1 The rational numbers

In this worksheet we define and work with the basic properties of the rational numbers.

Definition 1. *Let $S = \mathbb{Z} \times \mathbb{N}$ be the set of pairs (m, n) where m is an integer and n is a natural number. Define a relation on S by saying $(x, y) \sim (u, v)$ if $xv - yu = 0$.*

Proposition 2. *The relation in Definition 1 is an equivalence relation.*

Definition 3. *Let $a \in Z$ and $b \in \mathbb{N}$. The* rational number $\frac{a}{b}$ *is the equivalence class $[(a, b)]$ under the relation on S from Definition 1. The set of all such equivalence classes is denoted by $\mathbb{Q}$.*

Definition 4. *Let $\frac{a}{b}$ and $\frac{x}{y}$ be rational numbers. We define a new rational number as follows:*

$$\frac{a}{b} + \frac{x}{y} = \frac{ay + bx}{by}.$$

Notice that since both b and y are natural numbers, so is by. Similarly, $ax + by$ is an integer.

In terms of the relation, we have defined:

$$[(a, b)] + [(x, y)] = [(ay + bx, by)].$$

However, this makes clear that the definition depends not only on the equivalence class, but also on the representative of the equivalence class that we chose. Therefore, we would like to know that if we took different representatives of the same equivalence classes the answer (equivalence class) that we obtain would be the same. This is the content of the following result.

Theorem 5. *Suppose that $(a, b), (c, d), (m, n), (p, q) \in S$. If $(a, b) \sim (c, d)$ and $(m, n) \sim (p, q)$ then $(an + bm, bn) \sim (cq + dp, dq)$.*

We can also define multiplication.

Definition 6. *Let $\frac{a}{b}$ and $\frac{x}{y}$ be rational numbers. We define a new rational number as follows:*

$$\frac{a}{b} . \frac{x}{y} = \frac{ax}{by}.$$

We also need to know that this is *well-defined* (only depends on the equivalence class, and not on the choice of representative).

Theorem 7. *Suppose that $(a, b), (c, d), (m, n), (p, q) \in S$. If $(a, b) \sim (c, d)$ and $(m, n) \sim (p, q)$ then $(am, bn) \sim (cp, dq)$.*

Since we are working with fractions, we would like to know that a fraction can be written uniquely in simplest form.

Proposition 8. *Let $(a, b) \in S$. There exists a unique pair $(p, q) \in [(a, b)]$ so that for any $(x, y) \in [(a, b)]$ there is a natural number k so that $x = pk$ and $y = qk$.*

We have $[(a, b)] = \{(pk, qk) \mid k \in \mathbb{N}\}$.

We say that $\frac{p}{q}$ is in simplest form.

Lemma 9. *Suppose that $\frac{p}{q}$ is in simplest form. Then $\gcd(p, q) = 1$.*

Here are some basic results that we need about rational numbers.

Proposition 10. *Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{x}{y}$ be rational numbers. The following things are true.*

1. $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$
2. $\frac{a}{b} . \frac{c}{d} = \frac{c}{d} . \frac{a}{b}$
3. $\frac{1}{1} . \frac{a}{b} = \frac{a}{b}$
4. $\frac{a}{b} + \frac{0}{1} = \frac{a}{b}$
5. $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{x}{y} = \frac{a}{b} + \left(\frac{c}{d} + \frac{x}{y}\right)$
6. $\left(\frac{a}{b} . \frac{c}{d}\right) . \frac{x}{y} = \frac{a}{b} . \left(\frac{c}{d} . \frac{x}{y}\right)$
7. $\frac{a}{b} . \left(\frac{c}{d} + \frac{x}{y}\right) = \left(\frac{a}{b} . \frac{c}{d}\right) + \left(\frac{a}{b} . \frac{x}{y}\right)$.

We see in the above proposition that $\frac{0}{1}$ indeed acts like 0 (additively) and $\frac{1}{1}$ acts like 1 (multiplicatively).

Notation 11. *If n is an integer, we write $\frac{n}{1}$ as n.*

Exercise 12. *Let $\frac{a}{b}$ be a rational number. Define a rational number $-\left(\frac{a}{b}\right)$ so that*

$$\frac{a}{b} + \left(-\left(\frac{a}{b}\right)\right) = 0.$$

Let $\frac{a}{b}$ be a rational number which is not equal to 0. Define a rational number $\left(\frac{a}{b}\right)^{-1}$ so that

$$\frac{a}{b} . \left(\frac{a}{b}\right)^{-1} = 1.$$

The answers to the above exercise allow us to speak of subtraction and division in the rational numbers.