1

Approximation by rational numbers

Throughout the present Chapter, we are essentially concerned with the following problem: for which functions $\Psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ is it true that, for a given real number ξ, or for all real numbers ξ in a given class, the equation $|\xi - p/q| < \Psi(q)$ has infinitely many solutions in rational numbers p/q? We begin by stating the results on rational approximation obtained by Dirichlet and Liouville in the middle of the nineteenth century. In Section 1.2, we define the continued fraction algorithm and recall the main properties of continued fractions expansions. These are used in Section 1.3 to give a full proof of a metric theorem of Khintchine. The next two Sections are devoted to the Duffin–Schaeffer Conjecture and to some complementary results on continued fractions.

1.1 Dirichlet and Liouville

Every real number ξ can be expressed in infinitely many ways as the limit of a sequence of rational numbers. Furthermore, for any positive integer b, there exists an integer a with $|\xi - a/b| \leq 1/(2b)$, and one may hope that there are infinitely many integers b for which $|\xi - a/b|$ is in fact much smaller than $1/(2b)$. For instance, this is true when ξ is irrational, as follows from the theory of continued fractions. In order to measure the accuracy of the approximation of ξ by a rational number a/b (written in its lowest terms), we have to compare the difference $|\xi - a/b|$ with the *size*, or complexity, of a/b. A possible definition for this notion is, for example, the number of digits of a plus the number of digits of b. However, as usual, we define the size, or the height, of a/b as the maximum of the absolute values of its denominator and numerator: this definition is more relevant and can be easily extended (see Definition 2.1).

The first statement of Theorem 1.1 is often referred to as Dirichlet's Theorem, although it is not explicitly stated under this form in [196], a paper which appeared in 1842. However, it follows easily from the proof of the main result of [196], which actually provides an extension of the second assertion of Theorem 1.1 to linear forms and to systems of linear forms.

THEOREM 1.1. *Let* ξ *and* Q *be real numbers with* $Q \geq 1$. *There exists a rational number* p/q, *with* $1 \leq q \leq Q$, *such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{qQ}.$$

Furthermore, if ξ *is irrational, then there exist infinitely many rational numbers* p/q *such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}, \tag{1.1}$$

and if $\xi = a/b$ *is rational, then for any rational* $p/q \neq a/b$ *with* $q > 0$ *we have*

$$\left|\xi - \frac{p}{q}\right| \geq \frac{1}{|b|q}.$$

PROOF. Let t denote the integer part of Q. If ξ is the rational a/b, with a and b integers and $1 \leq b \leq t$, it is sufficient to set $p = a$ and $q = b$. Otherwise, the $t + 2$ points $0, \{\xi\}, \ldots, \{t\xi\}$, and 1 are pairwise distinct and they divide the interval $[0, 1]$ into $t + 1$ subintervals. Clearly, at least one of these has its length at most equal to $1/(t + 1)$. This means that there exist integers k, ℓ and m_k, m_ℓ with $0 \leq k < \ell \leq t$ and

$$|(\ell\xi - m_\ell) - (k\xi - m_k)| \leq \frac{1}{t+1} < \frac{1}{Q}.$$

We conclude by setting $p := m_\ell - m_k$ and $q := \ell - k$, and by noticing that q satisfies $1 \leq q \leq t \leq Q$. Instead of reasoning with the lengths of the intervals, we could as well use an argument dating back to Dirichlet [196], now called Dirichlet's *Schubfachprinzip* (or pigeon-hole principle, or *principe des tiroirs*, or *principio dei cassetti*, or *principio de las cajillas*, or *principiul cutiei*, or *skatulya-elv*, or *lokeroperiaate*, or *zasada pudełkowa*). It asserts that at least two among the $t + 2$ points $0, \{\xi\}, \ldots, \{t\xi\}$, and 1 lie in one of the $t + 1$ intervals $[j/(t+1), (j+1)/(t+1)]$, where $j = 0, \ldots, t$; hence, the existence of integers k, ℓ, m_k, and m_ℓ as above.

