

6

Fractions and Rational Numbers

- 6.1 ▶ The Basic Concepts of Fractions and Rational Numbers
- 6.2 ▶ Addition and Subtraction of Fractions
- 6.3 ▶ Multiplication and Division of Fractions
- 6.4 ▶ The Rational Number System



COOPERATIVE INVESTIGATION

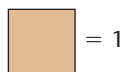
Folded Fractions

Materials Needed

Each student (or cooperating pair of students) should have about six paper squares (4- to 5-inch side length) and several colored pencils. Patty Paper (waxed paper that serves as meat-patty separators) works very well.

Example: Folding Quarters

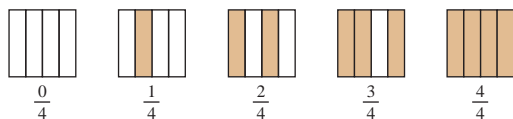
A *unit of area* is defined as the area of a square.



Fold the unit square in half twice vertically and then unfold. Since the four rectangles are congruent (i.e., have the same shape and size), each rectangle represents the fraction $\frac{1}{4}$.



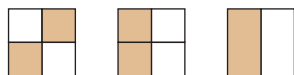
Coloring individual rectangles gives representations of the fractions $\frac{0}{4}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{4}{4}$. (Notice that we count regions, not folds or fold lines.)



Quarters can also be obtained by other folding procedures. For example, we can make a vertical half fold followed by a horizontal half fold to create four small squares within the unit square. This technique illustrates why $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.



If one colored fraction pattern can be rearranged and regrouped to cover the same region of the unit as another fraction pattern, we say that the two fractions are equivalent. For example, the following rearrangement and regrouping show that $\frac{2}{4}$ is equivalent to $\frac{1}{2}$:

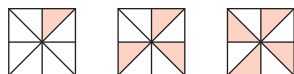


Therefore, we write $\frac{2}{4} = \frac{1}{2}$.

Fraction Folding Activities and Problems

- Eighths.** Fold a square into quarters as in the example, and also fold along the two diagonals.

(a) Identify these fractions:



- How many *different* ways can two of the eight regions be colored to give a representation of $\frac{2}{8}$?

Two colorings are considered identical if one pattern can be rotated to become identical to the second pattern.

- Sixths.** "Roll" the paper square into thirds, flatten, and then fold in half in the opposite direction to divide the square into six congruent rectangles.

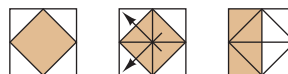
This procedure illustrates that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.



How many *different* ways can you represent the fraction $\frac{3}{6}$?

Remember that two patterns are different only if one cannot be rotated to become identical to another.

- Identifying fractions.** The leftmost shaded region shown in the figure is bounded by segments joining successive midpoints of the sides of the square. What is the area of the shaded region? The region can be divided into smaller regions that are easy to rearrange into a pattern that shows that $\frac{1}{2}$ of the square is shaded.



Use similar multiple representations of the same amount of shaded region to identify the fractions represented by the following colored regions:

- The shaded region obtained by joining the midpoints of the vertical edges of the square with the $\frac{1}{3}$ and $\frac{2}{3}$ points along the horizontal edges:



- The shaded region obtained by joining the $\frac{1}{3}$ and $\frac{2}{3}$ points along the vertical and horizontal edges:



- The shaded square whose corners are the intersections of the segments joining the corners of the unit square to the midpoint of an opposite side of the unit square. (Suggestion: Rearrange the unshaded regions to create squares, each congruent to the shaded square.)





In this chapter, we focus on the properties, computational procedures, and applications of fractions and rational numbers. We'll see that rational numbers help us solve problems that could not be answered in integers.

KEY IDEAS

- Fraction basics: unit, denominator, numerator
- Fraction models: colored regions (area model), fraction strips, fraction circles, set model, number-line model
- Equivalence of fractions and the fundamental law of fractions
- Simplest form
- Common denominator, least common denominator
- Order relation on the fraction
- Arithmetic operations on fractions and their illustration with fraction models
- Proper fractions and mixed numbers
- Reciprocals
- Algorithms for division, including the “invert-and-multiply” rule
- The meaning of a fraction as an operator
- Rational number system
- Properties of the rational number system
- Existence of multiplicative inverses of nonzero rational numbers
- Density property of the rational numbers
- Computations (exact, approximate, mental) with rational numbers
- Applications of rational numbers

6.1

The Basic Concepts of Fractions and Rational Numbers

Fractions first arise in measurement problems, where they can express a quantity that is less than a whole unit. The word *fraction* comes from the Latin word *fractio*, meaning “the act of breaking into pieces.”

Fractions indicate capacity, length, weight, area, or any quantity formed by dividing a unit of measure into b equal-size parts and then collecting a of those parts. To interpret the meaning of the fraction $\frac{a}{b}$, we must

- agree on the **unit** (such as a cup, an inch, the area of the hexagon in a set of pattern blocks, or a whole pizza);
- understand that the unit is **subdivided into b parts of equal size**; and
- understand that we are considering **a of the parts of the unit**.

For example, suppose the unit is one pizza, as shown in Figure 6.1. The pizza has been divided into 8 parts, and 3 parts have been consumed. Using fractions, we can say that $\frac{3}{8}$ of the pizza was eaten and $\frac{5}{8}$ of the pizza remains.

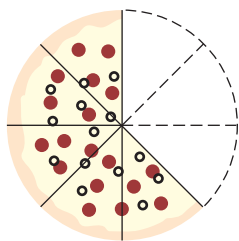


Figure 6.1

Choosing a whole pizza as the unit and dividing it into 8 equal parts shows that $\frac{3}{8}$ of the pizza has been eaten and $\frac{5}{8}$ remains

DEFINITION Fractions

A **fraction** is an ordered pair of integers a and b , $b \neq 0$, written $\frac{a}{b}$ or a/b . The integer a is called the **numerator** of the fraction, and the integer b is called the **denominator** of the fraction.

The word *numerator* is derived from the Latin *numeros*, meaning “number.” *Denominator* is from the Latin *denominare*, meaning “naming.”

This definition permits the numerator or denominator to be a negative integer and the numerator can be zero. For example, $\frac{-3}{-8}$, $\frac{-4}{12}$, $\frac{31}{-4}$, and $\frac{0}{1}$ are all fractions according to the definition just given.

Fraction Models

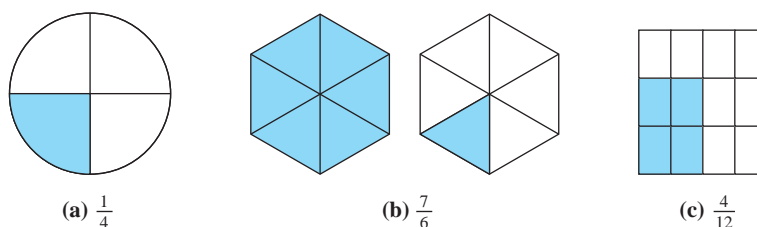
Some popular representations of fraction concepts are colored regions, fraction strips, fraction circles, the set model, and the number line. A representation of a fraction must clearly answer these three questions:

- What is the unit? (What is the “whole”?)
- Into how many equal parts (the denominator) has the unit been subdivided?
- How many of these parts (the numerator) are under consideration?

Errors and misconceptions about fractions suggest that at least one of these questions has not been properly answered or clearly considered. A common error is the failure to identify the fraction’s unit, since the notation $\frac{a}{b}$ makes the numerator and denominator explicit but leaves the unit unidentified.

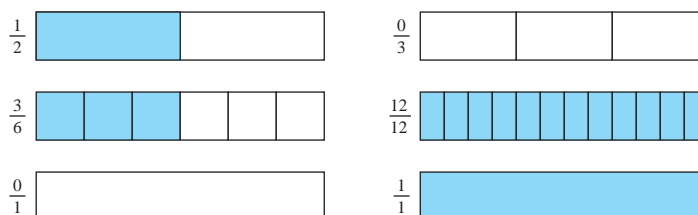
Colored Regions Choose a shape to represent the unit and then divide that shape into subregions of equal size. Represent a fraction by coloring some of the subregions. See Figure 6.2. Colored-region models are sometimes called *area models*.

Figure 6.2
Some colored-region
models for fractions



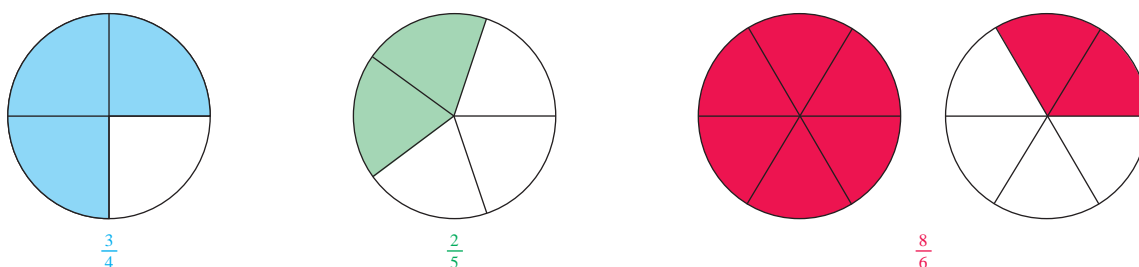
Fraction Strips Define the unit by a rectangular strip. Model a fraction such as $\frac{3}{6}$ by shading 3 of 6 equally sized subrectangles of the rectangle. See Figure 6.3. A set of fraction strips typically contains strips for the denominators 1, 2, 3, 4, 6, 8, and 12.

Figure 6.3
Fractions modeled by
fraction strips



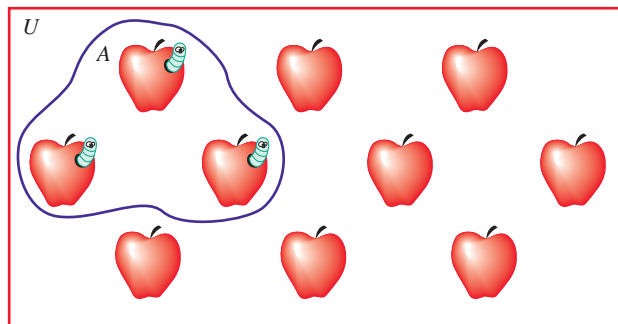
Fraction Circles Divide a circular region into uniformly spaced radial segments corresponding to the denominator. Shade the number of sectors in the numerator. This visual model is particularly effective, since children can easily draw and interpret it. See Figure 6.4.

Figure 6.4
Fractions
illustrated by
fraction circles



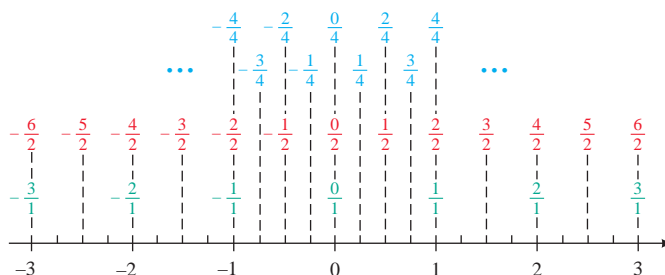
The Set Model Let the unit be a finite set U of objects. Recall that $n(U)$ denotes the number of elements in U . A subset A of U represents the fraction $\frac{n(A)}{n(U)}$. For example, the set of 10 apples shown in Figure 6.5 contains a subset of 3 that are wormy. Therefore, we would say that $\frac{3}{10}$ of the apples are wormy. In Chapter 14, we will see that the set model of fractions is particularly useful in probability. An apple drawn at random from the 10 apples has a $\frac{3}{10}$ probability of being wormy. Sometimes the set model is called the *discrete model*.

Figure 6.5
The set model showing
that $\frac{3}{10}$ of the apples
are wormy



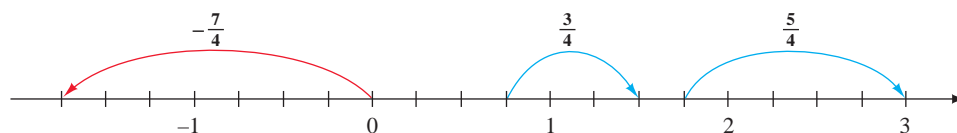
The Number-Line Model Assign the points 0 and 1 on a number line. Doing so determines all of the points corresponding to the integers. The unit is the length of the line segment from 0 to 1; it is also the distance between successive integer points. Assign a fraction such as $\frac{5}{2}$ to a point along the number line by subdividing the unit interval into two equal parts and then counting off five of these lengths to the right of 0. See Figure 6.6. Notice that you can name the same point on the fraction number line by different fractions. For example, $\frac{1}{2}$ and $\frac{2}{4}$ correspond to the same distance. The number-line model also illustrates negative fractions such as $-\frac{3}{4}$.

Figure 6.6
Fractions as points on
the number-line model



As we did with the number-line model for the whole numbers and the integers, we can represent fractions as “jumps” along the number line, using right-pointing or left-pointing arrows. See Figure 6.7.

Figure 6.7
Fractions as “jumps”
along the number line



Equivalent Fractions

We know that the same point on the number line corresponds to infinitely many different fractions.

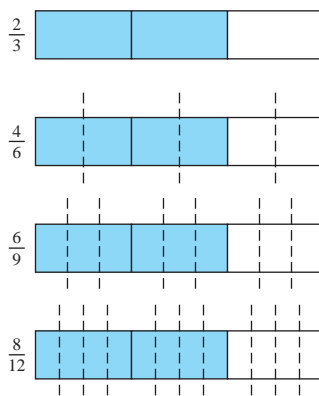
For example, the point one-half unit to the right of 0 corresponds to $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and so on. Similarly, in

Figure 6.8, the fraction strip representing $\frac{2}{3}$ is subdivided by vertical dashed lines to show that $\frac{4}{6}$, $\frac{6}{9}$,

and $\frac{8}{12}$ express the same shaded portion of a whole strip.

Figure 6.8

The fraction-strip model showing that $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{8}{12}$ are equivalent fractions



Fractions that express the same quantity are **equivalent fractions**. The equality symbol, $=$, signifies that fractions are equivalent, so we write

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12}.$$

We can use any number of additional dashed lines between each vertical pair of solid lines in Figure 6.8; therefore,

$$\frac{2}{3} = \frac{2 \cdot n}{3 \cdot n}$$

for natural number n . So given a fraction $\frac{a}{b}$, we obtain an infinite list, $\frac{2a}{2b}, \frac{3a}{3b}, \frac{4a}{4b}, \dots$, of equivalent fractions.

This idea leads to the following important property.

PROPERTY The Fundamental Law of Fractions

Let $\frac{a}{b}$ be a fraction. Then

$$\frac{a}{b} = \frac{an}{bn}, \text{ for any integer } n \neq 0.$$

Equivalent fractions can also be found by dividing both the numerator and denominator by a common factor. For example, the numerator and denominator of $\frac{35}{21}$ are each divisible by 7, so, by the fundamental law,

$$\frac{35}{21} = \frac{5 \cdot 7}{3 \cdot 7} = \frac{5}{3}.$$

Now suppose we are given two fractions, say, $\frac{3}{12}$ and $\frac{2}{8}$, and we wish to know if they are equivalent. From the fundamental law of fractions, we see that

$$\frac{3}{12} = \frac{3 \cdot 8}{12 \cdot 8} \quad \text{and} \quad \frac{2}{8} = \frac{2 \cdot 12}{8 \cdot 12}.$$

Since $\frac{3 \cdot 8}{12 \cdot 8}$ and $\frac{2 \cdot 12}{8 \cdot 12}$ have the same denominator, they are equivalent because they also have the same numerator: $24 = 3 \cdot 8 = 2 \cdot 12$.

Thus, $\frac{3}{12}$ and $\frac{2}{8}$ are equivalent fractions. More generally, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent to $\frac{ad}{bd}$ and $\frac{bc}{bd}$. Since the denominators are the same, the fractions are equivalent if and only if the numerators ad and bc are equal. This result, known as the **cross-product property**, is stated in the following theorem.

THEOREM The Cross-Product Property of Equivalent Fractions

The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are **equivalent** if and only if $ad = bc$. That is,

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc$$

EXAMPLE

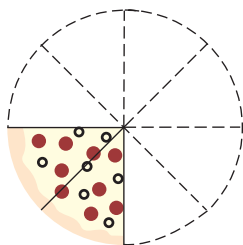
6.1

Examining Fraction Equivalence with Pizza Diagrams

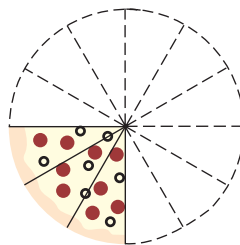
Enrique and Ana each ordered a large pizza. Enrique likes large pieces, so he requested that his pizza be cut into 8 pieces. Ana, however, asked that her pizza be cut into 12 pieces. Enrique ate 6 pieces of his pizza, and Ana ate 9 pieces. Enrique claims that he ate less than Ana, since he ate 3 fewer pieces. Use words and diagrams (similar to Figure 6.1) to see if Enrique is correct.

Solution

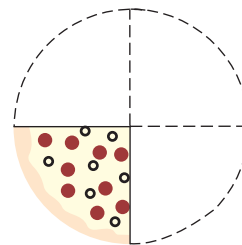
The following pizza diagrams show that both Enrique and Ana have eaten $\frac{3}{4}$ of their pizzas and both have $\frac{1}{4}$ of their pizza remaining:



Enrique's pizza



Ana's pizza



A pizza that is $\frac{3}{4}$ eaten, with $\frac{1}{4}$ remaining

The fundamental law of fractions gives the same result: $\frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4} = \frac{3}{4}$ and $\frac{9}{12} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{3}{4}$.

Alternatively, we can check that $\frac{6}{8} = \frac{9}{12}$ using the cross-product property: $6 \cdot 12 = 8 \cdot 9$.

Fractions in Simplest Form

Often, it is preferable to use the simplest equivalent form of a fraction, which is the one with the smallest positive denominator. For example, it may be best to use $\frac{2}{3}$ instead of $\frac{400}{600}$ and to use $\frac{-3}{4}$ instead of $\frac{75}{-100}$.

