MATH 162, SHEET 7: THE RATIONAL NUMBERS

In this script, we define the rational numbers $\mathbb{Q}$ and show that they form an ordered field. At the end of this script, we show that any ordered field contains a canonical copy of the rational numbers.

Definition 7.1. Let $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$. We define a relation $\sim$ on X as:

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc$$

Lemma 7.2. *The relation $\sim$ is reflexive. That is, $(a, b) \sim (a, b)$ for every $(a, b) \in X$.*

Lemma 7.3. *The relation $\sim$ is symmetric. That is, if $(a, b) \sim (c, d)$, then $(c, d) \sim (a, b)$.*

Lemma 7.4. *The relation $\sim$ is transitive. That is, if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $(a, b) \sim (e, f)$.*

Remark 7.5. A relation on any set that is reflexive, symmetric, and transitive is called an *equivalence relation*. Thus, the preceding lemmas show that $\sim$ is an equivalence relation on X.

Remark 7.6. A *partition* of a set is a collection of non-empty disjoint subsets whose union is the original set. Any equivalence relation on a set creates a partition of that set by collecting into subsets all of the elements that are equivalent (related) to each other. When the partition of a set arises from an equivalence relation in this manner, the subsets are referred to as *equivalence classes*.

Remark 7.7. If we think of the set X as representing the collection of all fractions whose denominators are not zero, then the relation $\sim$ may be thought of as representing the equivalence of two fractions.

Definition 7.8. As a set, the *rational numbers*, denoted $\mathbb{Q}$, are the equivalence classes in the set X under the equivalence relation $\sim$. If $(a, b) \in X$, we denote the equivalence class of this element as $[(a, b)]$. Thus,

$$\mathbb{Q} = \{\ [(a, b)] \mid (a, b) \in X\}.$$

Definition 7.9. We define addition and multiplication in $\mathbb{Q}$ as follows. If $[(a, b)]$ and $[(c, d)]$ are in $\mathbb{Q}$, then:

$$[(a, b)] +_{\mathbb{Q}} [(c, d)] = [(ad + bc, bd)]$$

$$[(a, b)] \cdot_{\mathbb{Q}} [(c, d)] = [(ac, bd)]$$

We use the notation $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ to represent addition and multiplication in $\mathbb{Q}$ so as to distinguish these operations from the usual addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{Z}$.

Theorem 7.10. *Addition in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(a',b')] +_{\mathbb{Q}} [(c',d')].$$

Theorem 7.11. *Multiplication in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$[(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(a',b')] \cdot_{\mathbb{Q}} [(c',d')].$$

Theorem 7.12. *The rational numbers $\mathbb{Q}$ satisfy axiom FA1 for fields.*

Theorem 7.13. *The rational numbers $\mathbb{Q}$ satisfy axiom FA2 for fields.*

Theorem 7.14. *The rational numbers $\mathbb{Q}$ satisfy axiom FA3 for fields, where the additive identity is the element $[(0,1)]$.*

Theorem 7.15. *The rational numbers $\mathbb{Q}$ satisfy axiom FA4 for fields, where the additive inverse of the element $[(a,b)]$ is given by $-[(a,b)] = [(-a,b)]$.*

Theorem 7.16. *The rational numbers $\mathbb{Q}$ satisfy axiom FA5 for fields.*

Theorem 7.17. *The rational numbers $\mathbb{Q}$ satisfy axiom FA6 for fields.*

Theorem 7.18. *The rational numbers $\mathbb{Q}$ satisfy axiom FA7 for fields, where the multiplicative identity is the element $[(1,1)]$.*

Theorem 7.19. *The rational numbers $\mathbb{Q}$ satisfy axiom FA8 for fields, where the multiplicative inverse of the element $[(a,b)] \neq [(0,1)]$ is given by $[(a,b)]^{-1} = [(b,a)]$.*

Theorem 7.20. *The rational numbers $\mathbb{Q}$ satisfy axiom FA9 for fields.*

Theorem 7.21. *The rational numbers $\mathbb{Q}$ satisfy axiom FA10 for fields.*

Remark 7.22. Thus, the rational numbers $\mathbb{Q}$ with $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ form a field.

Next, we define an ordering on $\mathbb{Q}$.

Definition 7.23. Let $[(a,b)], [(c,d)] \in \mathbb{Q}$. Assuming $b, d > 0$ in $\mathbb{Z}$, we say that $[(a,b)] <_{\mathbb{Q}} [(c,d)]$ if and only if $ad < bc$. Note that we use the symbol $<_{\mathbb{Q}}$ to denote our ordering relation in $\mathbb{Q}$ so as to distinguish it from the usual ordering $<$ in $\mathbb{Z}$. Note also that, given an element in $\mathbb{Q}$, it is always possible to choose a representative (a,b) with $b > 0$.

Theorem 7.24. *Ordering in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$[(a,b)] <_{\mathbb{Q}} [(c,d)] \quad \textit{if and only if} \quad [(a',b')] <_{\mathbb{Q}} [(c',d')].$$

Theorem 7.25. *The relation $<_{\mathbb{Q}}$ is an ordering on the rational numbers $\mathbb{Q}$.*

Theorem 7.26. *The rational numbers $\mathbb{Q}$ form an ordered field.*

There is an important sense in which every ordered field contains a "copy" of the rational numbers. Let F be an ordered field with additive identity 0_F and multiplicative identity 1_F. To prevent ambiguity, denote the additive identity in the rational numbers by $0_{\mathbb{Q}}$ and the multiplicative identity in the rational numbers by $1_{\mathbb{Q}}$.

Theorem 7.27. *Any ordered field F contains a copy of the rational numbers $\mathbb{Q}$ in the following sense. There exists an injective map $i : \mathbb{Q} \longrightarrow F$ that respects all of the axioms for an ordered field. In particular:*

- $i(0_{\mathbb{Q}}) = 0_F$
- $i(1_{\mathbb{Q}}) = 1_F$
- *If $a, b \in \mathbb{Q}$, then $i(a + b) = i(a) + i(b)$.*
- *If $a, b \in \mathbb{Q}$, then $i(a \cdot b) = i(a) \cdot i(b)$.*
- *If $a, b \in \mathbb{Q}$ and $a < b$, then $i(a) < i(b)$.*

Since we know that $\mathbb{Q}$ is an ordered field, this result says that $\mathbb{Q}$ is the "minimal" ordered field.