-1}^* = -s_{T-1}\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)\mathrm{E}(\mathbf{P}_{T-1})\, x_{T-1} + \left[\prod_{k=0}^{T-1} s_k x_0 + \frac{1}{2w\left(\prod_{k=0}^{T-1}(1-B_k)\right)}\right]\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)\mathrm{E}(\mathbf{P}_{T-1}). \tag{70}$$

Note that u_t^*, $t = 1, 2, \ldots, T-1$, is proportional to $\mathrm{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')\mathrm{E}(\mathbf{P}_t)$. This implies that each investor will spread his wealth among risky securities in the same relative proportions. On the other side, the ratio of the investment in risky assets to the investment in the riskless asset is determined at each time period by observing the realized value of his wealth and based on the investor's attitude toward risk. This result can be viewed as an extension of the well-known separation theorem in single-period portfolio selection to formulations in multiperiod portfolio selection.

The expected terminal wealth and the variance of the terminal wealth under the optimal portfolio policy $\mathbf{u}_t^*$ in the investment situations with a riskless asset are given as follows using (55), (56), (62), (63), (65), (66), and (67):

$$\mathrm{E}(x_T) = \prod_{t=0}^{T-1} s_t x_0 + \frac{\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)}{2w\left(\prod_{t=0}^{T-1}(1-B_t)\right)} \tag{71}$$

$$\mathrm{Var}(x_T) = \frac{\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)}{4w^2\prod_{t=0}^{T-1}(1-B_t)}. \tag{72}$$

The optimal portfolio policy for $(P1(\sigma))$ and $(P2(\epsilon))$ in the investment situations with a riskless asset is given as follows from (54), (24), (25), (68), (71), and (72),

$$\mathbf{u}_t^* = -s_t \mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) \mathrm{E}(\mathbf{P_t})\, x_t + \left[\prod_{k=0}^{T-1} s_k x_0 + \frac{1}{2w^*\left(\prod_{k=0}^{T-1}(1-B_k)\right)}\right]\left(\prod_{k=t+1}^{T-1}\frac{1}{s_k}\right)\mathrm{E}^{-1}\left(\mathbf{P_t}\mathbf{P}_t'\right)\mathrm{E}(\mathbf{P_t})$$

$$t = 0, 1, \ldots, T-2; \tag{73}$$

$$\mathbf{u}_{T-1}^* = -s_{T-1}\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)\mathrm{E}(\mathbf{P}_{T-1})\, x_{T-1} + \left[\prod_{k=0}^{T-1} k_t x_0 + \frac{1}{2w^*\left(\prod_{k=0}^{T-1}(1-B_k)\right)}\right]\mathrm{E}^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)\mathrm{E}(\mathbf{P}_{T-1}), \tag{74}$$

where

$$w^* = \begin{cases} \frac{1}{2}\sqrt{\frac{(1-\prod_{t=0}^{T-1}(1-B_t))}{\sigma\prod_{t=0}^{T-1}(1-B_t)}} & \text{when } (P1(\sigma)) \text{ is solved} \\ \frac{(1-\prod_{t=0}^{T-1}(1-B_t))}{2(\epsilon-\prod_{t=0}^{T-1}s_t x_0)(\prod_{t=0}^{T-1}(1-B_t))} & \text{when } (P2(\epsilon)) \text{ is solved.} \end{cases} \tag{75}$$

Finally, the analytical expression of the mean-variance efficient frontier in equation (27) can be reduced to the following simpler form for situations with a riskless asset using equations (62)–(67),

$$\text{(76)} \quad \mathrm{Var}(x_T) = \frac{\prod_{t=0}^{T-1}(1-B_t)}{1-\prod_{t=0}^{T-1}(1-B_t)}\left(\mathrm{E}(x_T) - x_0\prod_{t=0}^{T-1} s_t\right)^2 \quad \text{for } \mathrm{E}(x_T) \geq \prod_{t=0}^{T-1} s_t x_0.$$

Notice that one end point of the mean-variance efficient frontier is the point with $\mathrm{E}(x_T) = \prod_{t=0}^{T-1} s_t x_0$ and $\mathrm{Var}(x_T) = 0$ that is associated with the investment decision with which the investor keeps all his money in the riskless asset.