DEFINITION Fractions in Simplest Form

A fraction $\frac{a}{b}$ is in **simplest form** if a and b have no common divisor larger than 1 and b is positive.

A fraction in simplest form is also said to be in **lowest terms**. (We avoid using the term *reduced form* because it mistakenly suggests that the fraction is smaller than an equivalent one that is not in simplest form.)

There are several ways to determine the simplest form of a fraction $\frac{a}{b}$. Here are three of the most popular.

Method 1. Divide successively by common factors. Suppose we want to write $\frac{560}{960}$ in simplest form. Using the fundamental law of fractions, we repeatedly divide both numerator and denominator by common factors. Since 560 and 960 are both divisible by 10, it follows that

$$\text{Both 560 and 960 are divisible by: } \frac{560}{960} = \frac{56 \cdot 10}{96 \cdot 10} = \frac{56}{96}.$$

$$\text{But both 56 and 96 are even (that is, divisible by 2): } \frac{56}{96} = \frac{28 \cdot 2}{48 \cdot 2} = \frac{28}{48}.$$

$$\text{Both 28 and 48 are divisible by 4: } \frac{28}{48} = \frac{7 \cdot 4}{12 \cdot 4} = \frac{7}{12}.$$

Since 7 and 12 have no common factor other than 1, $\frac{7}{12}$ is the simplest form of $\frac{560}{960}$. Indeed, one might quickly and efficiently carry out this simplification as follows to obtain the desired result.

$$\frac{\overset{7}{\cancel{28}} \cancel{560}}{\underset{\underset{12}{\cancel{48}}}{\cancel{48}} \cancel{960}} \quad \text{or} \quad \frac{560}{960} = \frac{56}{96} = \frac{28}{48} = \frac{7}{12}.$$

Method 2. Divide by the common factors in the prime-power factorizations of a and b . Using this method, we have

$$\frac{560}{960} = \frac{2^4 \cdot 5 \cdot 7}{2^6 \cdot 3 \cdot 5} = \frac{7}{2^2 \cdot 3} = \frac{7}{12}.$$

Method 3. Divide a and b by GCD (a, b). Using the ideas of Section 4.3, calculate the greatest common divisor $\text{GCD}(560, 960) = 80$. Therefore, $\frac{560}{960} = \frac{560 \div 80}{960 \div 80} = \frac{7}{12}$.

EXAMPLE

6.2

Simplifying Fractions

Find the simplest form of each fraction.

(a) $\frac{240}{72}$

(b) $\frac{294}{84}$

(c) $\frac{48}{64}$

Solution

(a) By Method 1, $\frac{240}{72} = \frac{120}{36} = \frac{60}{18} = \frac{10}{3}$, where the successive common factors 2, 2, and 6 were divided into the numerator and denominator.

(b) Using Method 2 this time, we find that

$$\frac{294}{-84} = \frac{2 \cdot 3 \cdot 7^2}{(-2) \cdot 2 \cdot 3 \cdot 7} = \frac{7}{-2} = \frac{-7}{2}.$$

(c) By Method 3, $\text{GCD}(48, 64) = 16$, $\frac{48}{64} = \frac{48 \div 16}{64 \div 16} = \frac{3}{4}$.

Common Denominators

Fractions with the same denominator are said to have a **common denominator**. In working with fractions, it is often helpful to rewrite them with equivalent fractions that share a common denominator.

For example, $\frac{5}{8}$ and $\frac{7}{10}$ can each be rewritten by equivalent fractions with the common denominator $8 \cdot 10 = 80$, so

$$\frac{5}{8} = \frac{5 \cdot 10}{8 \cdot 10} = \frac{50}{80} \quad \text{and} \quad \frac{7}{10} = \frac{7 \cdot 8}{10 \cdot 8} = \frac{56}{80}.$$

In the same way, any two fractions $\frac{a}{b}$ and $\frac{c}{d}$ can be rewritten with the common denominator $b \cdot d$, since $\frac{a}{b} = \frac{a \cdot d}{b \cdot d}$ and $\frac{c}{d} = \frac{c \cdot b}{d \cdot b}$.

Sometimes we need to find the common positive denominator that is as small as possible. Assuming that $\frac{a}{b}$ and $\frac{c}{d}$ are in simplest form, we require a common denominator that is the least common multiple of both b and d . This denominator is called the **least common denominator**. For the example $\frac{5}{8}$ and $\frac{7}{10}$, we would calculate $\text{LCM}(8, 10) = 40$, so 40 is the least common denominator. Therefore,

$$\frac{5}{8} = \frac{5 \cdot 5}{8 \cdot 5} = \frac{25}{40} \quad \text{and} \quad \frac{7}{10} = \frac{7 \cdot 4}{10 \cdot 4} = \frac{28}{40}.$$

We can often find the least common denominator using mental arithmetic. Consider, $\frac{5}{6}$ and $\frac{3}{8}$, and notice that $4 \cdot 6 = 24$ and $3 \cdot 8 = 24$ is the least common denominator. Thus, $\frac{5}{6} = \frac{20}{24}$ and $\frac{3}{8} = \frac{9}{24}$ when written with the least common denominator.

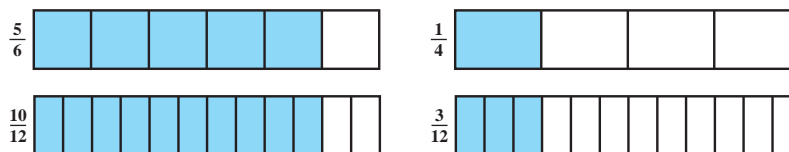
EXAMPLE 6.3**Finding Common Denominators**

Find equivalent fractions with a common denominator.

- (a) $\frac{5}{6}$ and $\frac{1}{4}$
- (b) $\frac{9}{8}$ and $\frac{-12}{7}$
- (c) $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{2}{3}$

Solution

- (a) By mental arithmetic, the least common denominator of $\frac{5}{6}$ and $\frac{1}{4}$ is 12. Therefore, $\frac{5}{6} = \frac{10}{12}$ and $\frac{1}{4} = \frac{3}{12}$. This can be shown with fraction strips:



- (b) Since 8 and 7 have no common factors other than 1, $8 \cdot 7 = 56$ is the least common denominator:

$$\frac{9}{8} = \frac{9 \cdot 7}{8 \cdot 7} = \frac{63}{56} \quad \text{and} \quad \frac{-12}{7} = \frac{-12 \cdot 8}{7 \cdot 8} = \frac{-96}{56}.$$

- (c) With three fractions, it is still possible to use the product of all of the denominators as a common denominator. Since $4 \cdot 8 \cdot 3 = 96$, we have

$$\frac{3}{4} = \frac{3 \cdot 8 \cdot 3}{4 \cdot 8 \cdot 3} = \frac{72}{96}, \quad \frac{5}{8} = \frac{5 \cdot 4 \cdot 3}{8 \cdot 4 \cdot 3} = \frac{60}{96}, \quad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 4 \cdot 8}{3 \cdot 4 \cdot 8} = \frac{64}{96}.$$

Alternatively, $\text{LCM}(4, 8, 3) = 24$, so

$$\frac{3}{4} = \frac{3 \cdot 6}{4 \cdot 6} = \frac{18}{24}, \quad \frac{5}{8} = \frac{5 \cdot 3}{8 \cdot 3} = \frac{15}{24}, \quad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 8}{3 \cdot 8} = \frac{16}{24}.$$

These equations express the three fractions as equivalent fractions with the least common denominator.

Rational Numbers

We have seen that different fractions can express the same number amount. For example, the fraction strips in Figure 6.8 show that $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{8}{12}$ represent the same number. In Figure 6.6, one point on the number line can be expressed by infinitely many fractions.

Any number that can be represented by a fraction *or* any of its equivalent fractions is called a **rational number**. Any one of these equivalent fractions can be used to represent that rational number. For example, $\frac{2}{3}$ and $\frac{10}{15}$ are different fractions, since their numerators and denominators are different. However, since $\frac{2}{3}$ and $\frac{10}{15}$ are equivalent fractions, they both represent the *same rational number*. When we say “the rational number $\frac{2}{3}$,” we really mean the rational number represented by the fraction $\frac{2}{3}$ or any fraction that is equivalent to $\frac{2}{3}$.

DEFINITION Rational Numbers

A **rational number** is a number that can be represented by a fraction $\frac{a}{b}$, where a and b are integers, $b \neq 0$.

Two rational numbers are **equal** if and only if they can be represented by equivalent fractions, and equivalent fractions represent the same rational number.

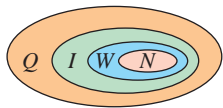


Figure 6.9
The Venn diagram of the number systems N , W , I , and Q

The **set of rational numbers** is denoted by Q . The letter Q reminds us that a fraction is represented as a quotient. A rational number such as $\frac{3}{4}$ can also be represented by $\frac{6}{8}$, $\frac{30}{40}$, or any other fraction equivalent to $\frac{3}{4}$. An integer $n \in I$ is identified with the rational number $\frac{n}{1}$, so the integers I are a proper subset of the rational numbers Q . Figure 6.9 shows the Venn diagram of the sets of natural numbers N , whole numbers W , integers I , and rational numbers Q .

EXAMPLE 6.4

Representing Rational Numbers

How many different rational numbers are given in this list of five fractions?

$$\frac{2}{5}, \quad 3, \quad \frac{-4}{-10}, \quad \frac{39}{13}, \quad \text{and} \quad \frac{7}{4}$$

Solution

Since $\frac{2}{5} = \frac{-4}{-10}$ and $\frac{3}{1} = \frac{39}{13}$, there are three different rational numbers: $\frac{2}{5}$, 3, and $\frac{7}{4}$.

Ordering Fractions and Rational Numbers

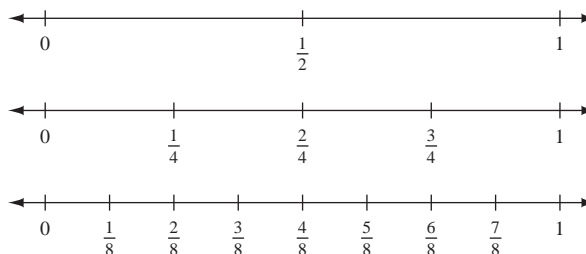
Two fractions with the same positive denominator are ordered by comparing their numerators. For example, $\frac{3}{7} < \frac{4}{7}$ and $\frac{-4}{5} < \frac{3}{5}$. To order two fractions with different denominators, we can rewrite them with a common positive denominator and then compare their numerators. For example, by rewriting $\frac{3}{4}$ and $\frac{5}{6}$ as $\frac{9}{12}$ and $\frac{10}{12}$, we see that $\frac{3}{4} < \frac{5}{6}$.

Sometimes it is helpful to compare two fractions with a third one. For example, to order $\frac{3}{7}$ and $\frac{5}{9}$, we see that $\frac{3}{7} < \frac{1}{2}$ and $\frac{1}{2} < \frac{5}{9}$. Therefore, $\frac{3}{7} < \frac{1}{2} < \frac{5}{9}$, and we conclude that $\frac{3}{7} < \frac{5}{9}$.

Manipulatives and pictorial representations of fractions also reveal the order relation. The following excerpt shows how parallel number lines help visualize the order relation:

FROM THE NCTM PRINCIPLES AND STANDARDS

During grades 3–5, students should build their understanding of fractions as parts of a whole and as division. They will need to see and explore a variety of models of fractions, focusing primarily on familiar fractions such as halves, thirds, fourths, fifths, sixths, eighths, and tenths. By using an area model in which part of a region is shaded, students can see how fractions are related to a unit whole, compare fractional parts of a whole, and find equivalent fractions. They should develop strategies for ordering and comparing fractions, often using benchmarks such as $\frac{1}{2}$ and 1. For example, fifth graders can compare fractions such as $\frac{2}{5}$ and $\frac{5}{8}$ by comparing each with $\frac{1}{2}$ —one is a little less than $\frac{1}{2}$, and the other is a little more. By using parallel number lines, each showing a unit fraction and its multiples, students can see fractions as numbers, note their relationship to 1, and see relationships among fractions, including equivalence.

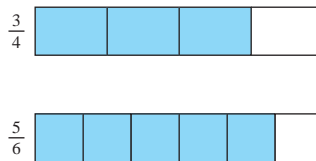


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We can also use fraction strips to order fractions. For example, by aligning the fraction strips for $\frac{3}{4}$ and $\frac{5}{6}$, as shown in Figure 6.10, we see that $\frac{3}{4}$ represents a smaller shaded portion of a whole strip than $\frac{5}{6}$ does.

Figure 6.10

Showing that $\frac{3}{4} < \frac{5}{6}$
with fraction strips



The ordering just described for fractions is used to define an ordering of rational numbers. For example, suppose we were told to make a right turn in three-quarters of a mile, and the odometer on our car shows we have gone six-tenths of a mile. Did we miss our turn, or do we need to drive further? To decide, we need to compare the rational number “six-tenths” with the rational number “three-quarters.”

We can represent these rational numbers by the fractions $\frac{6}{10}$ and $\frac{3}{4}$. To compare these two fractions, rewrite them with a common denominator, $\frac{6}{10} = \frac{6 \cdot 4}{10 \cdot 4}$ and $\frac{3}{4} = \frac{10 \cdot 3}{10 \cdot 4}$. Since $6 \cdot 4 < 10 \cdot 3$, we see that $\frac{6}{10} < \frac{3}{4}$.

To compare any two rational numbers, say, r and s , first write r and s as fractions $r = \frac{a}{b}$ and $s = \frac{c}{d}$, both with positive denominators b and d . Since $r = \frac{a}{b} = \frac{ad}{bd}$ and $s = \frac{c}{d} = \frac{bc}{bd}$, we see that $\frac{a}{b} < \frac{c}{d}$ if and only if the “cross products” ad and bc satisfy $ad < bc$. Earlier, we had seen that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

DEFINITION Ordering Rational Numbers

Let two rational numbers be represented by the fractions $\frac{a}{b}$ and $\frac{c}{d}$, with b and d positive. Then $\frac{a}{b}$ is **less than** $\frac{c}{d}$, written $\frac{a}{b} < \frac{c}{d}$, if and only if $ad < bc$.

The corresponding relations “less than or equal to,” \leq ; “greater than,” $>$; and “greater than or equal to,” \geq , are defined similarly.

It makes no difference which equivalent fractions are used to represent the rational numbers in the definition; the ordering is always the same. For example, if the rational numbers “six-tenths” and “three-quarters” are represented by $\frac{3}{5}$ and $\frac{9}{12}$, their cross products are $3 \cdot 12 = 36$ and $5 \cdot 9 = 45$. Since $36 < 45$, we still see that “six-tenths” is less than “three-quarters.”

EXAMPLE

6.5

Comparing Rational Numbers

Are the following relations correct? If not, what is the correct relation?

(a) $\frac{3}{4} > \frac{2}{5}$

(b) $\frac{15}{29} = \frac{6}{11}$

(c) $\frac{2106}{7047} = \frac{234}{783}$

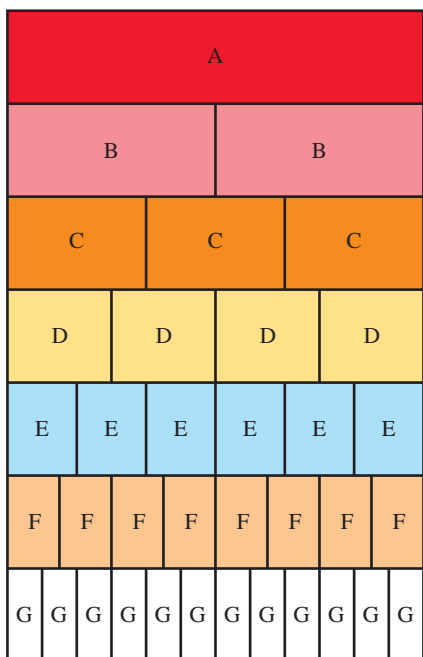
(d) $\frac{-10}{13} < \frac{-22}{29}$

Solution

- (a) Correct. Since $3 \cdot 5 = 15 > 8 = 2 \cdot 4$, we have $\frac{3}{4} > \frac{2}{5}$. Alternatively, $\frac{1}{2}$ can be used as a benchmark, with $\frac{3}{4} > \frac{1}{2}$ and $\frac{2}{5} < \frac{1}{2}$. Thus, $\frac{3}{4} > \frac{2}{5}$ by the transitive strategy.
- (b) Incorrect. Since $15 \cdot 11 = 165 < 174 = 6 \cdot 29$, we have $\frac{15}{29} < \frac{6}{11}$.
- (c) Using a calculator, we find that $2106 \cdot 783 = 1,648,998 = 7047 \cdot 234$. Thus, the two rational numbers are equal: $\frac{2106}{7047} = \frac{234}{783}$.
- (d) Since $-10 \cdot 29 = -290 < -286 = -22 \cdot 13$, we conclude that $\frac{-10}{13} < \frac{-22}{29}$.

COOPERATIVE INVESTIGATION**Exploring Fraction Concepts with Fraction Tiles**

The basic concepts of fractions are best taught with a variety of manipulative materials that can be purchased from education supply houses or, alternatively, can be homemade from patterns printed and cut from card stock. The commercially made manipulatives are usually colored plastic, with the unit a square, a circle, a rectangle, or some other shape that can be subdivided in various ways to give denominators such as 2, 3, 4, 6, 8, and 12. Usually, the shapes are also available in translucent plastic for demonstrations on an overhead projector. In the following activity, you will make and then work with a set of *fraction tiles*:

**Materials**

Each group of three or four students needs a set of fraction tiles, either a plastic set or a set cut from card stock using a pattern such as the one shown (at reduced scale) to the left. The pieces at the right are lettered A through G but can also be described by their color.

Directions

If a B tile is chosen as the unit, then the fraction $\frac{1}{2}$ is represented by a D tile, which can be verified by comparing two D tiles and one B tile. Similarly, an A tile represents 2, and $\frac{2}{3}$ is represented by two E tiles or by a C tile. Now use your fraction tile set to answer and discuss the questions that follow. Compare and discuss your answers within your group.