Suppose now that ξ is irrational and let Q_0 be a positive integer. By the first assertion of the theorem, there exists an integer q with $1 \leq q \leq Q_0$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{qQ_0} \leq \frac{1}{q^2}$$

holds for some integer p. We may assume that q is the smallest integer between 1 and Q_0 with this property. By the first assertion of the theorem applied with $Q = 1/|\xi - p/q|$, there exists a rational number p'/q' with $1 \le q' \le 1/|\xi - p/q|$ such that

$$\left|\xi - \frac{p'}{q'}\right| < \frac{1}{q'}\left|\xi - \frac{p}{q}\right| < \frac{1}{q' Q_0} \quad \text{and} \quad \left|\xi - \frac{p'}{p'}\right| < \frac{1}{q^2}.$$

Our choice of q ensures that q' is strictly larger than q and we proceed inductively to get an infinite sequence of distinct rational numbers satisfying (1.1), thus the second assertion is proved. The third one is immediate.

Theorem 1.1 provides a useful criterion of irrationality: a real number having infinitely many *good* rational approximants must be irrational.

Recall that a complex number ξ is an algebraic number if it is root of a non-zero integer polynomial $P(X)$. Otherwise, ξ is a transcendental number. In 1844, two years after Dirichlet's paper, Liouville [368, 369] was the first to prove that transcendental numbers exist, and, moreover, he constructed explicit examples of such numbers. Thirty years later, Cantor [152] gave an alternative proof of the existence of real transcendental numbers: he showed that the set of real algebraic numbers is countable and that, given a countable set of real numbers, any real interval of positive length contains points *not* belonging to that set. Cantor's proof, however, does not yield any explicit example of a real trancendental number.

A detailed proof of Theorem 1.2 is given in [370] and includes the case $n = 1$ (that is, the last assertion of Theorem 1.1). The main idea, however, already appeared in Liouville's note [369].

THEOREM 1.2. *Let ξ be a real root of an irreducible integer polynomial $P(X)$ of degree $n \ge 2$. There exists a positive constant $c_1(\xi)$ such that*

$$\left|\xi - \frac{p}{q}\right| \ge \frac{c_1(\xi)}{q^n} \tag{1.2}$$

for all rational numbers p/q. A suitable choice for $c_1(\xi)$ is

$$c_1(\xi) := \frac{1}{1 + \max_{|t-\xi|\le 1} |P'(t)|}.$$

PROOF. With $c_1(\xi)$ defined as above, inequality (1.2) is true when $|\xi - p/q| \ge 1$. Let p/q be a rational number satisfying $|\xi - p/q| < 1$. Since $P(X)$ is irreducible and has integer coefficients, we have $P(p/q) \ne 0$ and $|q^n P(p/q)| \ge 1$. By Rolle's Theorem, there exists a real number t lying between ξ and p/q such that

$$|P(p/q)| = |P(\xi) - P(p/q)| = |\xi - p/q| \times |P'(t)|.$$

Hence, we have $|t - \xi| \le 1$ and

$$\left|\xi - \frac{p}{q}\right| \ge \frac{1}{q^n\,|P'(t)|} \ge \frac{c_1(\xi)}{q^n},$$

as claimed.

COROLLARY 1.1. *The number* $\xi := \sum_{n\ge 1} 10^{-n!}$ *is transcendental.*

PROOF. Since its decimal expansion is not ultimately periodic, ξ is irrational. For any integer $n \ge 2$, set $q_n = 10^{(n-1)!}$ and $p_n = q_n(10^{-1!}+\ldots+10^{-(n-1)!})$. Then we have

$$\left|\xi - \frac{p_n}{q_n}\right| = \sum_{m\ge n} \frac{1}{10^{m!}} \le \frac{2}{10^{n!}} = \frac{2}{q_n^n}$$

and ξ is not algebraic of degree greater than or equal to 2, by Theorem 1.2. Consequently, ξ is a transcendental number.

Corollary 1.1 illustrates how Theorem 1.2 can be applied to prove the transcendence of a large class of real numbers, which are now called Liouville numbers.

DEFINITION 1.1. *Let* ξ *be a real number. If for any positive real number* w *there exists a rational number* p/q *such that*

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^w},$$

then ξ *is called a Liouville number.*

An easy modification of the proof of Corollary 1.1 shows that any real number $\sum_{n\ge 1} a_n 10^{-n!}$ with a_n in $\{1, 2\}$ is a Liouville number. Hence, there exist uncountably many Liouville numbers. Furthermore, Theorem 1.2 provides a useful transcendence criterion, see Exercise 1.1.