The optimal portfolio policy of multiperiod mean-variance formulation with one riskless asset, $\mathrm{E}(w)$, is generated by solving auxiliary problem $A(\lambda^*, w)$ where optimal parameter $\lambda^* = 2w\prod_{t=0}^{T-1} s_t x_0 + (1/\prod_{t=0}^{T-1}(1-B_t))$ as proven in (68). For a given wealth x_{T-1} at period $T-1$, the optimal portfolio policy u^*_{T-1} is obtained by minimizing $\mathrm{E}(-wx_T^2 + \lambda^* x_T)$. It can be demonstrated that for a given wealth x_t at period t, the optimal portfolio policy u^*_t is obtained by minimizing $\mathrm{E}[-w(s_{T-1}\ldots s_{t+1}x_{t+1})^2 + \lambda^* s_{T-1}\ldots s_{t+1}x_{t+1})$. This is a kind of property similar to the so-called partially myopic policy introduced in Mossin (1968) while maximizing expected utility functions of the terminal wealth. Notice that the objective function in the auxiliary problem, $A(\lambda^*, w)$, is of a quadratic form. The policy derived in this section, however, is not a myopic policy, as evidenced by the appearance of λ^* in the control policy at all time periods. Optimal parameter λ^* is determined by the initial wealth and the return statistics along the entire time horizon.

When setting $T = 1$ in our formulation, problems $(P1(\sigma))$ and $(P2(\epsilon))$ are reduced to the single-period mean-variance formulation (Markowitz 1959). It can be verified (Ng 1997) that the expressions of the efficient frontier are exactly the same as given in this paper and in equation (35) of Merton (1972) when setting $T = 1$. The work of multiperiod mean-variance approach presented in this paper can be viewed as a generalization of the analytical work of Merton (1972) in single-period mean-variance formulation.

The mean-variance hedging problem is studied in Duffie and Richardson (1991), Schäl (1994), and Schweizer (1995), where an optimal dynamic strategy is sought to hedge the derivatives under an imperfect market situation. The result in this section on multiperiod mean-variance formulation with one riskless asset can be also derived from the mean-variance hedging framework. The mean-variance hedging problem formulation (Schweizer 1995) considers a market with one risky asset and one riskless one. The investor optimizes the investment amount in the one risky asset and thus the amount in the riskless asset in order to minimize the gap between the contingent claim and the final wealth of his portfolio. An analytical solution is obtained for the mean-variance hedging problem formulation (Schweizer 1995) with one risky asset and one riskless one using the projection theorem. Integrated with the separation property mentioned earlier in this section, the analytical solution for the multiperiod mean-variance formulation with one riskless asset discussed in this section can be derived using the result of mean-variance hedging problem formulation.

6. MULTIPERIOD PORTFOLIO SELECTION VIA MAXIMIZING UTILITY FUNCTION $U(\mathrm{E}(x_T), \mathrm{VAR}(x_T))$

We consider in this section a more general problem formulation for multiperiod portfolio selection. The objective of an investor now is to maximize $U(\mathrm{E}(x_T), \mathrm{Var}(x_T))$, a utility

that is a function of the expected value and the variance of the terminal wealth x_T. Since investors always would like to maximize their final wealth with a low risk level, utility function $U\left(\mathrm{E}\left(x_T\right), \mathrm{Var}(x_T)\right)$ is assumed to satisfy the following:

$$\frac{\partial U\left(\mathrm{E}(x_T), \mathrm{Var}(x_T)\right)}{\partial \mathrm{E}(x_T)} > 0 \tag{77}$$

and

$$\frac{\partial U\left(\mathrm{E}(x_T), \mathrm{Var}(x_T)\right)}{\partial \mathrm{Var}(x_T)} < 0. \tag{78}$$

Kroll, Levy, and Markowitz (1984) found that solutions obtained by maximizing $U\left(\mathrm{E}(x_T), \mathrm{Var}(x_T)\right)$ and via direct utility maximization are highly correlated to each other.