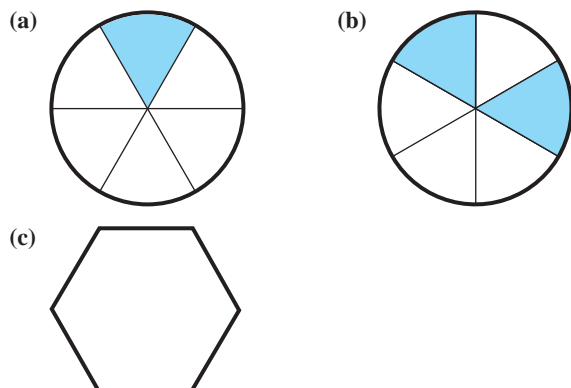
Questions

- If an A tile is the unit, find (one or more) tiles to represent $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, $\frac{2}{6}$, and $\frac{5}{12}$.
- If a C tile is $\frac{1}{6}$, what fractions are represented by each of the other tiles?
- If a B tile represents $\frac{3}{2}$, represent these fractions with tiles: $\frac{1}{2}$, $\frac{1}{1}$, $\frac{1}{4}$, $\frac{3}{1}$, and $\frac{3}{4}$.

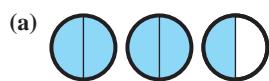
6.1 Problem Set

Understanding Concepts

1. What fraction is represented by the shaded portion of the following figures?



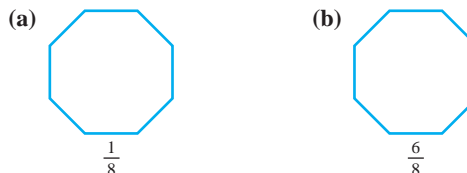
2. What fraction is represented by the shaded portion of the following figures?



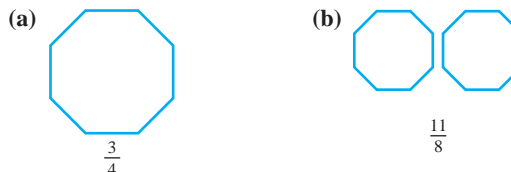
Unit = one disc



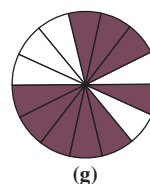
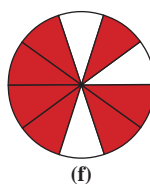
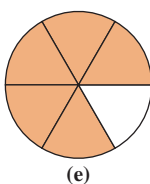
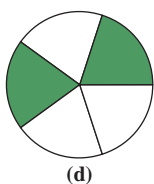
3. Subdivide and shade the unit octagons shown to represent the given fraction.



4. Subdivide and shade the unit octagons shown to represent the given fraction.



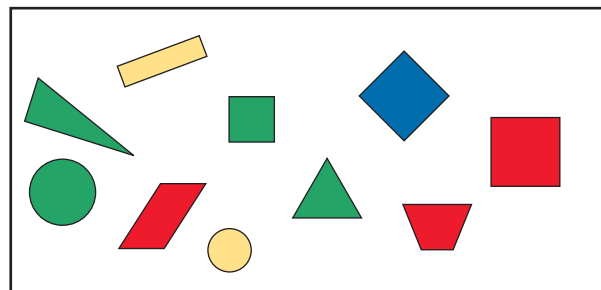
5. What fraction is illustrated by the shaded region of these rectangles and circles, each having one unit of area?



6. Draw fraction strips and fraction circle diagrams to represent these fractions:

(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{2}{8}$ (d) $\frac{7}{12}$

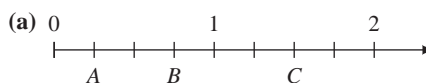
7. Consider the following set of objects:

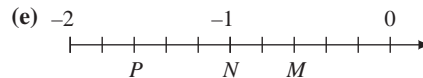
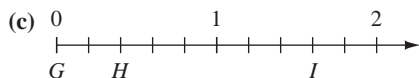
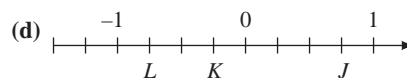
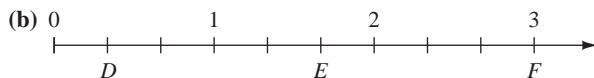


What is the fraction of the objects in the set that

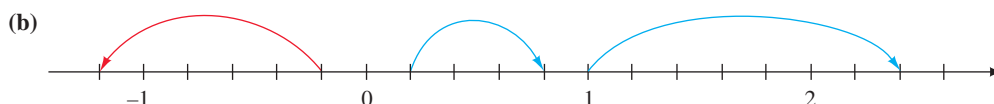
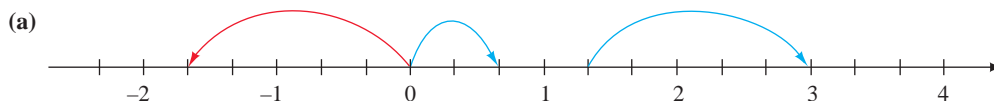
- (a) are green?
 (b) are squares?
 (c) are rectangles?
 (d) are quadrilaterals (four-sided polygons)?
 (e) have no mirror line of symmetry?

8. For each lettered point on the number lines shown, express its position by a corresponding fraction. Remember, it is the number of subintervals in a unit interval that determines the denominator, not the number of tick marks.





9. For each number-line diagram, list the fractions represented by the arrows:



10. Choose a fraction from the set $\left\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{6}{13}, \frac{17}{20}, \frac{17}{30}\right\}$ that best represents the shaded area of each of these unit rectangles.

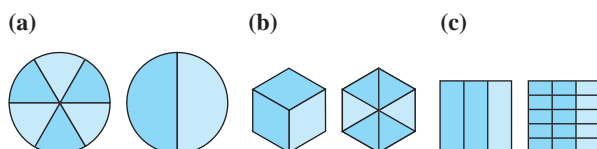


11. Depict the fraction $\frac{4}{6}$ with the following models:

- (a) Fraction-circle model (b) Set model
(c) Fraction-strip model (d) Number-line model

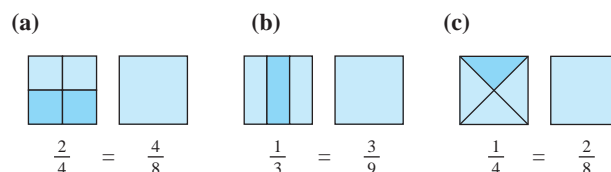
12. In Figure 6.8, fraction strips show that $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}$, and $\frac{8}{12}$ are equivalent fractions. Use a similar drawing of fraction strips to show that $\frac{3}{4}, \frac{6}{8}$, and $\frac{9}{12}$ are equivalent fractions.

13. What equivalence of fractions is shown in these pairs of colored-region models?



14. Find four different fractions equivalent to $\frac{4}{9}$.

15. Subdivide and shade the unit square on the right to illustrate that the given fractions are equivalent.



16. Fill in the missing integer to make the fractions equivalent.

- (a) $\frac{4}{5} = \frac{\quad}{30}$ (b) $\frac{6}{9} = \frac{2}{\quad}$
(c) $\frac{-7}{25} = \frac{\quad}{500}$ (d) $\frac{18}{3} = \frac{-6}{\quad}$

17. Determine whether each set of two fractions is equivalent by calculating equivalent fractions with a common denominator.

- (a) $\frac{18}{42}$ and $\frac{3}{7}$ (b) $\frac{18}{49}$ and $\frac{5}{14}$
(c) $\frac{9}{25}$ and $\frac{140}{500}$ (d) $\frac{24}{144}$ and $\frac{32}{96}$

18. Determine which of these pairs of fractions are equivalent:

- (a) $\frac{78}{24}$ and $\frac{546}{168}$ (b) $\frac{243}{317}$ and $\frac{2673}{3487}$
(c) $\frac{412}{-864}$ and $\frac{-308}{616}$

19. (a) Is it true that $\frac{4 \cdot 3}{9 \cdot 3} = \frac{12}{27}$?
(b) Is it true that $\frac{4 \cdot 3}{9 \cdot 3} = \frac{4}{9}$?
(c) Is it true that $\frac{4 + 3}{9 + 3} = \frac{7}{12}$?
(d) Is it true that $\frac{4 + 3}{9 + 3} = \frac{4}{9}$?

20. Rewrite the following fractions in simplest form:

- (a) $\frac{84}{144}$ (b) $\frac{208}{272}$
(c) $\frac{-930}{1290}$ (d) $\frac{325}{231}$

21. Find the prime factorizations of the numerators and denominators of these fractions, and use them to express the fractions in simplest form:

(a) $\frac{96}{288}$

(b) $\frac{247}{-75}$

(c) $\frac{2520}{378}$

22. For each of these sets of fractions, determine equivalent fractions with a common denominator:

(a) $\frac{3}{11}$ and $\frac{2}{5}$

(b) $\frac{5}{12}$ and $\frac{2}{3}$

(c) $\frac{4}{3}$, $\frac{5}{8}$, and $\frac{1}{6}$

(d) $\frac{1}{125}$ and $\frac{-3}{500}$

23. For each of these sets of fractions, determine equivalent fractions with the least common denominator.

(a) $\frac{3}{8}$ and $\frac{5}{6}$

(b) $\frac{1}{7}$, $\frac{4}{5}$, and $\frac{2}{3}$

(c) $\frac{17}{12}$ and $\frac{7}{32}$

(d) $\frac{17}{51}$ and $\frac{56}{42}$

24. Order each pair of fractions.

(a) $\frac{11}{7}$, $\frac{11}{9}$

(b) $\frac{3}{13}$, $\frac{4}{13}$

(c) $\frac{13}{18}$, $\frac{7}{12}$

(d) $\frac{9}{11}$, $\frac{8}{10}$

25. Order the rational numbers from least to greatest in each part.

(a) $\frac{2}{3}$, $\frac{7}{12}$

(b) $\frac{2}{3}$, $\frac{5}{6}$

(c) $\frac{5}{6}$, $\frac{29}{36}$

(d) $\frac{-5}{6}$, $\frac{-8}{9}$

(e) $\frac{2}{3}$, $\frac{5}{6}$, $\frac{29}{36}$, $\frac{8}{9}$

26. Find the numerator or denominator of these fractions to create a fraction that is close to $\frac{1}{2}$ but is slightly larger:

(a) $\frac{\quad}{20}$

(b) $\frac{7}{\quad}$

(c) $\frac{\quad}{17}$

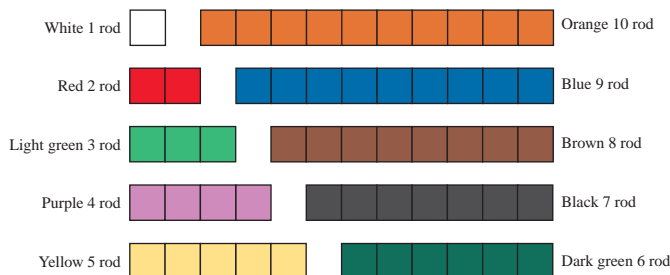
(d) $\frac{30}{\quad}$

27. How many different rational numbers are in this list?

$\frac{27}{36}$, 4, $\frac{21}{28}$, $\frac{24}{6}$, $\frac{3}{4}$, $\frac{-8}{-2}$

Into the Classroom

28. Number rods (or Cuisenaire® rods) can be used to illustrate the basic concepts of fractions. For example, suppose that the brown rod is the whole (that is, the unit). Then, since a train of two purple rods has the length of the brown rod, we see that the purple rod represents the fraction $\frac{1}{2}$:

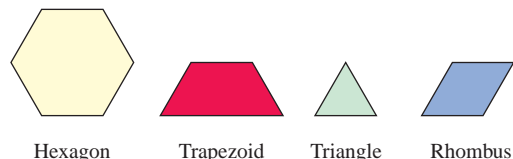


- (a) Which rod represents $\frac{1}{4}$? $\frac{3}{4}$?

- (b) What fraction does a train made from the red and orange rods laid end to end represent?

- (c) Using the discussion and questions in this problem as a model, write several more activities and accompanying questions that you feel would be effective in the elementary school classroom to teach basic fraction concepts with the use of number rods as a manipulative.

29. Pattern blocks are a popular and versatile manipulative that is used successfully in many elementary school classrooms. The following four shapes are included in any set.

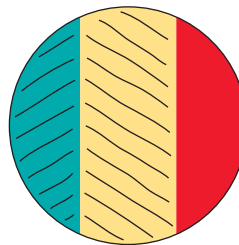


Use your pattern blocks to answer these questions:

- (a) Choose the hexagon as the unit. What fraction is each of the other three pattern blocks?
- (b) Choose the trapezoid as the unit. What fraction is each of the other three pattern blocks?
- (c) Choose the rhombus as the unit. What fraction is each of the other three pattern blocks?

Responding to Students

30. When asked to illustrate the concept of $\frac{2}{3}$ with a colored-region diagram, Shanti drew the figure shown. How would you respond to Shanti?



31. (Writing) Like most fifth graders, Dana likes pizza. When given the choice of $\frac{1}{4}$ or $\frac{1}{6}$ of a pizza, Dana says, "Since 6 is bigger than 4 and I'm really hungry, I'd rather have $\frac{1}{6}$ of the pizza." Write a dialogue, including useful diagrams, to clear up Dana's misconception about fractions.
32. Nicole claims that $\frac{3}{4} < \frac{5}{8}$ because $3 < 5$ and $4 < 8$. Is Nicole correct, or how can you help her understanding?
33. Miley has been asked to compare $\frac{3}{8}$ and $\frac{4}{13}$. She says that if she had three $\frac{3}{8}$, she would have $\frac{9}{8}$, a bit more than 1, but if she had

three $\frac{4}{13}$, she would have $\frac{12}{13}$, which is less than 1. Therefore, Miley claims that $\frac{3}{8} > \frac{4}{13}$. Is Miley's reasoning valid?

Thinking Critically

34. (a) The rectangle shown is $\frac{2}{3}$ of a unit. What is the unit?

(Suggestion: Divide the rectangle into two identical rectangles.)



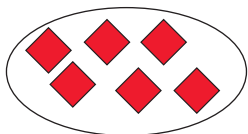
- (b) The rectangle shown is $\frac{5}{2}$ of a whole. What is the whole?

(Suggestion: Use the idea from part (a), but first decide the number of parts into which to subdivide the rectangle.)

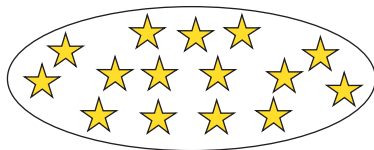


- (c) (Writing) Carefully explain how to find the unit if any rectangle has area $\frac{a}{b}$.

35. (a) The set shown is $\frac{3}{4}$ of a unit. What is the unit?



- (b) The set shown is $\frac{5}{3}$ of a unit. What is the unit?

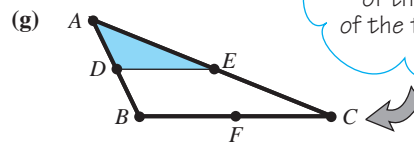
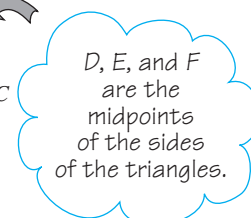
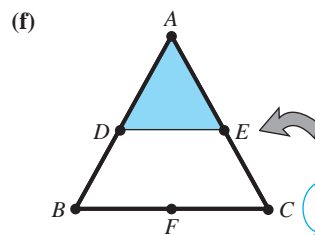
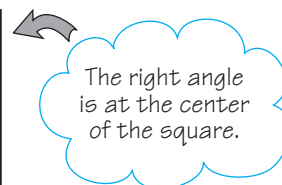
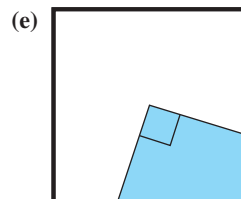
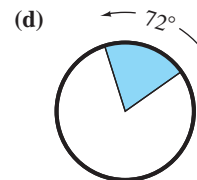
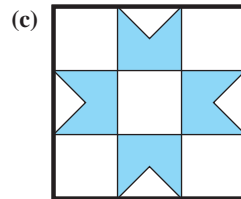
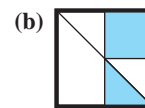
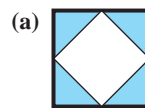
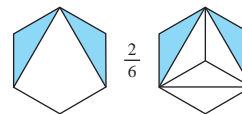


36. Decide whether each statement is *true* or *false*. Explain your reasoning in a brief paragraph.

- There are infinitely many ways to replace two fractions with two equivalent fractions that have a common denominator.
- There is a unique least common denominator for a given pair of fractions.
- There is a least positive fraction.
- There are infinitely many fractions between 0 and 1.

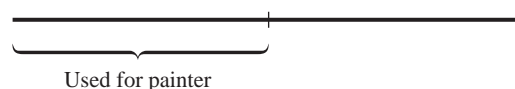
37. What fraction represents the part of the whole region that has been shaded? Draw additional lines to make your answer visually clear. For example, $\frac{2}{6}$ of the regular hexagon on the

left is shaded, since the entire hexagon can be subdivided into six congruent regions, as shown on the right.



38. Solve this fraction problem from ancient India: "In what time will four fountains, being let loose together, fill a cistern, which they would severally [i.e., individually] fill in a day, in half a day, in a quarter and in a fifth part of a day?" (Reference: *History of Hindu Mathematics*, by B. Datta and A. N. Singh. Bombay, Calcutta, New Delhi, Madras, London, New York: Asia Publishing House, 1962, p. 234.)

