Amortization and Sinking Funds methods

Problems from actuarial exams

1. A 1,000 loan is to be repaid with equal payments at the end of each year for 20 years. The principal portion of the 13th payment is 1.5 times the principal portion of the 5th payment.
Calculate the total amount of interest paid on the loan.

2. A loan of L is to be repaid with 40 payments of 100 at the end of each quarter. Interest on the loan is charged at a nominal rate of i ($0 < i < 1$), convertible monthly. The outstanding principals immediately after the 8th and 24th payments are 2308.15 and 1345.5 respectively. Find the amount of interest repaid in the 15th payment.

3. Donald takes a loan to be paid with annual payments of 500 at the end of each year for 2n years. The annual effective interest rate is 4.94%.
The sum of the interest paid in year 1 plus the interest paid in year $n+1$ is equal to 720.
Calculate the amount of interest paid in year 10.

4. Joe repays a loan of 10,000 by establishing a sinking fund and making 20 equal payments at the end of each year. The sinking fund earns 7% effective annually.
Immediately after the fifth payment, the yield on the sinking fund increases to 8% effective annually. At that time, Joe adjusts his sinking fund payment to X so that the sinking fund will accumulate to 10,000, 20 years after the original loan date.
Determine X.

5. Glenda repays a loan of 360,000 by making payments of 60,000 at the end of each year for 10 years as follows:
(i) she replaces the capital by means of a sinking fund; and
(ii) she pays interest on the loan at an annual effective rate of $8\frac{1}{3}\%$.
The effective annual interest rate earned by the sinking fund is i. Find i.

6. John wishes to borrow 10,000. Lenders X and Y offer the following terms.
Lender X would be repaid with 10 equal annual payments made at he end of each year at 8% interest effective annually.
Lender Y charges an annual effective interest rate of i with John accumulating the amount necessary to repay the loan by means of 10 annual deposits at the end of each year into a sinking fund earning 7% interest effective annually.
The total payment (principal and interest) is the same for lender X as for lender Y. Find i.

Solutions

1. Let X be the yearly payment.
Principal portion of 13th payment: $X\, v^{20-13+1}$
Principal portion of 5th payment: $X\, v^{20-5+1}$
Thus we get the equation:

$$X\, v^{8} = 1.5\, X\, v^{16},$$

which simplifies to

$$1 = 1.5\, v^{8} \qquad \text{or} \qquad (1+i)^{8} = 1.5$$

Thus $i = .051989 \sim .052$ Equation for X:

$$1000 = X\, a_{\overline{20}|.052} \qquad \text{or} \qquad X = 1000/a_{\overline{20}|.052} = 81.61.$$

Interest: $20\, 81.61 - 1000 = 632.2$

2. Outstanding principal=outstanding balance.
After 8 payments: $100\, a_{\overline{32}|}/s_{\overline{3}|} = 2308.15$
After 24 payments: $100\, a_{\overline{16}|}/s_{\overline{3}|} = 1345.50$
But $a_{\overline{32}|} = a_{\overline{16}|} + v^{16}\, a_{\overline{16}|}$ so we get the equation

$$1345.50 + 13454.50\, v^{16} = 2308.15$$

$$\text{so} \quad v^{16} = .7154 \qquad \text{and} \qquad v = .97929$$

The interest of the 15th payment is

$$100\,(1 - v^{40-15+1}) = 41.96$$

3. Interest of 1st payment: $500(1 - v^{2n})$
Interest of $n+1$th payment: $500(1 - v^{2n-(n+1)+1}) = 500(1 - v^{n})$
Thus we get the equation

$$\begin{aligned} 500(1 - v^{2n}) + 500(1 - v^{n}) &= 720 \\ 2 - v^{2n} - v^{n} &= 1.44 \\ v^{2n} + v^{n} - .56 &= 0 \end{aligned}$$

Solving this quadratic equation we get: $v^{n} = .4$
Interest of 10th payment:

$$500(1-v^{2n-10+1}) = 500(1-v^{2n}v^{-9}) = 500(1-(.4)^{2}(1+i)^{9}) = 500(1-.16(1.0494)^{9}) = 376.53$$

4. Let Y =payment into sinking fund at 7% for 20 years of a loan of 10,000.

$$10,000 = Y\, s_{\overline{20}|.07} \quad \text{which gives} \quad Y = 243.93$$

Let X be the adjusted payment. The accumulated amount in the fund at the end of 20 years is

$$X\, s_{\overline{15}|.08} + Y s_{\overline{5}|.07}\,(1.08)^{15} = 10,000$$

Solving for X we get $X = 204.41$.

5. Of the 60,000 payment, $30,000 = 360,000\,(.083333)$ is interest and 30,000 goes into the sinking fund. Thus, at the end of 10 years, the accumulation of the sinking fund is

$$30,000\, s_{\overline{10}|i} = 360,000 \quad \text{or} \quad s_{\overline{10}|i} = 12$$

We need to solve for i. This is a little complicated. Let's do some estimations:

$$\begin{aligned}(1+i)^{10} =& \sum_{k=0}^{10} \binom{10}{k} i^k = 1 + 10i + 45i^2 + 120i^3 + \text{ terms with higher orders of } i\\ \sim& 1 + 10i + 45i^2 + 120i^3\end{aligned}$$

So

$$12 = \frac{(1+i)^{10} - 1}{i} \sim 10 + 45i + 120i^2$$

thus we solve the equation

$$120i^2 + 45i - 2 = 0 \quad \text{which gives} \quad i \sim= .04$$

If you didn't know the formula for $(1+i)^{10}$, you could have tried Taylor's approximation of the function $f(x) = (1_x)^{10}$. Note that $f'(x) = 10(1+x)^9$, $f''(x) = 10\,9\,(1+x)^8 = 90(1+x)^8$, and $f'''(x) = 90\,8\,(1+x)^7 = 720(1+x)^7$. So, for x small

$$\begin{aligned}f(x) \sim& f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3\\ =& 1 + \frac{10}{1}x + \frac{90}{2}x^2 + \frac{720}{6}x^3\\ =& 1 + 10x + 45x^2 + 120x^3\end{aligned}$$

which is the same as I used before.

6. Payment to X is $10,000/a_{\overline{10}|.08} = 1490.2949$
Payment to Y is

$$10,000/a_{\overline{10}|i\&.07} = 10,000[\frac{1}{s_{\overline{10}|.07}} + i]$$

Since the payments are the same

$$i = \frac{1490.2949}{10,000} - \frac{1}{s_{\overline{10}|.07}} = .14902949 - .0723775 = .07665 = .0767$$