The following multiperiod portfolio selection problem is formulated:

$$\begin{aligned} (U): \quad & \max \quad U\left(\mathrm{E}(x_T), \mathrm{Var}(x_T)\right) \\ & \text{s.t. } x_{t+1} = e_t^0 x_t + \mathbf{P_t}'\mathbf{u}_t \qquad t = 0, 1, 2, \ldots, T-1. \end{aligned} \tag{79}$$

Define Π_U to be the set of the optimal solutions of problem (U); that is,

$$\Pi_U = \{\pi \,|\, \pi \text{ is the maximizer of } (U)\}. \tag{80}$$

Problem formulation (U) covers a general class of multiperiod portfolio selection problems. A utility function, in general, can be nonlinear with respect to $\mathrm{E}(x_T)$ and $\mathrm{Var}(x_T)$. The multiperiod mean-variance formulation discussed in the previous sections can be seen as a special case of problem formulation (U) where the utility function is linear with respect to $\mathrm{E}(x_T)$ and $\mathrm{Var}(x_T)$.

LEMMA 1. *If $\pi^* \in \Pi_U$, then there exists a $w > 0$ such that $\pi^* \in \Pi_{\mathrm{E}}(w)$.*

Proof. Since U is an increasing function of $\mathrm{E}(x_T)$ and a decreasing function of $\mathrm{Var}(x_T)$, the optimal solution of (U) must be on the mean-variance efficient frontier in the $\{\mathrm{E}(x_T), \mathrm{Var}(x_T)\}$ space. It is known from equation (27) that $\mathrm{Var}(x_T)$ is a convex function of $\mathrm{E}(x_T)$ on the efficient frontier. Thus, supporting lines exist everywhere on the efficient frontier in the $\{\mathrm{E}(x_T), \mathrm{Var}(x_T)\}$ space. In other words, every efficient solution, including $\pi^* \in \Pi_U$, can be generated by the auxiliary problem $(\mathrm{E}(w))$. □

Define the following

$$U_{\mathrm{E}}(\pi) = \frac{\partial U\left(\mathrm{E}(x_T), \mathrm{Var}(x_T)\right)}{\partial \mathrm{E}(x_T)}\Big|_{\pi} \tag{81}$$

$$U_V(\pi) = \frac{\partial U\left(\mathrm{E}(x_T), \mathrm{Var}(x_T)\right)}{\partial \mathrm{Var}(x_T)}\Big|_{\pi} \tag{82}$$

THEOREM 3. *Assume $\pi^* \in \Pi_{\mathrm{E}}(w^*)$. A necessary condition for $\pi^* \in \Pi_U$ is $w^* = -\left(U_V(\pi^*) / U_{\mathrm{E}}(\pi^*)\right)$.*

Proof. The efficient frontier in the $\{\mathrm{E}(x_T), \mathrm{Var}(x_T)\}$ space can be parameterized by the coefficient w. In other words, each point on the efficient frontier can be represented

by $(\mathrm{E}(x_T(w)), \mathrm{Var}(x_T(w)))$. Since $\Pi_U \subseteq \cup_{w\geq 0}\Pi_{\mathrm{E}}(w)$, problem (U) can be reduced in abstract to the following equivalent form:

$$\max_{w\geq 0} \quad U\left(\mathrm{E}(x_T(w)), \mathrm{Var}(x_T(w))\right). \tag{83}$$

A first-order necessary condition for optimum $w^* > 0$ is

$$U_{\mathrm{E}}\left(\pi^*\right)\frac{\partial \mathrm{E}(x_T(w^*))}{\partial w} + U_V\left(\pi^*\right)\frac{\partial \mathrm{Var}(x_T(w^*))}{\partial w} = 0. \tag{84}$$

On the other hand, when $\pi^* \in \Pi_{\mathrm{E}}(w^*)$, we have from Reid and Citron (1971):

$$\frac{\partial \mathrm{E}(x_T(w^*))}{\partial w} - w^*\frac{\partial \mathrm{Var}(x_T(w^*))}{\partial w} = 0. \tag{85}$$

Thus vector $\left[U_{\mathrm{E}}(\pi^*), U_V(\pi^*)\right]$ is proportional to $\left[1, -w^*\right]$. We must have $w^* = -\left(U_V(\pi^*)/U_{\mathrm{E}}(\pi^*)\right)$. □