Combining the theorems of Liouville and Dirichlet, we see that the problem of rational approximation of real quadratic numbers is, in some sense, solved.

COROLLARY 1.2. *Let* ξ *be a real quadratic algebraic number. Then there exists a positive real number* $c_2(\xi)$ *such that*

$$\left|\xi - \frac{p}{q}\right| \ge \frac{c_2(\xi)}{q^2} \quad \text{for all rationals } p/q \tag{1.3}$$

whereas

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2} \quad \text{for infinitely many rationals } p/q.$$

One may ask whether there exist real numbers, other than quadratic irrationalities, for which the property (1.3) is satisfied. The answer is affirmative, and a way to prove it is to use the theory of continued fractions.

1.2 Continued fractions

Let $x_0, x_1, \ldots$ be real numbers with $x_1, x_2, \ldots$ positive. A *finite continued fraction* denotes any expression of the form

$$[x_0; x_1, x_2, \ldots, x_n] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ldots + \cfrac{1}{x_n}}}}.$$

More generally, we call any expression of the above form or of the form

$$[x_0; x_1, x_2, \ldots] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ldots}}} = \lim_{n \to +\infty} [x_0; x_1, x_2 \ldots, x_n]$$

a *continued fraction*, provided that the limit exists.

The aim of this Section is to show how a real number ξ can be expressed as $\xi = [x_0; x_1, x_2, \ldots]$, where x_0 is an integer and x_n a positive integer for any $n \geq 1$. We first deal with the case of a rational number ξ, then we describe an algorithm which associates to any irrational ξ an infinite sequence of integers $(a_n)_{n \geq 0}$, with $a_n \geq 1$ for $n \geq 1$, and we show that the sequence of rational numbers $[a_0; a_1, a_2, \ldots, a_n]$ converges to ξ.

For more results on continued fractions or/and different points of view on this theory, the reader may consult, for example, a text of Van der Poorten [462] and the books of Cassels [155], Dajani and Kraaikamp [174], Hardy and Wright [271], Iosifescu and Kraaikamp [286], Perron [454], Rockett and Szüsz [474], Schmidt [512], and Schweiger [519].

LEMMA 1.1. *Any rational number r has exactly two different continued fraction expansions. These are $[r]$ and $[r-1; 1]$ if r is an integer and, otherwise, one of them has the form $[a_0; a_1, \ldots, a_{n-1}, a_n]$ with $a_n \geq 2$, and the other one is $[a_0; a_1, \ldots, a_{n-1}, a_n - 1, 1]$.*

PROOF. Let r be a rational number and write $r = u/v$ with v positive and u and v coprime. We argue by induction on v. If $v = 1$, then $r = u = [u]$, and if $r = [a_0; a_1, \ldots, a_n]$ with $n \geq 1$, we have $r = a_0 + 1/[a_1; a_2, \ldots, a_n]$.

Since $a_1 \geq 1$ and r is an integer, we deduce that $n = 1$ and $a_1 = 1$, thus $r = a_0 + 1 = [r - 1; 1]$.

We assume $v \geq 2$ and that the lemma holds true for any rational of denominator positive and at most equal to $v - 1$. Performing the Euclidean division of u by v, there exist integers q and c with $u = qv + c$ and $1 \leq c \leq v - 1$. Thus, $u/v = q + c/v$ and, by our inductive hypothesis, the rational v/c has exactly two expansions in continued fractions, which we denote by $[a_1; a_2, \ldots, a_{n-1}, a_n]$ with $a_n \geq 2$, and $[a_1; a_2, \ldots, a_{n-1}, a_n - 1, 1]$. Setting a_0 equal to q, the desired result follows for u/v.

Unless otherwise explicitly stated, by 'the' continued fraction expansion of a rational number p/q, we mean [1] if $p/q = 1$, and if not, the unique expansion which does not end with 1.