39. Andrei bought a length of rope at the hardware store. He used half of it to make a bow painter (a rope located at the front of the canoe) for his canoe and then used a third of the remaining piece to tie up a roll of carpet. He now has 20 feet of rope left. What was the length of rope Andrei purchased? Since it is often helpful to visualize a problem, obtain your answer pictorially by adding additional marks and labels to the following drawing:



40. Use a diagram similar to the previous problem to answer this question. Andrei used one-third of the rope to make a leash for his dog and one-fourth of the remaining rope to tie up a bundle of stakes. This left 15 feet of the rope unused. What was the original length of the rope?
41. (Writing) The following row of Pascal's triangle (see Chapter 1) has been separated by a vertical bar drawn between the 15 and the 20, with 3 entries of the row to the left of the bar and 4 entries to the right of the bar:

1 6 15 | 20 15 6 1

Notice that $\frac{15}{20} = \frac{3}{4}$. That is, the two fractions formed by the

entries adjacent to the dividing bar and by the number of entries to the left and the right of the bar are equivalent fractions. Was this equivalence an accident, or is it a general property of Pascal's triangle? Investigate the property further by placing the dividing bar in new locations and examining other rows of Pascal's triangle. Write a report summarizing your calculations.

Making Connections

42. Express the following quantities by a fraction placed in the blank space:

- (a) 20 minutes is _____ of an hour.
 (b) 30 seconds is _____ of a minute.
 (c) 5 days is _____ of a week.
 (d) 25 years is _____ of a century.
 (e) A quarter is _____ of a dollar.
 (f) 3 eggs is _____ of a dozen.
 (g) 2 feet is _____ of a yard.
 (h) 3 cups is _____ of a quart.

43. Francisco's pickup truck has a 24-gallon gas tank and an accurate fuel gauge. Estimate the number of gallons in the tank at these readings:



- (a) (b) (c)

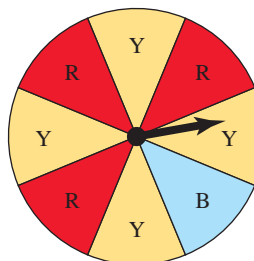
44. If 153 of the 307 graduating seniors go on to college, it is likely that a principal would claim that $\frac{1}{2}$ of the class is college bound. Give simpler convenient fractions that approximately express the data in these situations:

- (a) Estebán is on page 310 of a 498-page novel. He has read _____ of the book.
 (b) Myra has saved \$73 toward the purchase of a \$215 plane ticket. She has saved _____ of the amount she needs.
 (c) Nine students in Ms. Evaldo's class of 35 students did perfect work on the quiz. _____ of the class scored 100% on the quiz.
 (d) The Math Club has sold 1623 of the 2400 raffle tickets. It has sold _____ of the available tickets.

45. **Fractions in Probability.** If a card is picked at random from an ordinary deck of 52 playing cards, there are 4 ways it can be an ace, since it could be the ace of hearts, diamonds, clubs, or spades. To measure the chances of drawing an ace, it is common to give the probability as the rational number $\frac{4}{52}$.

In general, if n equally likely outcomes are possible and m of these outcomes are successful for an event to occur, then the probability of the event is $\frac{m}{n}$. As another example, $\frac{5}{6}$ is the probability of rolling a single die and having more than one spot appear. Give fractions that express the probability of the following events:

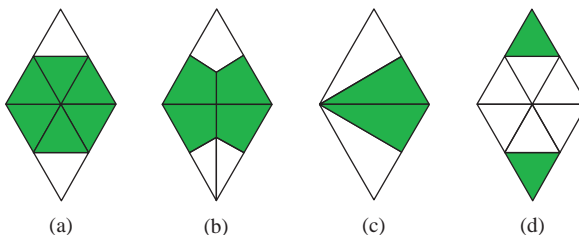
- (a) Getting a head in the flip of a fair coin
 (b) Drawing a face card from a deck of cards
 (c) Rolling an even number on a single die
 (d) Drawing a green marble from a bag that contains 20 red, 30 blue, and 25 green marbles
 (e) Drawing either a red or a blue marble from the bag of marbles described in part (d)
46. What fraction represents the probability that the spinner shown comes up (a) yellow? (b) red? (c) blue? (d) not blue?



State Assessments

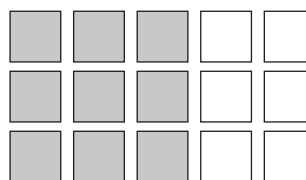
47. (Grade 5)

Which figure has $\frac{2}{3}$ of the area of the diamond shape shaded?



48. (Writing) (Grade 5)

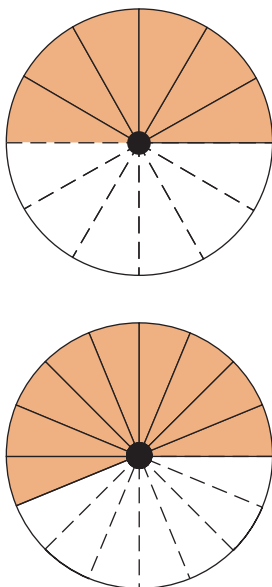
LuAnn thinks that $\frac{3}{5}$ of the squares are shaded and Jim thinks that $\frac{9}{15}$ of the squares are shaded.



- (a) Who is correct—LuAnn, Jim, or both?
 (b) In words and diagrams, explain how you answered part (a).

49. (Grade 4)

Two pizzas were partially eaten at the party, as shown below.

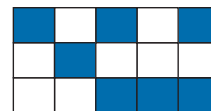


Which of these fraction inequalities compares the portion of pizza that was not eaten?

- A. $\frac{5}{12} < \frac{9}{16}$ B. $\frac{9}{12} < \frac{5}{16}$
 C. $\frac{5}{12} < \frac{9}{12}$ D. $\frac{5}{16} < \frac{9}{12}$

50. (Grade 5)

A dart is thrown repeatedly at this rectangle. How likely is it that it lands in a shaded region?



- A. Rarely
 B. Most of the time
 C. Not quite half the time
 D. Over half the time

51. (Grade 5)

Which of the following lists the mixed numbers in order from **least to greatest**?

- A. $7\frac{1}{5}$ $4\frac{2}{5}$ $2\frac{3}{5}$ $1\frac{4}{5}$ B. $5\frac{2}{5}$ $6\frac{3}{5}$ $6\frac{4}{5}$ $7\frac{1}{5}$
 C. $3\frac{3}{5}$ $2\frac{3}{5}$ $3\frac{4}{5}$ $4\frac{1}{5}$ D. $6\frac{4}{5}$ $4\frac{2}{5}$ $3\frac{2}{5}$ $1\frac{1}{5}$

52. (Massachusetts, Grade 6)

Which point on the number line shown below appears to be located at $1\frac{3}{8}$?



6.2

Addition and Subtraction of Fractions

The geometric and physical models of fractions motivate the definitions of the addition and subtraction of fractions. Since rational numbers are represented with fractions, we are simultaneously defining sums and differences of rational numbers.

Addition of Fractions

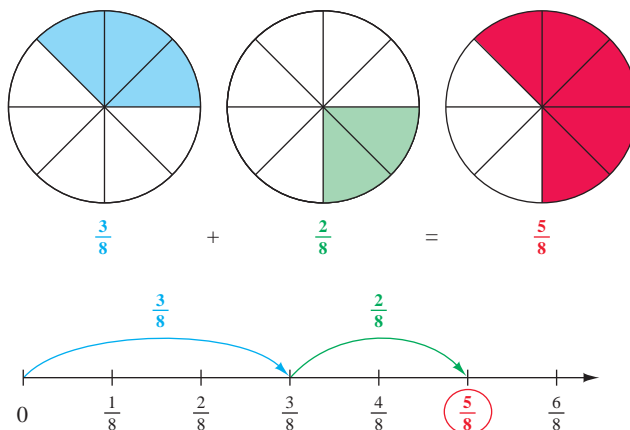
The sum of $\frac{3}{8}$ and $\frac{2}{8}$ is illustrated in two ways in Figure 6.11. The fraction-circle and number-line models both show that $\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$. The models suggest that the sum of two fractions with a common denominator is found by adding the two numerators. This motivates the following definition:

DEFINITION Addition of Fractions with a Common Denominator

Let two fractions $\frac{a}{b}$ and $\frac{c}{b}$ have a common denominator. Then their **sum** is the fraction

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

Figure 6.11
Showing that
 $\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$ with the
fraction-circle and
number-line models



To add fractions with unlike denominators, we first rewrite the fractions with a common denominator. For example, to add $\frac{1}{4}$ and $\frac{2}{3}$, rewrite them with 12 as a common denominator:

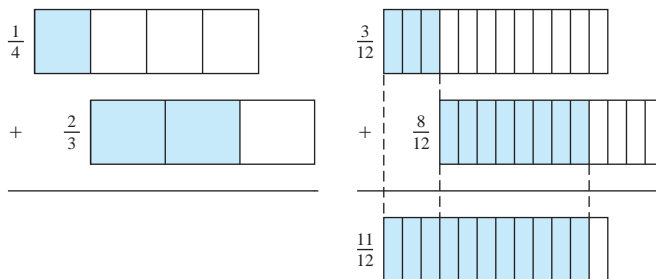
$$\frac{1}{4} = \frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12} \quad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}.$$

According to the preceding definition, we then have

$$\frac{1}{4} + \frac{2}{3} = \frac{1 \cdot 3}{4 \cdot 3} + \frac{2 \cdot 4}{3 \cdot 4} = \frac{1 \cdot 3 + 2 \cdot 4}{4 \cdot 3} = \frac{3 + 8}{12} = \frac{11}{12}.$$

The procedure just followed can be modeled with fraction strips, as shown in Figure 6.12. It is important to see how the fraction strips are aligned.

Figure 6.12
The fraction-strip
model showing that
 $\frac{1}{4} + \frac{2}{3} =$
 $\frac{3}{12} + \frac{8}{12} = \frac{11}{12}$



The same procedure can be followed to add any two fractions $\frac{a}{b}$ and $\frac{c}{d}$. We have

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{c \cdot b}{d \cdot b} = \frac{ad + bc}{bd}.$$

↑ Rewrite the fractions with a common denominator.

← Add the numerators and retain the common denominator.

FORMULA Addition of Fractions with Arbitrary Denominators

Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. Then their sum is the fraction

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

There are two important observations about this formula:

- The formula also applies to the addition of fractions with common denominators, since

$$\frac{a}{b} + \frac{c}{b} = \frac{ab + bc}{bb} = \frac{(a + c)b}{bb} = \frac{a + c}{b}.$$

- The formula is a useful computational rule, but it obscures the conceptual meaning of the addition of two numbers. Therefore, it is often better to continue to use the common denominator approach to the addition of fractions and to model fraction addition with manipulatives and pictorial representations.

EXAMPLE 6.6**Adding Fractions**

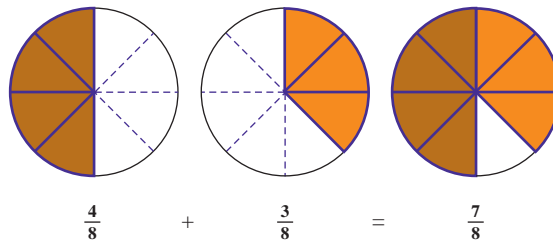
Show how to compute each of the sums of fractions that follow. Imagine that you are giving a careful explanation to a fifth grader, using an appropriate representation of fraction addition to illustrate how you obtain the answer.

(a) $\frac{1}{2} + \frac{3}{8}$ (b) $\frac{3}{8} + \frac{-7}{12}$ (c) $\left(\frac{3}{4} + \frac{5}{6}\right) + \frac{2}{3}$ (d) $\frac{3}{4} + \left(\frac{5}{6} + \frac{2}{3}\right)$

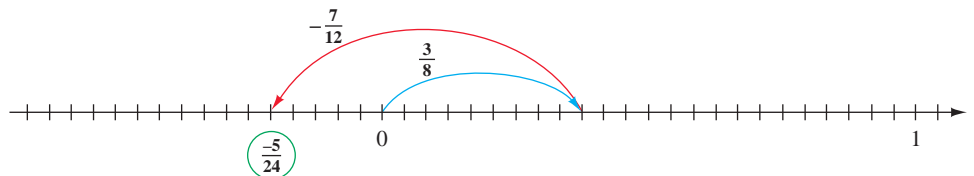
Solution

- (a) Since the two fractions have different denominators, we first look for a common denominator. One choice is $2 \cdot 8 = 16$, but it's even simpler to use the least common denominator, 8.

Then $\frac{1}{2} + \frac{3}{8} = \frac{4}{8} + \frac{3}{8} = \frac{7}{8}$. This relationship can also be shown with fraction circles:



- (b) We could use the common denominator $8 \cdot 12 = 96$, but since 4 is a common divisor of 8 and 12, we can also use the common denominator $96 \div 4 = 24$. This is the least common denominator, and we get $\frac{3}{8} + \frac{-7}{12} = \frac{9}{24} + \frac{-14}{24} = \frac{9 + (-14)}{24} = \frac{-5}{24}$. The following number-line diagram shows the same result:



- (c) The parentheses tell us first to compute

$$\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}.$$

Then we compute

$$\left(\frac{3}{4} + \frac{5}{6}\right) + \frac{2}{3} = \frac{19}{12} + \frac{2}{3} = \frac{19}{12} + \frac{8}{12} = \frac{19+8}{12} = \frac{27}{12},$$

which simplifies to $\frac{9}{4}$.

(d) The sum in parentheses is

$$\frac{5}{6} + \frac{2}{3} = \frac{5}{6} + \frac{4}{6} = \frac{9}{6} = \frac{3}{2}.$$

Then we have

$$\frac{3}{4} + \left(\frac{5}{6} + \frac{2}{3}\right) = \frac{3}{4} + \frac{3}{2} = \frac{3}{4} + \frac{6}{4} = \frac{9}{4}.$$

Parts (c) and (d) of Example 6.6 show that

$$\left(\frac{3}{4} + \frac{5}{6}\right) + \frac{2}{3} = \frac{3}{4} + \left(\frac{5}{6} + \frac{2}{3}\right),$$

since each side is $\frac{9}{4}$. This result is a particular example of the associative property for the addition of fractions. The properties of addition and subtraction of fractions and, therefore, of rational numbers will be explored in Section 6.4.

INTO THE CLASSROOM

Ann Hlabangana-Clay Discusses the Addition of Fractions



I use red shoelace licorice to introduce adding fractions. It is flexible and tangible for small fingers to demonstrate whole to part. To start the lesson, I give each student one whole red shoelace licorice. I ask them to spread it out from end to end and use it to measure a starting line and an ending line. To find $\frac{1}{2} + \frac{3}{4}$, I give each student a $\frac{1}{2}$ length shoelace and have them measure it against the whole. Each student also gets a $\frac{3}{4}$ length shoelace to measure against the whole and to compare to the $\frac{1}{2}$. After comparing the $\frac{1}{2}$ and the $\frac{3}{4}$, I have the students connect the two shoelaces together and share their findings with their partner.

Proper Fractions and Mixed Numbers

The sum of a natural number and a positive fraction is most often written as a **mixed number**. For example, $2 + \frac{3}{4}$ is written $2\frac{3}{4}$ and is read “two and three-quarters.” It is important to realize that it is the addition symbol, +, that is suppressed, since the common notation xy for multiplication might suggest, incorrectly, that $2\frac{3}{4}$ is $2 \cdot \frac{3}{4}$. Thus, $2\frac{3}{4} = 2 + \frac{3}{4}$, not $\frac{6}{4}$.

Mixed numbers have a hidden + sign:

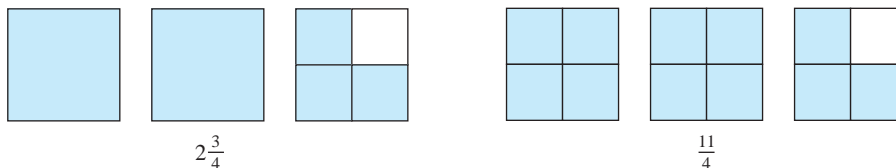
$$A\frac{b}{c} = A + \frac{b}{c}$$

A mixed number can always be rewritten in the standard form $\frac{a}{b}$ of a fraction. For example,

$$2\frac{3}{4} = 2 + \frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}.$$

Thus, $-2\frac{3}{4} = -\left(2\frac{3}{4}\right) = \frac{-11}{4}.$

Mixed numbers and their equivalent forms as a fraction can be visualized this way:



Since $A\frac{b}{c} = \frac{Ac}{c} + \frac{b}{c} = \frac{Ac + b}{c}$, it is an easy calculation to write a mixed number as a fraction:

$$A\frac{b}{c} = \frac{Ac + b}{c}.$$

A fraction $\frac{a}{b}$ for which $0 \leq |a| < b$ is called a **proper fraction**. For example, $\frac{2}{3}$ is a proper fraction, but $\frac{3}{2}$, $\frac{-8}{5}$, and $\frac{6}{6}$ are not proper fractions. It is common, though not necessary, to rewrite

fractions that are not proper as mixed numbers. For example, to express $\frac{439}{19}$ as a mixed number, we first use division with a remainder to find that $439 = 23 \cdot 19 + 2$. Then we have

$$\frac{439}{19} = \frac{23 \cdot 19 + 2}{19} = \frac{23}{1} + \frac{2}{19} = 23 + \frac{2}{19} = 23\frac{2}{19}.$$

In mixed-number form, it is obvious that $23\frac{2}{19}$ is just slightly larger than 23; this fact was not evident in the original fraction form $\frac{439}{19}$. Nevertheless, it is perfectly acceptable to express rational numbers

as “improper” fractions. In general, the fractional form $\frac{a}{b}$ is the more convenient form for arithmetic and algebra, and the mixed-number form is easiest to understand for practical applications. For example, it would be more common to buy $2\frac{1}{4}$ yards of material than to request $\frac{9}{4}$ yards.

EXAMPLE 6.7

Working with Mixed Numbers

(a) Give an improper fraction for $3\frac{17}{120}$.

(b) Give a mixed number for $\frac{355}{113}$.

(c) Give a mixed number for $\frac{-15}{4}$.

(d) Compute $2\frac{3}{4} + 4\frac{2}{5}$.

Solution

(a) $3\frac{17}{120} = \frac{3}{1} + \frac{17}{120} = \frac{3 \cdot 120 + 1 \cdot 17}{120} = \frac{360 + 17}{120} = \frac{377}{120}.$

This rational number was given by Claudius Ptolemy around A.D. 150 to approximate π , the ratio of the circumference of a circle to its diameter.