Lemma 1 implies that problem (U) can be embedded into problem $\mathrm{E}(w)$. Theorem 3 gives a necessary condition for a solution of $\mathrm{E}(w)$ to attain the optimum of (U). Problem $(\mathrm{E}(w))$ can be further embedded into the auxiliary problem $(A(\lambda, w))$ as we know from the previous sections. Thus we can conclude that a multiperiod portfolio problem of maximizing $U(\mathrm{E}(x_T), \mathrm{Var}(x_T))$ can be also embedded into $(A(\lambda, w))$. The following theorem gives the condition for optimal parameters with which the solution of $(A(\lambda, w))$ attains the optimal point of (U).

THEOREM 4. *Assume $\pi^* \in \Pi_A(\lambda^*, w^*)$. Necessary conditions for $\pi^* \in \Pi_U$ are*

$$w^* = -\frac{U_V(\pi^*)}{U_{\mathrm{E}}(\pi^*)} \quad \textit{and} \quad \lambda^* = 1 - 2\frac{U_V(\pi^*)}{U_{\mathrm{E}}(\pi^*)}\mathrm{E}(x_T)\,|_{\pi^*}.$$

Proof. The theorem can be easily proven by combining Theorems 2 and 3. □

The optimal solution for problem $(A(\lambda, w))$ is provided for given $\gamma = \lambda/w$. The computational procedure to obtain the optimal γ^* can be constructed by studying the derivative of U with respect to γ. The derivative of the utility function with respect to γ can be obtained using the following formula:

$$\frac{dU}{d\gamma} = \left(\frac{\partial U}{\partial \mathrm{E}(x_T)} - 2\mathrm{E}(x_T)\frac{\partial U}{\partial \mathrm{Var}(x_T)}\right)\nu + \frac{\partial U}{\partial \mathrm{Var}(x_T)}\nu\gamma, \tag{86}$$

where

$$\frac{d\mathrm{E}(x_T)}{d\gamma} = \nu \text{ and } \frac{d\mathrm{E}\left(x_T^2\right)}{d\gamma} = \nu\gamma$$

are used in the above equation based on equations (49) and (50).

By setting $dU/d\gamma$ in (86) equal to zero, we have the following necessary optimum condition for γ:

$$\left(\frac{\partial U}{\partial \mathrm{E}(x_T)} - 2\mathrm{E}(x_T)\frac{\partial U}{\partial \mathrm{Var}(x_T)}\right) + \frac{\partial U}{\partial \mathrm{Var}(x_T)}\gamma = 0; \tag{87}$$

that is,

$$\gamma^* = 2\mathrm{E}(x_T) - \frac{\partial U}{\partial \mathrm{E}(x_T)} \Big/ \frac{\partial U}{\partial \mathrm{Var}(x_T)}. \tag{88}$$

As the derivative of $dU/d\gamma$ is obtainable for given γ, a numerical search method using gradient information, such as the gradient method or the false position method, can be adopted to update the value of γ in $(A(\lambda, w))$. The search process for optimal γ continues until the stopping condition (88) is satisfied. Notice that both $\mathrm{E}(x_T)$ and $\mathrm{Var}(x_T)$ are dependent on parameter γ. Substituting the optimal value of γ^* into (40) yields the optimal portfolio policy for problem (U). The search algorithm is straightforward and only involves a one-dimensional search.

7. ILLUSTRATIVE CASES

Three cases are given in this section to demonstrate the adoption of the multiperiod mean-variance formulations and the efficiency of the solution methods derived in this paper.