The following algorithm allows us to associate to any irrational real number ξ an infinite sequence of integers. Let us define the integer a_0 and the real number $\xi_1 > 1$ by

$$a_0 = [\xi] \quad \text{and} \quad \xi_1 = 1/\{\xi\}.$$

We then have $\xi = a_0 + 1/\xi_1$. For any positive integer n, we define inductively the integer a_n and the real number $\xi_{n+1} > 1$ by

$$a_n = [\xi_n] \quad \text{and} \quad \xi_{n+1} = 1/\{\xi_n\},$$

and we observe that $\xi_n = a_n + 1/\xi_{n+1}$. We point out that the algorithm does not stop since ξ is assumed to be irrational. Thus, we have associated to any irrational real number ξ an infinite sequence of integers $a_0, a_1, a_2, \ldots$ with a_n positive for all $n \geq 1$.

If ξ is rational, the same algorithm terminates and associates to ξ a *finite* sequence of integers. Indeed, the ξ_js are then rational numbers and, if we set $\xi_j = u_j/v_j$, with u_j, v_j positive and $\gcd(u_j, v_j) = 1$, an easy induction shows that we get $u_j > u_{j+1}$, for any positive integer j with $v_j \neq 1$. Consequently, there must be some index n for which $v_{n-1} \neq 1$ and ξ_n is an integer. Thus, $a_{n+1}, a_{n+2}, \ldots$ are not defined. We have $\xi = [a_0; a_1, \ldots, a_n]$, and this corresponds to the Euclidean algorithm.

DEFINITION 1.2. *Let ξ be an irrational number (resp. a rational number). Let $a_0, a_1, \ldots$ (resp. $a_0, a_1, \ldots, a_N$) be the sequence of integers associated to ξ by the algorithm defined above. For any integer $n \geq 1$ (resp. $n = 1, \ldots, N$), the rational number*

$$\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n]$$

is called the n-th convergent of ξ and a_n is termed the n-th partial quotient of ξ. Further, for any integer $n \geq 1$ (resp. $n = 1, \ldots, N-1$), there exists a real number η_n in $]0, 1[$ such that

$$\xi = [a_0; a_1, \ldots, a_{n-1}, a_n + \eta_n].$$

We observe that the real numbers η_n occurring in Definition 1.2 are exactly the real numbers $1/\xi_{n+1}$ given by the algorithm.

In all of what follows until the end of Theorem 1.5, unless otherwise explicitly stated, we assume that ξ is a real irrational number and we associate to ξ the sequences $(a_n)_{n\geq 0}$ and $(p_n/q_n)_{n\geq 1}$ as given by Definition 1.2. However, the statements below remain true for rational numbers ξ provided that the a_ns and the p_n/q_ns are well-defined.

The integers p_n and q_n can be easily expressed in terms of a_n, p_{n-1}, p_{n-2}, q_{n-1}, and q_{n-2}.

THEOREM 1.3. *Setting*

$$p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = a_0, \quad \textit{and} \quad q_0 = 1,$$

we have, for any positive integer n,

$$p_n = a_n p_{n-1} + p_{n-2} \quad \textit{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

PROOF. We proceed by induction. Since $p_1/q_1 = a_0 + 1/a_1 = (a_0a_1 + 1)/a_1$, the definitions of p_{-1}, q_{-1}, p_0, and q_0 show that the theorem is true for $n = 1$. Assume that it holds true for a positive integer n and denote by $p'_0/q'_0, \ldots, p'_n/q'_n$ the convergents of the rational number $[a_1; a_2, \ldots, a_{n+1}]$. For any integer j with $0 \leq j \leq n+1$ we have

$$\frac{p_j}{q_j} = [a_0; a_1, \ldots, a_j] = a_0 + \frac{1}{[a_1; a_2, \ldots, a_j]} = a_0 + \frac{q'_{j-1}}{p'_{j-1}},$$

thus

$$p_j = a_0 p'_{j-1} + q'_{j-1} \quad \text{and} \quad q_j = p'_{j-1}. \tag{1.4}$$

It follows from (1.4) with $j = n+1$ and the inductive hypothesis applied to the rational $[a_1; a_2, \ldots, a_{n+1}]$ that

$$\begin{aligned} p_{n+1} &= a_0(a_{n+1}p'_{n-1} + p'_{n-2}) + a_{n+1}q'_{n-1} + q'_{n-2} \\ &= a_{n+1}(a_0 p'_{n-1} + q'_{n-1}) + a_0(p'_{n-2} + q'_{n-2}) \end{aligned}$$

and

$$q_{n+1} = a_{n+1}p'_{n-1} + p'_{n-2},$$

whence, by (1.4) with $j = n$ and $j = n - 1$, we get $q_{n+1} = a_{n+1} q_n + q_{n-1}$ and $p_{n+1} = a_{n+1} p_n + p_{n-1}$, as claimed.