It has better accuracy than $3\frac{1}{7}$, the value proposed by Archimedes in about 240 B.C.

(b) Using the division algorithm, we calculate that $355 = 3 \cdot 113 + 16$. Therefore,

$$\frac{355}{113} = \frac{3 \cdot 113 + 16}{113} = \frac{3}{1} + \frac{16}{113} = 3\frac{16}{113},$$

which corresponds to a point somewhat to the right of 3 on the number line. The value $\frac{355}{113}$ was used around A.D. 480 in China to approximate π ; as a decimal number, it is correct to six places!

(c) $\frac{-15}{4} = \frac{-(3 \cdot 4 + 3)}{4} = -\left(3 + \frac{3}{4}\right) = -3\frac{3}{4}.$

(d) $2\frac{3}{4} + 4\frac{2}{5} = 2 + 4 + \frac{3}{4} + \frac{2}{5} = 6 + \frac{15}{20} + \frac{8}{20} = 6 + \frac{23}{20} = 7\frac{3}{20}.$

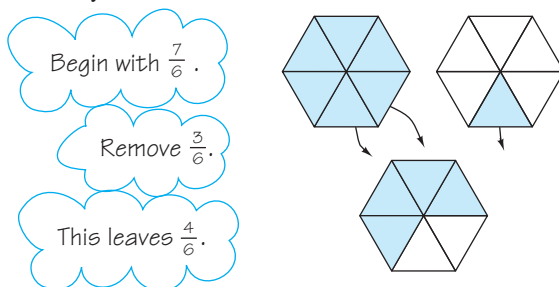
Subtraction of Fractions

Figure 6.13 shows how the take-away, measurement, and missing-addend conceptual models of the subtraction operation can be illustrated with colored regions, the number line, and fraction strips.

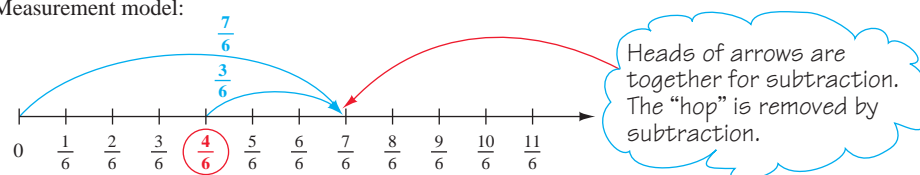
In each case, we see that $\frac{7}{6} - \frac{3}{6} = \frac{4}{6}.$

Figure 6.13
Models showing that
 $\frac{7}{6} - \frac{3}{6} = \frac{4}{6}$

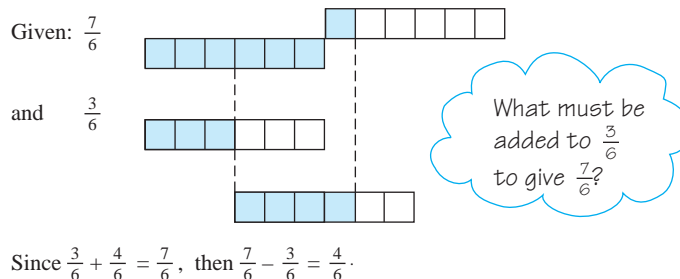
Take-away model:



Measurement model:



Missing-addend model:



Subtraction of whole numbers and integers was defined on the basis of the missing-addend approach, which emphasizes that subtraction is the inverse operation to addition. Subtraction of fractions is defined in the same way.

DEFINITION Subtraction of Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. The $\frac{a}{b} - \frac{c}{d} = \frac{e}{f}$ if and only if $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$.

For two fractions $\frac{a}{b}$ and $\frac{c}{b}$ with the same denominator, we see that $\frac{c}{b} + \frac{a-c}{b} = \frac{c + (a-c)}{b} = \frac{a}{b}$ and, therefore, from the definition, $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$. To subtract fractions $\frac{a}{b}$ and $\frac{c}{d}$ with arbitrary denominators, we can rewrite them with common denominators as $\frac{ad}{bd}$ and $\frac{bc}{bd}$ to obtain the formula $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$. However, just as with the addition, the formula obscures the meaning of the subtraction of fractions. Therefore, it is important to explain subtraction using common denominators, manipulatives, and pictorial representations. These approaches emphasize the meaning of the subtraction operation.

EXAMPLE 6.8**Subtracting Fractions**

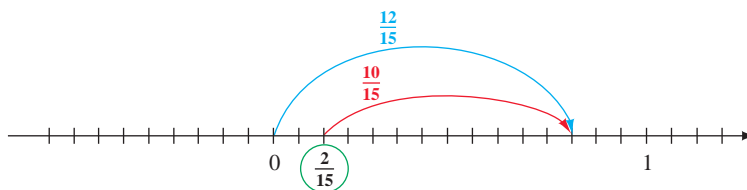
Show, as you might to a student, the steps to compute these differences:

(a) $\frac{4}{5} - \frac{2}{3}$ (b) $\frac{103}{24} - \frac{-35}{16}$ (c) $4\frac{1}{4} - 2\frac{2}{3}$

Solution

(a) $\frac{4}{5} - \frac{2}{3} = \frac{4 \cdot 3}{5 \cdot 3} - \frac{2 \cdot 5}{3 \cdot 5} = \frac{12}{15} - \frac{10}{15} = \frac{12 - 10}{15} = \frac{2}{15}$.

The subtraction can also be shown with a number-line diagram. Since both fractions are positive, both arrows point to the right, and since this is a subtraction, the arrowheads are together. The diagram shows that removing $\frac{10}{15}$ from $\frac{12}{15}$ results in $\frac{2}{15}$.



(b) Since $\text{LCM}(24, 16) = 48$, the least common denominator, 48, can be used to give

$$\frac{103}{24} - \frac{-35}{16} = \frac{206}{48} - \frac{-105}{48} = \frac{206 - (-105)}{48} = \frac{311}{48}.$$

(c) The mixed forms can be converted to standard form, showing that

$$4\frac{1}{4} - 2\frac{2}{3} = \frac{17}{4} - \frac{8}{3} = \frac{17 \cdot 3}{4 \cdot 3} - \frac{4 \cdot 8}{4 \cdot 3} = \frac{51}{12} - \frac{32}{12} = \frac{51 - 32}{12} = \frac{19}{12} = 1\frac{7}{12}.$$

Alternatively, subtraction of mixed numbers can follow the familiar regrouping algorithm:

$$\begin{array}{r} 4\frac{1}{4} \\ - 2\frac{2}{3} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{r} 4\frac{3}{12} \\ - 2\frac{8}{12} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{r} 3\frac{15}{12} \\ - 2\frac{8}{12} \\ \hline 1\frac{7}{12} \end{array}$$

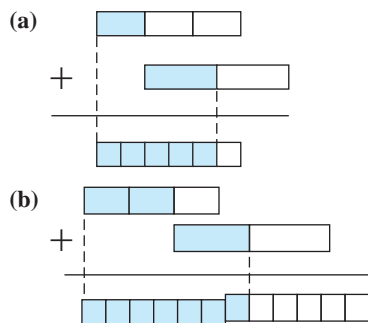
$4\frac{3}{12} = 3 + \frac{12}{12} + \frac{3}{12} = 3\frac{15}{12}$

6.2

Problem Set

Understanding Concepts

1. What addition fact is illustrated by the following fraction-strip models?



2. (a) Illustrate $\frac{1}{6} + \frac{1}{4}$ with the fraction-strip model.
 (b) Illustrate $\frac{2}{3} + \frac{3}{4}$ with the fraction-strip model. (The sum will require two strips.)

3. Use fraction circles to illustrate these sums:

(a) $\frac{2}{5} + \frac{6}{5}$ (b) $\frac{1}{4} + \frac{1}{2}$ (c) $\frac{2}{3} + \frac{1}{4}$

4. The points A, B, C, . . . , G, and H are equally spaced along the number line:



- (a) What rational number corresponds to point G?
 (b) What point corresponds to $\frac{1}{2}$?
 (c) Are there lettered points that correspond to $\frac{1}{4}$ to $\frac{1}{6}$?
 (d) Which lettered points are nearest to either side of $\frac{4}{7}$?
5. Represent each of these sums with a number-line diagram:
- (a) $\frac{1}{8} + \frac{3}{8}$ (b) $\frac{1}{4} + \frac{5}{4}$ (c) $\frac{3}{4} + \frac{-2}{4}$
6. Use the number-line model to illustrate the given sums. Recall that negative fractions are represented by arrows that point to the left.
- (a) $\frac{2}{3} + \frac{1}{2}$ (b) $\frac{-3}{4} + \frac{2}{4}$ (c) $\frac{-3}{4} + \frac{-1}{4}$

7. Perform the given additions. Express each answer in simplest form.

(a) $\frac{2}{7} + \frac{3}{7}$ (b) $\frac{6}{5} + \frac{4}{5}$
 (c) $\frac{3}{8} + \frac{11}{24}$ (d) $\frac{6}{13} + \frac{2}{5}$

8. Perform the given additions. Express each answer in simplest form.

(a) $\frac{5}{12} + \frac{17}{20}$ (b) $\frac{6}{8} + \frac{-25}{100}$
 (c) $\frac{-57}{100} + \frac{13}{10}$ (d) $\frac{213}{450} + \frac{12}{50}$

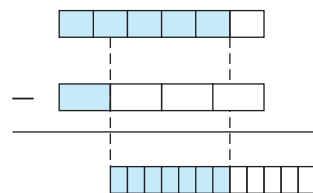
9. Express these fractions as mixed numbers:

(a) $\frac{9}{4}$ (b) $\frac{17}{3}$ (c) $\frac{111}{23}$ (d) $\frac{3571}{-100}$

10. Express these mixed numbers as fractions:

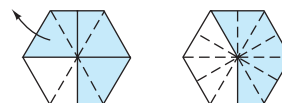
(a) $2\frac{3}{8}$ (b) $15\frac{2}{3}$ (c) $111\frac{2}{5}$ (d) $-10\frac{7}{9}$

11. (a) What subtraction fact is illustrated by this fraction-strip model?



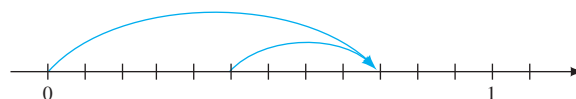
- (b) Use the fraction-strip model to illustrate $\frac{2}{3} - \frac{1}{4}$.

12. (a) What subtraction fact is illustrated by this colored-region model?



- (b) Use a fraction-circle model to illustrate $\frac{2}{3} - \frac{1}{4}$.

13. (a) What subtraction fact is illustrated by this number-line model?



- (b) Illustrate $\frac{2}{3} - \frac{1}{4}$ with the number-line model.

14. Compute these differences, expressing each answer in simplest form:

(a) $\frac{5}{8} - \frac{2}{8}$ (b) $\frac{3}{5} - \frac{2}{4}$
 (c) $2\frac{2}{3} - 1\frac{1}{3}$ (d) $4\frac{1}{4} - 3\frac{1}{3}$

15. Compute these differences, expressing each answer in simplest form.

(a) $\frac{6}{8} - \frac{5}{12}$ (b) $\frac{1}{4} - \frac{14}{56}$

(c) $\frac{137}{214} - \frac{-1}{3}$

(d) $\frac{-23}{100} - \frac{198}{1000}$

16. Use mental arithmetic to estimate the following sums and differences to the nearest integer:

(a) $\frac{19}{22} + \frac{31}{15}$

(b) $1\frac{5}{11} + 4\frac{1}{2}$

(c) $7\frac{53}{97} - 2\frac{5}{9}$

(d) $6\frac{2}{3} + 8\frac{11}{32} - 2\frac{1}{29}$

17. An alternative but equivalent definition of rational number inequality is the following:

$$\frac{a}{b} < \frac{c}{d} \quad \text{if and only if} \quad \frac{c}{d} - \frac{a}{b} > 0.$$

Use the alternative definition to verify these inequalities.

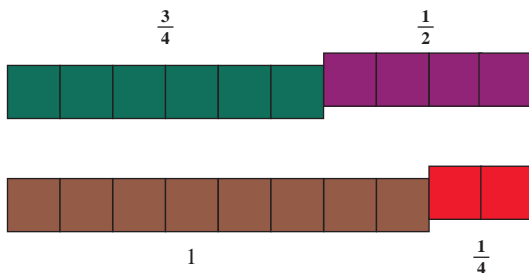
(a) $\frac{2}{3} < \frac{3}{4}$

(b) $\frac{4}{5} < \frac{14}{17}$

(c) $\frac{19}{10} < \frac{99}{50}$

Into the Classroom

18. In the number-rod diagram shown, the brown rod has been adopted as the unit. Therefore, the red rod is $\frac{1}{4}$, the purple rod is $\frac{1}{2}$, and the dark-green rod is $\frac{3}{4}$. Since the train formed with the purple and dark-green rods has the length of the brown-plus-red train, we have $\frac{1}{2} + \frac{3}{4} = 1\frac{1}{4}$.



- (a) Use number rods to create another illustration of the addition of fractions. Use both words and diagrams to describe your example clearly.
- (b) Use number rods to create an illustration of the subtraction of fractions.
19. **(Writing)** Create instructions for an elementary school class activity in which students use strips of paper 1" wide that are cut into smaller lengths and then measured to the nearest $\frac{1}{2}$ ", $\frac{1}{4}$ ", or $\frac{1}{8}$ ", and the length is written on the strip. For addition, two strips are chosen and their lengths added. Then the strips are laid end-to-end and measured to see if the total length agrees with the result of the addition calculation. A similar procedure for subtraction should be described in both words and diagrams.

Responding to Students

The "pizza stories" discussed in problems 20–22 suggest common misconceptions students have about fractions and their arithmetic. Use words and diagrams to formulate helpful responses for each situation described.

20. The Estevez family ordered a large pepperoni pizza and a medium tomato-and-green-pepper pizza. Since $\frac{1}{3}$ of the pepperoni pizza was not eaten and $\frac{1}{2}$ of the other pizza remains, Paco claims that $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ of a pizza remains. Is Paco correct?
21. After the pizza party, it was discovered that just 2 pieces of one of the pizzas were consumed, leaving 6 more slices. Dannea was told that she could have half of the leftover pizza but to leave the rest for her brother. Dannea reasoned that the whole pizza was 8 slices, so that half of the pizza was 4 slices. Does her brother have a reason to complain when he discovered that Dannea left him just 2 slices? In particular, what principle of fractions has Dannea violated?
22. Katrina claims that since $\frac{2}{6}$ of one pizza and $\frac{3}{4}$ of a second pizza were not eaten, " $\frac{2}{6} + \frac{3}{4} = \frac{5}{10} = \frac{1}{2}$," of a pizza is left over for tomorrow's lunch. Respond to Katrina in a helpful way.
23. Melanie got 1 hit in 3 times at bat in the first baseball game, giving her a hitting average of $\frac{1}{3}$. In the second game, her average was $\frac{2}{5}$, since she had 2 hits in 5 times at bat. She then claims that her two-game average is $\frac{1}{3} + \frac{2}{5} = \frac{3}{8}$, because she had 3 hits in her 8 times at bat. Melanie's brother John objects, since he knows that $\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$ and $\frac{11}{15} \neq \frac{3}{8}$. Who is right, Melanie or John?

Thinking Critically

24. **(Writing)** A worm is on page 1 of Volume 1 of a set of encyclopedias neatly arranged on a shelf. He (or she—how do you tell?) eats straight through to the last page of Volume 2. If the covers of each volume are $\frac{1}{8}$ " thick and the pages are a total of $\frac{3}{4}$ " thick in each of the volumes, how far does the worm travel?
25. **(Writing)** When the sultan died, he left a stable with 17 horses. His will stipulated that half of his horses should go to his oldest son, a third to his middle son, and a ninth to his youngest son. The executor of the estate decided to make the distribution easier by contributing his own horse to the estate. With 18 horses now available, he gave 9 horses (a half) to the oldest son, 6 horses (a third) to the middle son, and 2 horses (a ninth) to the youngest son. This satisfied the terms of the will, and since only $9 + 6 + 2 = 17$ horses were given to the sons, the executor was able to retain his own horse. Explain what has happened.

26. Find the missing fractions in the following Magic Fraction Squares so that the entries in every row, column, and diagonal add to 1:

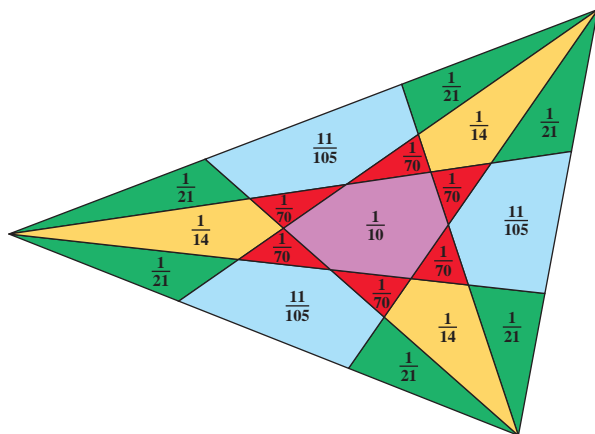
(a)





| | | |
|---------------|----------------|--|
| $\frac{1}{2}$ | $\frac{1}{12}$ | |
| $\frac{1}{4}$ | | |
| | | |

(b)

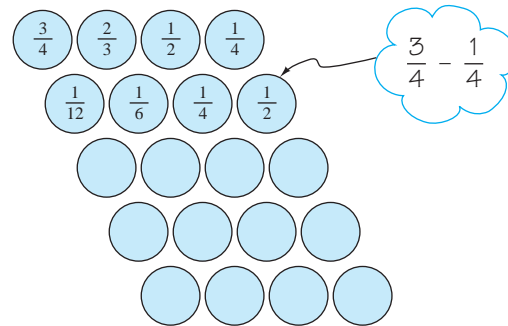
| | | |
|--|---------------|----------------|
| | | |
| | $\frac{1}{3}$ | $\frac{3}{5}$ |
| | | $\frac{2}{15}$ |

27. Start with any triangle, and let its area be the unit. Next, trisect each side of the triangle and join these points to the opposite vertices. This creates a dissection of the unit triangle, as illustrated in the accompanying figure. Amazingly, the areas of the regions shown are precisely the fractions shown in the figure.
- For example, the inner lavender hexagon has area $\frac{1}{10}$, and the area of each green triangle is $\frac{1}{21}$.