EXAMPLE 1. Consider the case study in Chapter 7 of Sharpe, Alexander, and Bailey (1995) by assuming a stationary multiperiod process with $T = 4$. An investor has one unit of wealth at the very beginning of the planning horizon. The investor is trying to find the best allocation of his wealth among three risky securities, A, B, and C in order to maximize $\mathrm{E}(x_4)$ while keeping his risk not exceeding 2; that is, $\sigma = 2$. The expected returns for risky securities, A, B, and C are $\mathrm{E}(e_t^A) = 1.162$, $\mathrm{E}(e_t^B) = 1.246$, and $\mathrm{E}(e_t^C) = 1.228$, $t = 0, 1, 2, 3$. The covariance of $\mathbf{e_t} = [e_t^A, e_t^B, e_t^C]'$ is

$$\mathrm{Cov}(\mathbf{e_t}) = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}, \qquad t = 0, 1, 2, 3.$$

Take security A as the reference asset. Thus,

$$\begin{aligned}
\mathrm{E}(\mathbf{P_t}) &= \mathrm{E}[e_t^B - e_t^A, e_t^C - e_t^A]' \\
&= [0.084, 0.066]', t = 0, 1, 2, 3, \\
\mathrm{E}(\mathbf{P_t}\mathbf{P_t'}) &= \mathrm{E}\begin{bmatrix} (e_t^B)^2 - 2e_t^A e_t^B + (e_t^A)^2 & e_t^B e_t^C - e_t^A e_t^C - e_t^A e_t^B + (e_t^A)^2 \\ e_t^B e_t^C - e_t^A e_t^C - e_t^A e_t^B + (e_t^A)^2 & (e_t^C)^2 - 2e_t^A e_t^C + (e_t^A)^2 \end{bmatrix} \\
&= \begin{bmatrix} 0.0697 & -0.0027 \\ -0.0027 & 0.0189 \end{bmatrix}, \qquad t = 0, 1, 2, 3. \\
\mathrm{E}(e^A\mathbf{P_t'}) &= \left[\mathrm{E}\left(e_t^A e_t^B\right) - \mathrm{E}\left((e_t^A)^2\right), \mathrm{E}\left(e_t^A e_t^C\right) - \mathrm{E}\left((e_t^A)^2\right)\right] \\
&= [0.1017, 0.0766], \qquad t = 0, 1, 2, 3.
\end{aligned}$$

Furthermore, we have $B_t = 0.3566$, $A_t^1 = 0.7424$, $A_t^2 = 0.8711$, $t = 0, 1, 2, 3$, $\mu = 0.3038$, $\nu = 0.4077$, $a = 0.0376$, $b = 3.2933$, and $c = 0.0754$.

The mean-variance efficient frontier in this example problem is given as follows using (27):

$$\mathrm{Var}(x_4) = 0.2262[\mathrm{E}(x_4) - 1.6465]^2 + 0.0754,$$

where $\mathrm{E}(x_4) \geq 1.6465$.

From (26), the corresponding w^* in $(\mathrm{E}(w))$ is 0.75773. The associated optimal portfolio policy is given as follows using equations (24) and (25):

$$u_t^* = -\mathbf{K}_t x_t + \mathbf{v}_t,$$

where

$$\mathbf{K}_t = \begin{bmatrix} 1.6238 \\ 4.2907 \end{bmatrix}, \qquad t = 0, 1, 2, 3, \qquad \mathbf{v}_0 = \begin{bmatrix} 4.3548 \\ 11.9327 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 5.1094 \\ 14.0004 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 5.9948 \\ 16.4263 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 7.0335 \\ 19.2726 \end{bmatrix}.$$

The investment in the first security, asset A, at period t is equal to $\left(x_t - \sum u_t^i\right)$. The corresponding expected terminal wealth and the risk level are given by $\mathrm{E}(x_4) = 4.5632$ and $\mathrm{Var}(x_4) = 2$, respectively, using equations (55) and (56).

EXAMPLE 2. Consider now a modified version of Example 1. In addition to the three risky assets, A, B, and C, there exists a riskless asset with a sure return rate of 1.04. Suppose this time that the investor seeks an efficient portfolio policy with a desired trade-off between the expected terminal wealth and risk, $\partial\mathrm{E}(x_4)/\partial\mathrm{Var}(x_4) = 2$. More directly, the investor would like to maximize $\mathrm{E}(x_4) - 2\mathrm{Var}(x_4)$.