THEOREM 1.4. *For any non-negative integer n, we have*

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \tag{1.5}$$

and, for all $n \geq 1$,

$$q_n p_{n-2} - p_n q_{n-2} = (-1)^{n-1} a_n. \tag{1.6}$$

PROOF. These equalities are clearly true for $n = 0$ and $n = 1$. It then suffices to argue by induction, using Theorem 1.3.

LEMMA 1.2. *For any irrational number ξ and any non-negative integer n, the difference $\xi - p_n/q_n$ is positive if, and only if, n is even.*

PROOF. We easily check that this is true for $n = 0, 1$, and 2 and we proceed by induction. Let $n \geq 4$ be an even integer. Then $\xi = [a_0; a_1, [a_2; a_3, \ldots, a_n + \eta_n]]$ and the inductive hypothesis implies that $[a_2; a_3, \ldots, a_n + \eta_n] > [a_2; a_3, \ldots, a_n]$. Since $[a_0; a_1, u] > [a_0; a_1, v]$ holds for all positive real numbers $u > v$, we get that $\xi > [a_0; a_1, [a_2; a_3, \ldots, a_n]] = p_n/q_n$. We deal with the case n odd in exactly the same way.

As a corollary of Lemma 1.2, we get a result of Vahlen [575].

COROLLARY 1.3. *Let p_n/q_n and p_{n+1}/q_{n+1} be two consecutive convergents of the continued fraction expansion of an irrational number ξ. Then at least one of them satisfies*

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

PROOF. We infer from Lemma 1.2 that ξ is an inner point of the interval bounded by p_n/q_n and p_{n+1}/q_{n+1}. Thus, using (1.5) and the inequality $a^2 + b^2 > 2ab$, valid for any distinct real numbers a and b, we get

$$\frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} > \frac{1}{q_n q_{n+1}} = \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \left| \xi - \frac{p_n}{q_n} \right| + \left| \xi - \frac{p_{n+1}}{q_{n+1}} \right|,$$

and the claimed result follows.

The next two theorems show that the real irrational numbers are in a one-to-one correspondence with the set of integer sequences $(a_i)_{i \geq 0}$ with a_i positive for $i \geq 1$.

THEOREM 1.5. *The convergents of even order of any real irrational ξ form a strictly increasing sequence and those of odd order a strictly decreasing sequence. The sequence of convergents $(p_n/q_n)_{n\geq 0}$ converges to ξ, and we set*

$$\xi = [a_0; a_1, a_2, \ldots].$$

Any irrational number has a unique expansion in continued fractions.

PROOF. It follows from (1.6) that for any integer n with $n \geq 2$ we have

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}a_n}{q_n q_{n-2}},$$

and, since the a_ns are positive, we deduce that the convergents of even order of the real irrational ξ form a strictly increasing sequence and those of odd order a strictly decreasing sequence. To conclude, we observe that, by Lemma 1.2, we have $p_{2n}/q_{2n} < \xi < p_{2n+1}/q_{2n+1}$ for all $n \geq 0$, and, by (1.5), the difference $p_{2n}/q_{2n} - p_{2n+1}/q_{2n+1}$ tends to 0 when n tends to infinity. Uniqueness is clear. Indeed, if $(b_i)_{i\geq 0}$ is a sequence of integers with b_i positive for $i \geq 1$ and such that $\lim_{n\to+\infty} [b_0; b_1, b_2, \ldots]$ exist, then this limit cannot be equal to $\lim_{n\to+\infty} [a_0; a_1, a_2, \ldots]$ as soon as there exists a non-negative integer i with $a_i \neq b_i$.