- (a) Verify that the large triangle is a unit triangle by summing the areas of the regions within the triangle.
- (b) What is the area of the three-pointed star composed of the hexagon, the six red triangles, and the three yellow quadrilaterals?
- (c) What is the area of the triangle composed of the hexagon and the three red triangles on every other side of the hexagon?
28. With the exception of $\frac{2}{3}$, which was given the hieroglyph , the ancient Egyptians attempted to express all fractions as a sum of different fractions, each with 1 as the numerator.
- (a) Verify that $\frac{23}{25} = \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{1}{50}$.
- (b) Verify that $\frac{7}{29} = \frac{1}{6} + \frac{1}{24} + \frac{1}{58} + \frac{1}{87} + \frac{1}{232}$.
- (c) Verify that $\frac{7}{29} = \frac{1}{5} + \frac{1}{29} + \frac{1}{145}$.
- (d) If  represents $\frac{1}{2}$, and  is interpreted as “one over,” does the symbol  seem reasonable for $\frac{2}{3}$?

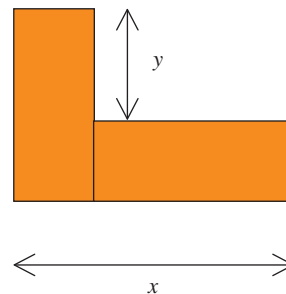
29. **Diffy with Fractions.** We previously played the Diffy game, described in Section 2.3, with whole numbers. Shown here are the beginning lines of Diffy when the entries are fractions. A new line is formed by subtracting the smaller fraction from the larger.



- (a) Fill in additional lines of the Diffy array. Does it terminate?
- (b) Try fraction Diffy with these fractions in your first row: $\frac{2}{7}$, $\frac{4}{5}$, $\frac{3}{2}$, and $\frac{5}{6}$.
- (c) **(Writing)** Suppose you know that Diffy with whole-number entries always terminates with 0, 0, 0, and 0. Does it necessarily follow that Diffy with fractions must terminate? Explain your reasoning carefully.

Making Connections

30. A “2 by 4” piece of lumber is planed from a rough board to a final size of $1\frac{1}{2}$ ” by $3\frac{1}{2}$ ”. Find the dimensions x and y of the shape created with two such boards.

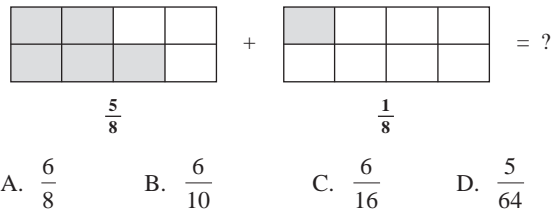


31. A wall in a house is made with sheets of $\frac{5}{8}$ -inch drywall screwed to the opposite short edges of the vertical 2-by-4 studs within the wall. (See problem 30). How thick is the resulting wall?
32. A $1\frac{1}{4}$ ” drywall screw is countersunk (i.e., screwed until it is below the surface of the drywall) by $\frac{1}{16}$ ” to fasten a $\frac{5}{8}$ ” sheet of drywall to the ceiling joist.
- (a) How much of the screw extends into the joist?
- (b) How much of the screw is within the drywall?
33. A board $10\frac{1}{2}$ inches long is sawn off a board that is 2 feet long. If the width of the saw cut is $\frac{1}{16}$ ”, what is the length of the remaining piece?

34. A picture is printed on an $8\frac{1}{2}$ "-by-11" piece of photographic paper and placed symmetrically in a frame whose opening is 8" by 10". What are the dimensions of the strips on the sides of the picture that are hidden by the frame?

State Assessments

35. (Massachusetts, Grade 4)
What is the solution to the problem shown below?



36. (Massachusetts, Grade 6)

Henry had a piece of rope that was $23\frac{1}{2}$ inches long. He cut the rope into two pieces so that one piece was $8\frac{1}{4}$ inches long. What was the length of the other piece of rope?

- A. $15\frac{1}{4}$ B. $15\frac{1}{2}$ C. $31\frac{1}{3}$ D. $31\frac{3}{4}$

37. (Grade 5)

Abe, Bert, and Carol shared a pizza. Abe ate $\frac{3}{8}$ of the pizza, Bert ate $\frac{1}{4}$ of the pizza, and Carol ate the rest. How much of the pizza did Carol eat?

- A. $\frac{1}{2}$ B. $\frac{3}{4}$ C. $\frac{5}{8}$ D. $\frac{3}{8}$

6.3

Multiplication and Division of Fractions

As a pleasant surprise, multiplication and division of fractions are computationally easier than addition and subtraction. For example, we will discover that the product of $\frac{3}{4}$ and $\frac{2}{5}$ is computed simply by multiplying the respective numerators and denominators to get $\frac{6}{20}$. Division is only slightly

trickier: To compute $\frac{2}{5} \div \frac{3}{4}$, we can use the “invert and multiply rule” to get the answer, $\frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$.

However, even though multiplication and division of fractions are easy computationally, these operations are quite difficult to understand conceptually. To successfully impart a deep understanding of the multiplication and division of fractions, teachers need to be able to answer these questions:

- How can multiplication be illustrated with manipulatives? with pictorial representations?
- Why is multiplication defined with such a simple formula?
- How is a fraction interpreted as a multiplicative operator?
- In what way can division still be viewed as repeated subtraction or as a missing factor?
- What justifies the “invert and multiply rule”?
- What real-life problems require the multiplication or division of fractions?

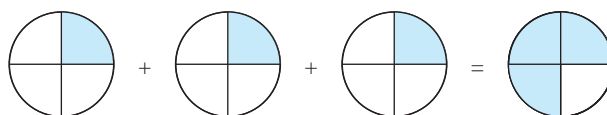
It is important to understand that the multiplication of fractions is an extension of the multiplication operation on the integers. It is helpful to take a stepwise approach in which first a fraction is multiplied by an integer and then an integer is multiplied by a fraction. These two preliminary steps prepare the student for the general definition of how two fractions are multiplied.

Similarly, division of fractions is made meaningful by continuing to view division as repeated subtraction (grouping), as a partition (sharing), and as a missing factor.

Multiplication of a Fraction by an Integer

Three family-size pizzas were ordered for a party, and a quarter of each pizza was not consumed by the partygoers. Viewing multiplication by a positive integer as repeated addition, we see from the fraction-circle diagram in Figure 6.14 that $3 \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$ of a pizza is left over.

Figure 6.14
A fraction-circle diagram showing that $3 \cdot \frac{1}{4} = \frac{3}{4}$



EXAMPLE

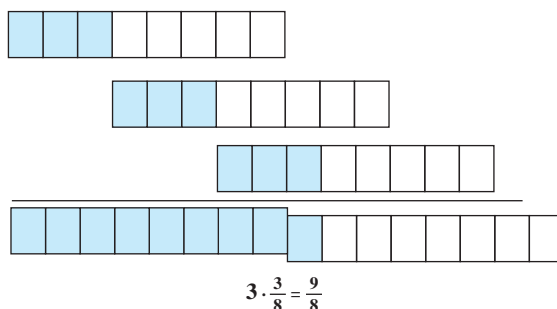
6.9

Multiplying a Fraction by an Integer

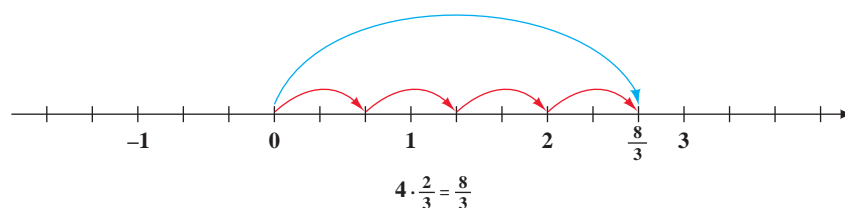
- (a) Use fraction strips to show that $3 \cdot \frac{3}{8} = \frac{9}{8}$.
- (b) Use a number line to show that $4 \cdot \frac{2}{3} = \frac{8}{3}$.
- (c) Use a number line to show that $-3 \cdot \frac{5}{6} = \frac{-15}{6}$.

Solution

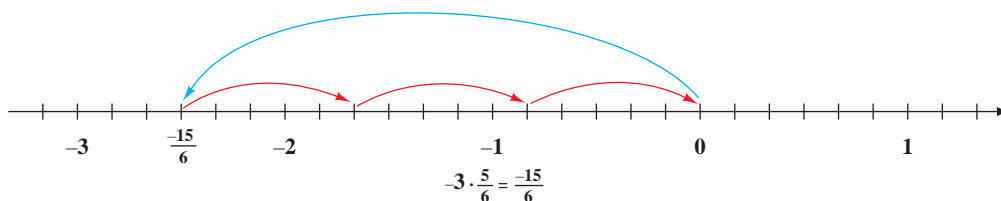
- (a) Three fraction strips, each representing $\frac{3}{8}$, are aligned to show that their combined shaded region represents $\frac{9}{8}$:



- (b) Four jumps of length $\frac{2}{3}$ arrive at the point $\frac{8}{3}$:



- (c) The repeated subtraction of three jumps, each of length $\frac{5}{6}$, takes us to the point $\frac{-15}{6}$ on the number line:



Since $\frac{5}{6}$ is positive, it is represented by a right-pointing arrow. Removing three of these arrows takes us to the point $\frac{-15}{6}$; this is equivalent to a leftward jump to $\frac{-15}{6}$, which is shown by a left-pointing arrow.

Multiplication of an Integer by a Fraction

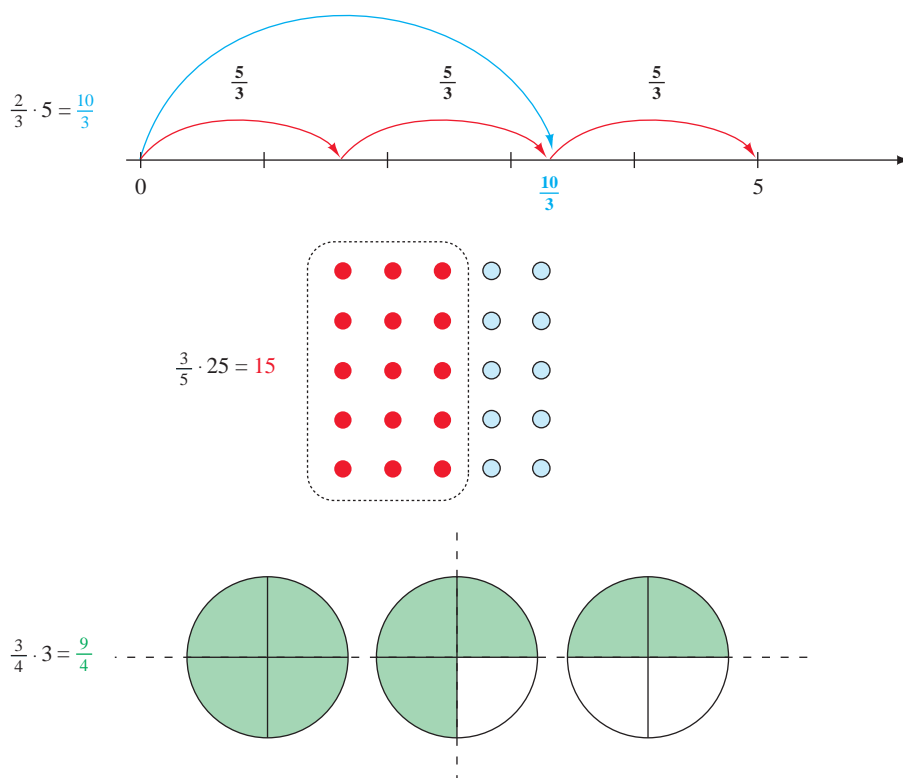
Imagine that we are taking a 5-mile hike and we meet someone on the trail who estimates that we have come two-thirds of the way to our destination. How many miles have we covered? This is an easy question to answer: One-third of the 5-mile trail is $\frac{5}{3}$ miles long, and we have covered twice this distance, so we have hiked $\frac{10}{3}$ miles so far.

We see that “two-thirds of” gives meaning to $\frac{2}{3}$ as an **operator**: To take two-thirds of 5, first divide 5 into 3 equal parts and then take 2 of those parts. That is, $\frac{2}{3} \cdot 5 = 2 \cdot \frac{5}{3} = \frac{10}{3}$. Here are some other examples related to the hike:

- If the entire hike will take 6 hours, we had been hiking $\frac{2}{3} \cdot 6 = \frac{12}{3} = 4$ hours when we met the other hiker.
- If we brought 2 liters of water for the hike, we might estimate that we have used $\frac{2}{3} \cdot 2 = \frac{4}{3}$ liters of our water.

Additional examples of how a fraction is used as a multiplicative operator are shown in Figure 6.15. In each case, we multiply by a fraction $\frac{a}{b}$ by first making a partition into b equal parts and then taking a of those parts. That is, $\frac{a}{b} \cdot n = a \cdot \frac{n}{b} = \frac{an}{b}$.

Figure 6.15
Depicting the product
of a fraction and an
integer



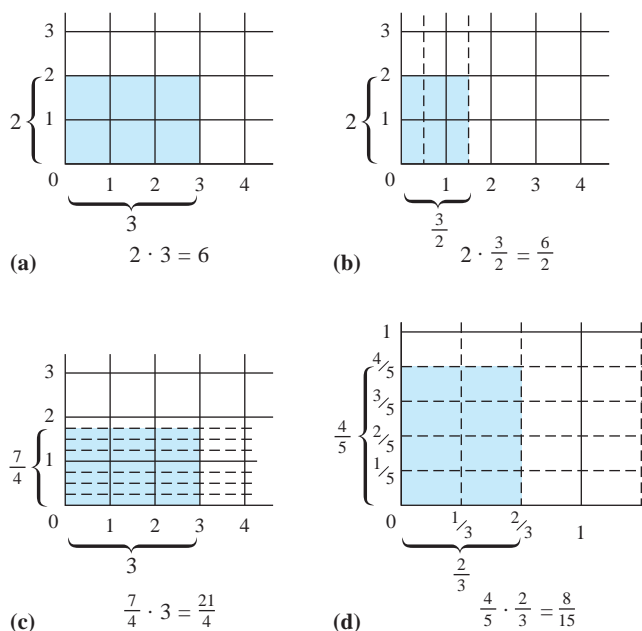
Multiplication of a Fraction by a Fraction

In earlier chapters, the rectangular area model provided a useful visualization of multiplication for the whole numbers and integers. This model works equally well to motivate the general definition of the multiplication of two fractions.

In Figure 6.16a, there is a reminder of how the rectangular area model is used with the whole numbers: We simply create a shaded rectangle that is 2 units high and 3 units long and notice that 6 unit squares fill the rectangle. That is, $2 \cdot 3 = 6$. In Figure 6.16b, the rectangular area model reconfirms the result $2 \cdot \frac{3}{2} = \frac{2 \cdot 3}{2} = \frac{6}{2}$ for how to multiply a fraction by a whole number. Similarly,

Figure 6.16c shows us that the product of the whole number 3 and the fraction $\frac{7}{4}$ is given by $\frac{7}{4} \cdot 3 = \frac{7 \cdot 3}{4} = \frac{21}{4}$.

Figure 6.16
Extending the area
model of multiplication
to fractions



Finally, in Figure 6.16d, we see that the product of the fractions $\frac{4}{5}$ and $\frac{2}{3}$ is given by $\frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$, since 8 rectangles are shaded and there are 15 congruent rectangles that fill a 1-by-1 unit square. The product of the numerators, $4 \cdot 2 = 8$, gives us the number of small shaded rectangles and forms the numerator of the answer. The denominator of the answer is the product $5 \cdot 3 = 15$ of the denominators and counts the number of small rectangles in a unit square.

DEFINITION Multiplication of Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. Then their **product** is given by

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Often a product, say, $\frac{4}{5} \cdot \frac{2}{3}$, is read as four-fifths “of” two-thirds, which emphasizes how $\frac{4}{5}$ is an operator applied to the fraction $\frac{2}{3}$. The association between “of” and “times” is natural for multiplication by whole numbers and extends to multiplication of fractions. For example, “I’ll buy three of the half-gallon-size bottles” is equivalent to buying $3 \cdot \frac{1}{2} = \frac{3}{2} = 1\frac{1}{2}$ gallons.