We can calculate

$$\mathrm{E}(\mathbf{P_t}) = \mathrm{E}[e_t^A - s_t, e_t^B - s_t, e_t^C - s_t]' = [0.122, 0.206, 0.188]', \qquad t = 0, 1, 2, 3,$$

and

$$\mathrm{E}(P_t P_t') = \mathrm{Cov}(e) + \mathrm{E}(P_t)\mathrm{E}(P_t') = \begin{bmatrix} 0.0295 & 0.0438 & 0.0374 \\ 0.0438 & 0.1278 & 0.0491 \\ 0.0374 & 0.0491 & 0.0642 \end{bmatrix}, \qquad t = 0, 1, 2, 3.$$

Furthermore, we have $B_t = \mathrm{E}(P_t')\mathrm{E}^{-1}(P_t P_t')\mathrm{E}(P_t) = 0.593817$, $t = 0, 1, 2, 3$.

The mean-variance efficient frontier in this case is given as follows by using equation (76):

$$\mathrm{Var}(x_4) = 0.02798[\mathrm{E}(x_4) - 1.1699]^2,$$

where $\mathrm{E}(x_4) \geq 1.1699$.

The associated optimal portfolio policy is given as follows using equations (69) and (70):

$$u_t^* = -\mathbf{K}_t x_t + \mathbf{v}_t,$$

where

$$\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}, \qquad t = 0, 1, 2, 3, \qquad \mathbf{v}_0 = \begin{bmatrix} 3.5440 \\ 5.7494 \\ 20.4751 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 3.6858 \\ 5.9794 \\ 21.2941 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3.8332 \\ 6.2185 \\ 22.1459 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 3.9865 \\ 6.4673 \\ 23.0317 \end{bmatrix}.$$

The investment in the riskless asset at period t is equal to $\left(x_t - \sum u_t^i\right)$. The corresponding expected terminal wealth and the risk level are $\mathrm{E}(x_4) = 10.1043$ and $\mathrm{Var}(x_4) = 2.2336$, respectively, using equations (71) and (72).

EXAMPLE 3. Consider Example 2 again, but this time the investor seeks an optimal portfolio policy that maximizes the following utility function

$$U\left(\mathrm{E}(x_4), \mathrm{Var}(x_4)\right) = \mathrm{E}^2(x_4) - \exp\left[\mathrm{Var}(x_4)\right].$$

The derivative of U with respect to γ can be obtained from equations (86) and (63),

$$\frac{dU}{d\gamma} = 0.97278\ \mathrm{E}(x_4)\left[1 + \exp\left(\mathrm{Var}(x_4)\right)\right] - 0.48639\ \exp\left(\mathrm{Var}(x_4)\right)\gamma.$$

Adopting the false position method, the optimal value of γ^* is found to be equal to 25.8965 at which $dU/d\gamma = 0$ and U attains its maximum of 120.0707. The associated optimal portfolio policy is given by

$$u_t^* = -\mathbf{K}_t x_t + \mathbf{v}_t,$$

where

$$\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}, \qquad t = 0, 1, 2, 3, \qquad \mathbf{v}_0 = \begin{bmatrix} 4.4318 \\ 7.1897 \\ 25.6044 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 4.6091 \\ 7.4773 \\ 26.6286 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 4.7935 \\ 7.7764 \\ 27.6937 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 4.9852 \\ 8.0874 \\ 28.8015 \end{bmatrix}.$$

The corresponding expected terminal wealth and the risk are $\mathrm{E}(x_4) = 12.6276$ and $\mathrm{Var}(x_4) = 3.6734$, respectively.

8. CONCLUSIONS

The Markowitz mean-variance approach has been extended in this paper to multiperiod portfolio selection problems. With a solution scheme using embedding, an analytical solution has been derived for the multiperiod mean-variance formulation that is non-tractable in its original setting. The derived analytical expression of the efficient frontier for the multiperiod portfolio selection will definitely enhance investors' understanding of

the trade-off between the expected terminal wealth and the risk. At the same time, the derived analytical optimal multiperiod portfolio policy provides investors with the best strategy to follow in a dynamic investment environment. The multiperiod mean-variance formulation in continuous-time is studied in Zhou and Li (1999). A future research subject is investigation of an efficient solution methodology for the constrained multiperiod mean-variance formulation using the separation property.

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