THEOREM 1.6. *Let $a_0, a_1, \ldots$ be integers with $a_1, a_2, \ldots$ positive. Then the sequence of rational numbers $[a_0; a_1, \ldots, a_i]$, $i \geq 1$, converges to the irrational number whose partial quotients are precisely $a_0, a_1, \ldots$*

PROOF. For any positive integer n, denote by p_n/q_n the rational number $[a_0; a_1, \ldots, a_n]$. The recurrence relations obtained in Theorems 1.3 and 1.4 hold true in the present context. As in the proof of Theorem 1.5, we deduce from (1.5) and (1.6) that the sequences $(p_{2n}/q_{2n})_{n\geq 1}$ and $(p_{2n+1}/q_{2n+1})_{n\geq 1}$ are adjacent. Hence, they converge to the same limit, namely to the irrational number $[a_0; a_1, a_2, \ldots]$, whose partial quotients are precisely $a_0, a_1, \ldots$, by Theorem 1.5.

We observe that for any irrational number ξ the sequences $(a_n)_{n\geq 0}$ and $(\xi_n)_{n\geq 1}$ given by the algorithm defined below Lemma 1.1 satisfy $\xi_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$ for all positive integers n.

THEOREM 1.7. *Let n be a positive integer and $\xi = [a_0; a_1, a_2 \ldots]$ be an irrational number. We then have*

$$\xi = [a_0; a_1, \ldots, a_n, \xi_{n+1}] = \frac{p_n\xi_{n+1} + p_{n-1}}{q_n\xi_{n+1} + q_{n-1}}$$

and

$$q_n\xi - p_n = \frac{(-1)^n}{q_n\xi_{n+1} + q_{n-1}} = \frac{(-1)^n}{q_n} \cdot \frac{1}{\xi_{n+1} + [0; a_n, a_{n-1}, \ldots, a_1]}.$$

Furthermore, the set of real numbers having a continued fraction expansion whose $n+1$ first partial quotients are $a_0, a_1, \ldots, a_n$ is precisely the closed interval bounded by $(p_{n-1}+p_n)/(q_{n-1}+q_n)$ and p_n/q_n, which are equal to $[a_0; a_1, \ldots, a_n, 1]$ and $[a_0; a_1, \ldots, a_n]$, respectively.

PROOF. We proceed by induction, using Theorem 1.3 and noticing that we have $\xi_n = a_n + 1/\xi_{n+1}$ and $q_n/q_{n-1} = [a_n; a_{n-1}, \ldots, a_1]$ for all positive integers n. The last assertion of the theorem follows immediately, since the admissible values of ξ_{n+1} run exactly through the interval $]1, +\infty[$.

COROLLARY 1.4. *For any irrational number ξ and any non-negative integer n, we have*

$$\frac{1}{q_n(q_n + q_{n+1})} < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

PROOF. Writing $\xi = [a_0; a_1, a_2, \ldots]$ and $\xi_{n+1} = [a_{n+1}; a_{n+2}, \ldots]$, we observe that $a_{n+1} < \xi_{n+1} < a_{n+1} + 1$, and we get from Theorem 1.7 that

$$\frac{1}{q_n\big((a_{n+1}+1)q_n + q_{n-1}\big)} < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n(a_{n+1}q_n + q_{n-1})}.$$

The corollary follows then from Theorem 1.3.

The following result of Legendre [359] provides a partial converse to Corollary 1.3.

THEOREM 1.8. *Let ξ be a real number. Any non-zero rational number a/b with*

$$\left|\xi - \frac{a}{b}\right| < \frac{1}{2b^2}$$

is a convergent of ξ.

PROOF. We assume that $\xi \neq a/b$ and we write $\xi - a/b = \varepsilon\theta/b^2$, with $\varepsilon = \pm 1$ and $0 < \theta < 1/2$. By Lemma 1.1, setting $a_{n-1} = 1$ if necessary, we may write $a/b = [a_0; a_1, \ldots, a_{n-1}]$, with n given by $(-1)^{n-1} = \varepsilon$, and we denote by $p_1/q_1, \ldots, p_{n-1}/q_{n-1}$ the convergents of a/b. Let ω be such that

$$\xi = \frac{p_{n-1}\omega + p_{n-2}}{q_{n-1}\omega + q_{n-2}} = [a_0; a_1, \ldots, a_{n-1}, \omega].$$