EXAMPLE 6.10

Calculating Products of Fractions

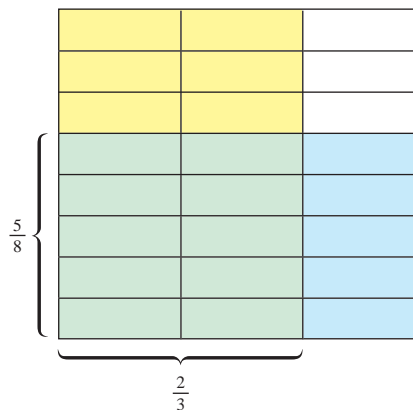
Illustrate these two products with the area model:

(a) $\frac{5}{8} \cdot \frac{2}{3}$ (b) $3\frac{1}{7} \cdot 5\frac{1}{4}$

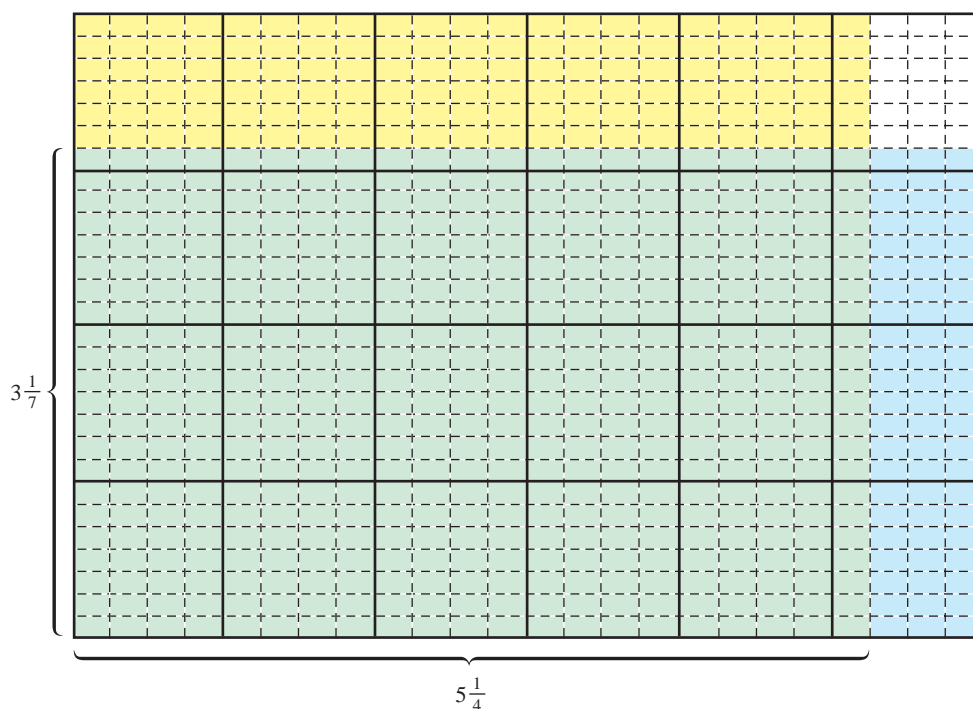
Solution

(a) Let a rectangle denote a whole unit of area. Next, use vertical lines to divide the unit into 3 equal parts, so that shading the 2 leftmost rectangular columns represents $\frac{2}{3}$. Similarly, use horizontal lines to divide the unit rectangle into 8 equal parts, and shade the lower 5 rectangular rows to

represent $\frac{5}{8}$ of the whole rectangle. The doubly shaded part of the whole represents $\frac{5}{8}$ of $\frac{2}{3}$ of the whole. Observing that the whole rectangle has been divided into 24 small rectangles, and 10 of these are within the overlapped doubly shaded region, we see that $\frac{5}{8} \cdot \frac{2}{3} = \frac{10}{24}$. We also see that the product of the numerators, $5 \cdot 2 = 10$, gives the number of small rectangles and is the numerator of our answer. Also, the product of the denominators, $8 \cdot 3 = 24$, gives the number of small rectangles in a unit rectangle and is the denominator of the fraction that represents the product.



- (b) Equivalently, we want to model the product $\frac{22}{7} \cdot \frac{21}{4}$ by a rectangular area diagram. The doubly shaded (green) region contains $22 \cdot 21$ small rectangles, where each unit square contains $4 \cdot 7 = 28$ small rectangles. Thus, $3\frac{1}{7} \cdot 5\frac{1}{4} = \frac{22 \cdot 21}{7 \cdot 4}$, which simplifies to $\frac{22 \cdot 21}{4 \cdot 7} = \frac{11 \cdot 3}{2 \cdot 1} = \frac{33}{2} = 16\frac{1}{2}$. This answer can also be seen in the figure by noticing that the green region contains 15 unit squares and 42 small rectangles. Since $42 = 28 + 14$, the small rectangles cover $1\frac{1}{2}$ unit squares. Altogether, we again arrive at the final answer of $16\frac{1}{2}$.

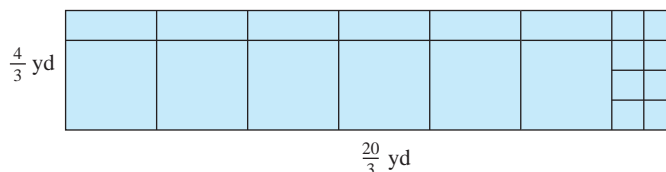


EXAMPLE 6.11**Computing the Area and Cost of a Carpet**

The hallway in the Bateks' house is a rectangle 4 feet wide and 20 feet long; that is, it measures $\frac{4}{3}$ yards by $\frac{20}{3}$ yards. What is the area of the hallway in square yards? Mrs. Batek wants to know how much she will pay to buy carpet priced at \$18 per square yard to carpet the hall.

Solution

Since $\frac{4}{3} \cdot \frac{20}{3} = \frac{80}{9} = 8\frac{8}{9}$, the area is $8\frac{8}{9}$ square yards, or nearly 9 square yards. This can be seen in the accompanying diagram, which shows the hallway divided into six full square yards, six $\frac{1}{3}$ -square-yard rectangular regions, and eight square regions that are each $\frac{1}{9}$ of a square yard. Thus, the total area is $6 + \frac{6}{3} + \frac{8}{9} = 8\frac{8}{9}$ square yards. At \$18 per square yard, the cost of the carpet will be $8\frac{8}{9} \cdot 18 = \frac{80}{9} \cdot 18 = 160$ dollars.



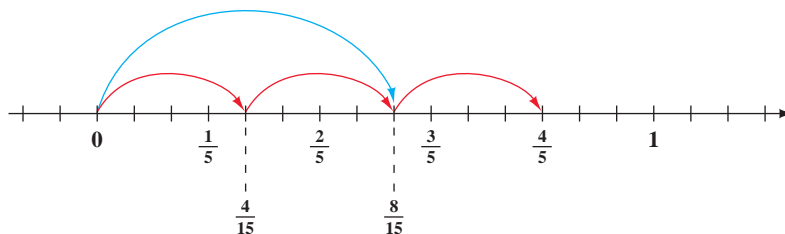
Multiplication of fractions can also be illustrated on the number line, as shown in the next example.

EXAMPLE 6.12**Multiplying Fractions on the Number Line**

Illustrate why $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$ with a number-line diagram.

Solution

It's helpful to view $\frac{2}{3}$ as a multiplicative operator. That is, first partition $\frac{4}{5}$ into three equal intervals, each of length $\frac{4}{15}$. Since two jumps of length $\frac{4}{15}$ arrive at $\frac{8}{15}$, the diagram confirms that $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$.

**Division of Fractions**

Consider the division problem $\frac{4}{3} \div \frac{1}{6}$. Viewing division as repeated subtraction, we want to know this: "How many one-sixths are in $\frac{4}{3}$?" Since $\frac{4}{3} = \frac{8}{6} = 8 \cdot \frac{1}{6}$, there are eight one-sixths in $\frac{4}{3}$. That is,

$$\frac{4}{3} \div \frac{1}{6} = 8, \text{ since } \frac{4}{3} = 8 \cdot \frac{1}{6}.$$

Let's try another example: Consider the division problem $\frac{7}{12} \div \frac{1}{6}$. We want to determine how many one-sixths are in $\frac{7}{12}$. Since $\frac{7}{12} = \frac{7}{2 \cdot 6} = \frac{7}{2} \cdot \frac{1}{6}$, there are $\frac{7}{2}$, or $3\frac{1}{2}$, one-sixths in $\frac{7}{12}$. That is, $\frac{7}{12} \div \frac{1}{6} = \frac{7}{2}$, since $\frac{7}{12} = \frac{7}{2} \cdot \frac{1}{6}$.

In general, to find the fraction $\frac{m}{n}$ that solves the division $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$, we must find the missing factor $\frac{m}{n}$ in the multiplication formula $\frac{a}{b} = \frac{m}{n} \cdot \frac{c}{d}$. The missing-factor interpretation is used to define the division of fractions.

DEFINITION Division of Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions, where $\frac{c}{d}$ is not zero. Then $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$ if and only if

$$\frac{a}{b} = \frac{m}{n} \cdot \frac{c}{d}.$$

This definition stresses that division is the inverse operation of multiplication. However, other conceptual models of division—as a measurement (that is, as a repeated subtraction or grouping) and as a partition (or sharing)—continue to be important. These models can be represented with manipulatives and diagrams, enabling the teacher to convey a deep understanding of what division of fractions really means and how it is used to solve problems.

This is reflected in the first of the Standards of Mathematical Practice (SMP 1) from the **Common Core:**

Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?”

The next two problems are solved by reasoning directly from the concept of division either as a grouping or as a sharing. No formula or computational rule is required to obtain the answer. It is very important to keep the conceptual notion of division in mind and not rely simply on rules and formulas that have no intrinsic meaning.



SMP 1 Make sense of problems and persevere in solving them.

EXAMPLE 6.13

Seeding a Lawn: Division of Fractions by Grouping

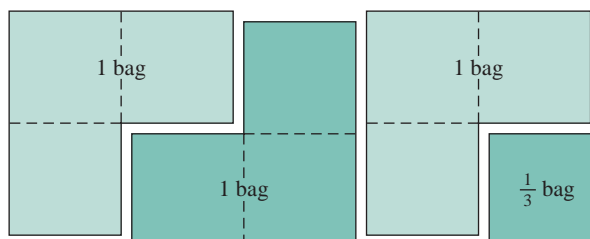
The new city park will have a $2\frac{1}{2}$ -acre grass playfield. Grass seed can be purchased in large bags, each sufficient to seed $\frac{3}{4}$ of an acre. How many bags are needed? Will there be some grass seed left over to keep on hand for reseeding worn spots in the field?



Solution

We need to determine the number of $\frac{3}{4}$ -acre regions in the $2\frac{1}{2}$ -acre field. That is, we must compute $2\frac{1}{2} \div \frac{3}{4}$. In the diagram that follows, the field is grouped into four regions. Three of these regions

each cover $\frac{3}{4}$ of an acre and, therefore, require a whole bag of seed. There is also a $\frac{1}{4}$ -acre region that will use $\frac{1}{3}$ of a bag. Thus, a total of $2\frac{1}{2} \div \frac{3}{4} = 3\frac{1}{3}$ bags of seed are needed for the playfield. We conclude that four bags of seed should be ordered, which is enough for the initial seeding and leaves $\frac{2}{3}$ of a bag on hand for reseeding.



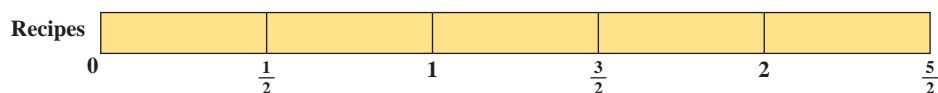
EXAMPLE 6.14

Making Cookies: Division of Fractions by Sharing

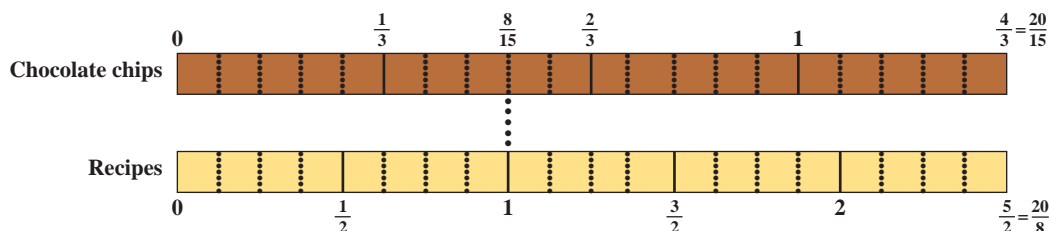
Jesse has enough flour to make $2\frac{1}{2}$ recipes of chocolate chip cookies and $1\frac{1}{3}$ cups of chocolate chips. How many cups of chocolate chips will be in each recipe?

Solution

The chocolate chips and recipes are shown on the following two fraction strips, where each small brown rectangle represents a third of a cup of chocolate chips and each small yellow rectangle represents a half recipe of cookies:



If we subdivide each small brown rectangle into 5 parts and each small yellow rectangle into 4 parts, then both the amounts of chocolate chips and recipes are divided into 20 parts.



Since 15 smallest brown squares correspond to a cup of chocolate chips, each of these represents $\frac{1}{15}$ of a cup of chips. Therefore, $\frac{8}{15}$ of a cup of chips corresponds to one recipe, and we conclude that $\frac{4}{3} \div \frac{5}{2} = \frac{8}{15}$.

Algorithms for Calculating the Division of Fractions

There are three ways to simplify the calculation of a quotient of fractions:

- **Division by finding common numerators**

In Example 6.14, it was shown that $\frac{4}{3} \div \frac{5}{2} = \frac{20}{15} \div \frac{20}{8} = \frac{8}{15}$.

To verify the formula $\frac{a}{b} \div \frac{a}{c} = \frac{c}{b}$, notice that $\frac{a}{b} = \frac{a}{c} \cdot \frac{c}{b}$.

- **Division by finding common denominators**

We showed earlier that $\frac{7}{12} \div \frac{1}{6} = \frac{7}{12} \div \frac{2}{12} = \frac{7}{2}$.

To verify the formula $\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$, we simply observe that $\frac{a}{b} = \frac{c}{b} \cdot \frac{a}{c}$.

- **Division by the “invert and multiply” rule**

In Example 6.13, we showed that $\frac{5}{2} \div \frac{3}{4} = \frac{20}{6}$, since $\frac{20}{6} = \frac{10}{3} = 3\frac{1}{3}$.

This calculation suggests that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, where the divisor $\frac{c}{d}$ is “inverted” and becomes a factor $\frac{d}{c}$ that is multiplied by $\frac{a}{b}$. This “invert and multiply” formula may be verified by observing that $\left(\frac{a}{b} \cdot \frac{d}{c}\right) \cdot \frac{c}{d} = \frac{adc}{bcd} = \frac{a}{b}$.

The following theorem summarizes these useful formulas for the division of fractions:

THEOREM Formulas for the Division of Fractions

Common-numerator rule $\frac{a}{b} \div \frac{a}{c} = \frac{c}{b}$

Common-denominator rule $\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$

Invert-and-multiply rule $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

EXAMPLE 6.15

Dividing Fractions

Compute these division problems:

(a) $\frac{3}{4} \div \frac{1}{8}$ (b) $\frac{2}{5} \div \frac{2}{3}$ (c) $3 \div \frac{4}{3}$

(d) $39 \div 13$ (e) $13 \div 39$ (f) $4\frac{1}{6} \div 2\frac{1}{3}$

Solution

(a) $\frac{3}{4} \div \frac{1}{8} = \frac{6}{8} \div \frac{1}{8} = 6$, or $\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \cdot \frac{8}{1} = \frac{24}{4} = 6$.

(b) $\frac{2}{5} \div \frac{2}{3} = \frac{3}{5}$, or $\frac{2}{5} \div \frac{2}{3} = \frac{2}{5} \cdot \frac{3}{2} = \frac{2 \cdot 3}{5 \cdot 2} = \frac{3}{5}$.

(c) $3 \div \frac{4}{3} = \frac{3}{1} \cdot \frac{3}{4} = \frac{9}{4}$.

(d) $39 \div 13 = \frac{39}{1} \cdot \frac{1}{13} = \frac{39}{13} = 3$.

(e) $13 \div 39 = \frac{13}{1} \cdot \frac{1}{39} = \frac{13}{39} = \frac{1}{3}$.

(f) $4\frac{1}{6} \div 2\frac{1}{3} = \frac{25}{6} \div \frac{7}{3} = \frac{25}{6} \cdot \frac{3}{7} = \frac{25 \cdot 3}{6 \cdot 7} = \frac{25 \cdot 1}{2 \cdot 7} = \frac{25}{14}$.

Reciprocals as Multiplicative Inverses in the Rational Numbers

The “invert and multiply” rule, though not the definition of division, has an important consequence: The division of any rational by a nonzero rational number has a unique quotient that is also a rational number. That is, the nonzero rational numbers are closed under division. This was not true of the system of integers. For example, $3 \div 8$ cannot be written as any integer. But when 3 and 8 are interpreted as the fractions $\frac{3}{1}$ and $\frac{8}{1}$, then we see that $3 \div 8 = \frac{3}{1} \div \frac{8}{1} = \frac{3}{8}$.

More precisely stated, the nonzero fraction $\frac{c}{d}$ is “inverted” by forming its **reciprocal**.

DEFINITION Reciprocal of a Fraction

The **reciprocal** of a nonzero fraction $\frac{c}{d}$ is the fraction $\frac{d}{c}$.

Since $\frac{c}{d} \cdot \frac{d}{c} = \frac{d}{c} \cdot \frac{c}{d} = 1$, and 1 is the multiplicative identity, we say that the reciprocal of a nonzero fraction is its **multiplicative inverse**. Multiplicative inverses are very useful for solving equations, as shown in the next two examples.

EXAMPLE 6.16

Using the Reciprocal

One morning at the preschool for 3- and 4-year-olds, $\frac{2}{3}$ of the 3-year-old children and $\frac{5}{6}$ of the 4-year-old children went on a walk. For safety, the children were paired so that each 3-year-old child held the hand of a 4-year-old. Among the children who did not go on the walk, there were 9 more 3-year-olds than 4-year-olds. What is the enrollment of the school?

Solution

Understand the Problem

Information is given about the number of 3- and 4-year-olds. Our goal is to determine the number of children in the school.

Devise a Plan

Although we are asked only for the total number of children, it is probably helpful to determine both the number of 3-year-olds and the number of 4-year-olds. Summing these two numbers will give us our answer. Thus, we can introduce variables, use the given information to form equations, and finally solve these equations to find our answer. Since the given information involves fractions, we expect that we need to be skillful working with the arithmetic of fractions.

Carry Out the Plan

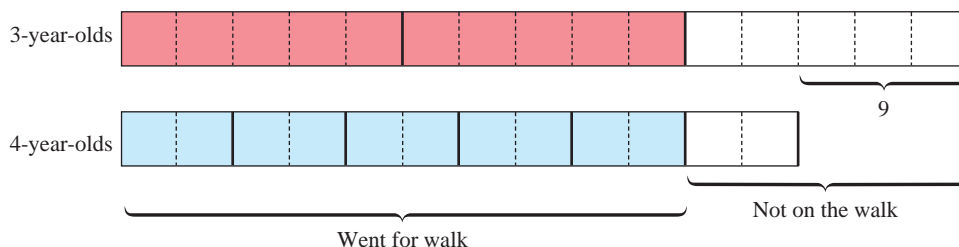
Let the variables T and F represent the respective number of 3- and 4-year-old children. The information given is equivalent to the equations $\frac{2}{3}T = \frac{5}{6}F$ and $T = F + 9$. The first equation can be solved for T by multiplying both sides by the reciprocal of $\frac{2}{3}$, namely, $\frac{3}{2}$, so

$$T = \left(\frac{3}{2} \cdot \frac{2}{3}\right) \cdot T = \frac{3}{2} \cdot \left(\frac{2}{3} \cdot T\right) = \frac{3}{2} \cdot \left(\frac{5}{6} \cdot F\right) = \left(\frac{3}{2} \cdot \frac{5}{6}\right) \cdot F = \frac{5}{4} \cdot F.$$

Comparing this equation with $T = F + 9$, we obtain the new relation $\frac{5}{4}F = F + 9$. Subtracting F from both sides gives $\frac{1}{4} \cdot F = 9$. To solve for F , we multiply this equation by the reciprocal of $\frac{1}{4}$, which is 4, to get $F = 4 \cdot 9 = 36$. Therefore, $T = 36 + 9 = 45$. The school has $36 + 45 = 81$ children in all.

Look Back

It is always a good idea to ask if a problem can be solved differently. The figure that follows shows clearly all of the information given in the problem: $\frac{2}{3}$ of the 3-year-olds walk with $\frac{5}{6}$ of the 4-year-olds, and there are 9 more older children. The dotted lines have been added to divide each region into rectangles of equal size.



Evidently, 3 small rectangles represent 9 children, so each small rectangle represents 3 children. The diagram now makes it clear there are $3 \cdot 3 \cdot 5 = 45$ children of age 3, $3 \cdot 6 \cdot 2 = 36$ of age 4, and 81 children in all.

The missing-factor model of division is used to solve the real-life problem in the next example. The reciprocal, being a multiplicative inverse, allows us to solve for the missing factor.

EXAMPLE 6.17**Bottling Root Beer: Division of Fractions with the Missing-Factor Model**

Ari is making homemade root beer. The recipe he followed nearly fills a 5-gallon glass jug, and he estimates that it contains $4\frac{3}{4}$ gallons of root beer. He is now ready to bottle his root beer. How many $\frac{1}{2}$ -gallon bottles can he fill?

Solution

Let x denote the number of half-gallon bottles required, where x will be allowed to be a fraction, since we expect that some bottle may be only partially filled. We must then solve the equation

$$x \cdot \frac{1}{2} = 4\frac{3}{4},$$

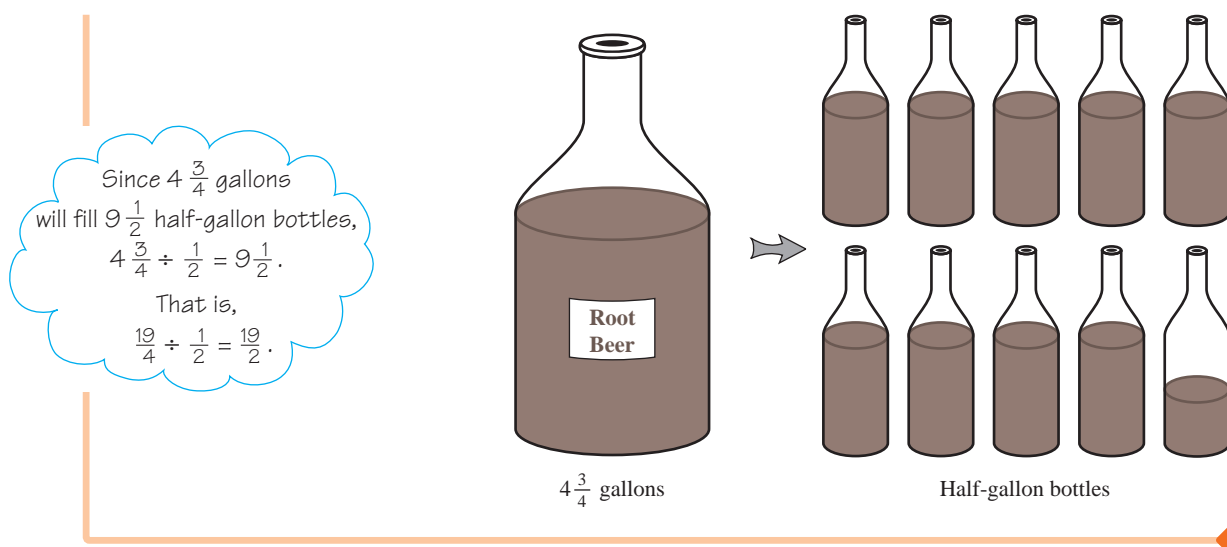
which is the missing-factor problem that is equivalent to the division problem $4\frac{3}{4} \div \frac{1}{2}$.

Since

$$4\frac{3}{4} = \frac{19}{4} = \frac{19}{2} \cdot \frac{1}{2},$$

it follows that the missing factor is $x = \frac{19}{2} = 9\frac{1}{2}$.

The situation is shown in the accompanying figure. Ari will need 9 half-gallon bottles, and he will probably see if he can also find a quart bottle to use.



INTO THE CLASSROOM

Teachers' Understandings of the Division of Fractions in the United States and China

As part of her comparative study of the differences in mathematical knowledge between teachers in the United States and China, Liping Ma asked teachers in both countries to calculate the division $1\frac{3}{4} \div \frac{1}{2}$. She reported that just 43% of the U.S. teachers were successful, and none showed an understanding of the rationale of the algorithm that was used. This was in sharp contrast to the teachers in China, all of whom computed the division correctly. The Chinese teachers were not satisfied just to obtain the answer but also enjoyed presenting and commenting upon various ways to do the division. They showed they had a deep conceptual understanding of all of the operations on fractions. Liping Ma's book* makes the case that elementary school teachers must acquire a "profound understanding of fundamental mathematics" to support the ways they present mathematics to their students, particularly the ways demanded by the standards set forth by the National Council of Teachers of Mathematics and by the Common Core State Standards in Mathematics.

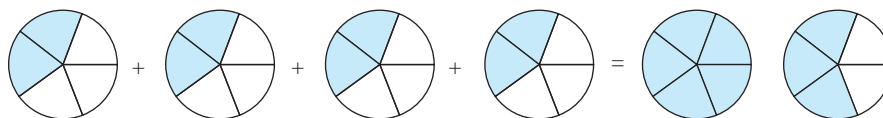
*From *Knowing and Teaching Elementary Mathematics: Teacher's Understanding of Fundamental Mathematics in China and in the United States*, © 1999, by Liping Ma.

6.3

Problem Set

Understanding Concepts

1. What fraction multiplication is illustrated with this fraction-circle diagram?



2. Draw a fraction-circle diagram that illustrates $5 \cdot \frac{3}{4}$.

3. Draw number-line diagrams that illustrate these products:

(a) $4 \cdot \frac{3}{8}$

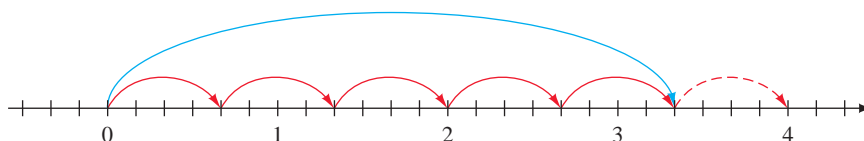
(b) $-4 \cdot \frac{3}{8}$

4. Draw number-line diagrams that illustrate these products.

(a) $4 \cdot \left(-\frac{3}{8}\right)$

(b) $-4 \cdot \left(-\frac{3}{8}\right)$

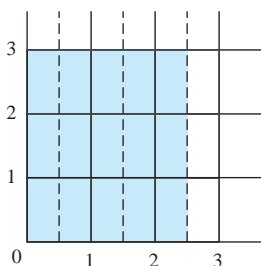
5. What fraction multiplication is illustrated by this number-line diagram?



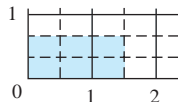
6. Draw a number-line diagram to illustrate that $\frac{3}{4} \times 5 = \frac{15}{4} = 3\frac{3}{4}$.

7. What multiplication facts are illustrated by these rectangular area models?

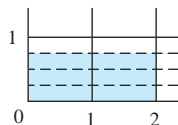
(a)



(b)



(c)



8. Illustrate these multiplications with the rectangular area model.

(a) $2 \times \frac{3}{5}$

(b) $\frac{3}{2} \times \frac{3}{4}$

(c) $1\frac{2}{3} \times 2\frac{1}{4}$

9. A rectangular plot of land is $2\frac{1}{4}$ miles wide and $3\frac{1}{2}$ miles long. What is the area of the plot in square miles? Draw a sketch that verifies your answer.

10. Find the reciprocals of the following fractions:

(a) $\frac{3}{8}$

(b) $\frac{4}{3}$

(c) $2\frac{1}{4}$

11. Find the reciprocals of the following fractions:

(a) $\frac{1}{8}$

(b) 5

(c) 1

12. Compute each of these products, expressing each answer in lowest terms:

(a) $\frac{5}{4} \times \frac{12}{25}$

(b) $\frac{7}{8} \times \frac{2}{21}$

(c) $\frac{-3}{5} \times \frac{10}{21}$

(d) $\frac{4}{7} \times \frac{21}{16}$

13. Use the common-numerator and common-denominator rules to compute these divisions, expressing each answer in lowest terms:

(a) $\frac{5}{4} \div \frac{3}{4}$

(b) $\frac{7}{8} \div \frac{7}{11}$

(c) $\frac{2}{5} \div \frac{7}{5}$

(d) $\frac{4}{7} \div \frac{4}{21}$

14. Compute these divisions, expressing each answer in simplest form:

(a) $\frac{2}{5} \div \frac{3}{4}$

(b) $\frac{6}{11} \div \frac{4}{3}$

(c) $\frac{100}{33} \div \frac{10}{3}$

(d) $2\frac{3}{8} \div 5$

(e) $3 \div 5\frac{1}{4}$

(f) $\frac{21}{25} \div \frac{7}{25}$

15. Compute the fraction with the simplest form that is equivalent to the given expression.

(a) $\frac{2}{3} \cdot \left(\frac{3}{4} + \frac{9}{12}\right)$

(b) $\left(\frac{3}{5} - \frac{3}{10}\right) \div \frac{6}{5}$

(c) $\left(\frac{2}{5} \div \frac{4}{15}\right) \cdot \frac{2}{3}$

16. Set up and evaluate expressions to solve these map problems:

(a) Each inch on a map represents an actual distance of $2\frac{1}{2}$ miles. If the map shows Helmer as being $3\frac{3}{4}$ inches due east of Deary, how far apart are the two towns?

(b) A map shows that Spokane is 60 miles north of Colfax. A ruler shows that the towns are $3\frac{3}{4}$ inches apart on the map. How many miles are represented by each inch on the map?

17. Carefully describe each step in the arithmetic of fractions that is required to solve these equations:

(a) $\frac{2}{5}x - \frac{3}{4} = \frac{1}{2}$

(b) $\frac{2}{3}x + \frac{1}{4} = \frac{3}{2}x$

18. I am a fraction whose product with 7 is the same as my sum with 7. Who am I?

Into the Classroom

The division of fractions is one of the most difficult concepts to teach in the elementary school curriculum. In each problem that follows, provide pictures and word descriptions that you believe would effectively convey the central ideas to a youngster. Some of the examples in the section can be used as models.

19. (Writing) Sean has a job mowing grass for the city, using a riding mower. He can mow $\frac{5}{6}$ of an acre per hour. How long will he need to mow the 3-acre city park?
20. (Writing) Angie, Bree, Corrine, Dot, and Elaine together picked $3\frac{3}{4}$ crates of strawberries. How many crates should each be allotted in order to distribute the berries evenly?
21. (Writing) Gerry is making a pathway out of concrete stepping-stones. The path is 25 feet long, and each stone extends $1\frac{2}{3}$ of a foot along the path. By letting x denote the number of stones, Gerry knows that he needs to solve the equation $x \cdot 1\frac{2}{3} = 25$, but he isn't sure how to solve for x . Provide a careful explanation.

Responding to Students

22. (Writing) Respond to Eva's statement:

I know that multiplication by $\frac{3}{4}$ means splitting into 4 equal parts and then taking 3 of the parts. I also know that division is the inverse of multiplication, so division by $\frac{3}{4}$ means splitting into 3 equal parts and then taking 4 of the parts. So, instead of dividing by $\frac{3}{4}$, I might just as well multiply by $\frac{4}{3}$.

Children (and adults!) often have difficulties relating fractions and their arithmetic to practical situations. In problems 23–25, respond to the children who have been asked to pose a multiplication problem that requires the answer to $\frac{1}{4}$ of $\frac{2}{3}$.

23. There is $\frac{2}{3}$ of a pizza left. Suzanne then ate a quarter of a pizza. How much pizza is left?
24. There is $\frac{2}{3}$ of a pizza left. Jamie ate $\frac{1}{4}$ of the remaining pieces. How many pieces of pizza now remain?

25. There is $\frac{2}{3}$ of a cake left. One-fourth of Miguel's classmates want some of the cake. How much cake is each child given?

In problems 26 and 27, respond to the children who have been asked to pose a problem that corresponds to the division of $1\frac{3}{4}$ by $\frac{1}{2}$.

26. There is $1\frac{3}{4}$ of a pizza left, and Kelly needs to share it equally with her brother. How much pizza does each child get?
27. There are $1\frac{3}{4}$ cups of chocolate chips in a bag. If a cookie recipe calls for a half-cup of chocolate chips, how many recipes can be made?

Thinking Critically

28. Jason and Kathy are driving in separate cars from home to Grandma's house, each driving the same route that passes through Midville. They both travel at 50 miles per hour, and Kathy left 4 hours ahead of Jason. They have the following conversation on their cell phones:

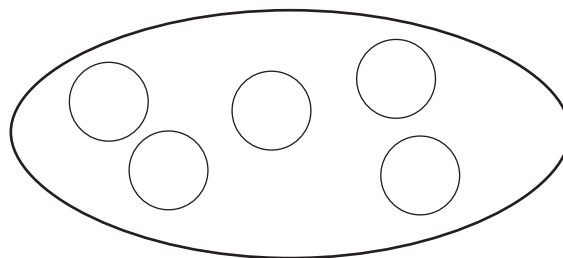
Jason: "Home is half as far away as Midville."

Kathy: "Grandma's house is half as far away as Midville."

How far is Grandma's from home? (*Suggestion:* Draw a picture.)

29. A bag contains red, green, and blue marbles. One-fifth of the marbles are red, there are $\frac{1}{3}$ as many blue marbles as green marbles, and there are 10 fewer red marbles than green marbles. Determine the number of marbles in the bag in these two approaches.

- (a) Use algebraic reasoning (i.e., introduce variables, form equations, and then solve the equations).
- (b) Use the set model of fractions, labeling the following Venn diagram:



30. The positive rational numbers 3 and $1\frac{1}{2}$ are an interesting pair because their sum is equal to their product: $3 + \frac{3}{2} = \frac{6}{2} + \frac{3}{2} = \frac{9}{2}$ and $3 \cdot \frac{3}{2} = \frac{9}{2}$.

- (a) Show that $3\frac{1}{2}$ and $1\frac{2}{5}$ have the same sum and product.
- (b) Show that $2\frac{3}{5}$ and $1\frac{5}{8}$ have the same sum and product.

- (c) If two positive rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ have the same sum and product, what must be true of the sum of their reciprocals?

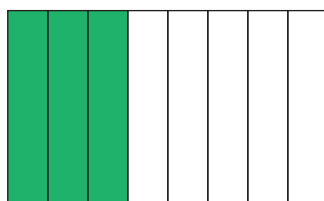
31. Three children had just cut their rectangular cake into three equal parts, as shown, to share, when a fourth friend joined them. Describe how to make one additional straight cut through the cake so that all four can share the cake equally.



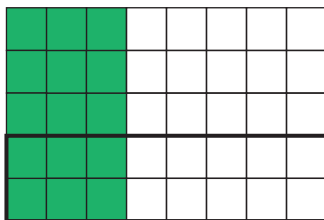
32. (a) Verify that $\frac{21}{8} \div \frac{7}{4} = \frac{21 \div 7}{8 \div 4} = \frac{3}{2}$.
 (b) Show that if $a \div c = m$ and $b \div d = n$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d} = \frac{m}{n}$.

33. **Illustrating Fraction Division with the Rectangular Area Model.** The following sequence of diagrams illustrates why $\frac{3}{8} \div \frac{2}{5} = \frac{15}{16}$, since it shows that the number of two-fifths in three-eighths is $\frac{15}{16}$.

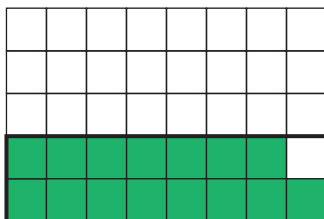
- A. Shade $\frac{3}{8}$ of a unit rectangle, using vertical lines.



- B. Use horizontal lines to create a rectangle of area $\frac{2}{5}$.



- C. Move the shaded boxes into the $\frac{2}{5}$ rectangle.



Make similar diagrams that illustrate each of these division problems:

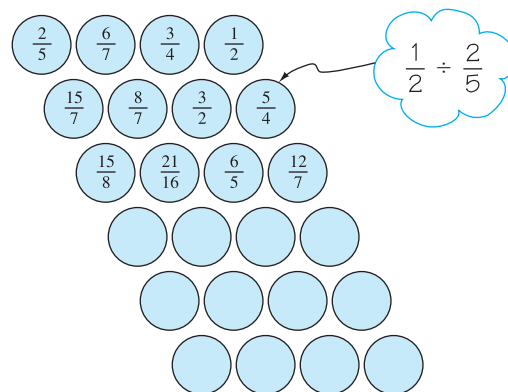
- (a) $\frac{2}{5} \div \frac{3}{4}$ (b) $\frac{3}{5} \div \frac{5}{6}$
 (c) $\frac{7}{8} \div \frac{1}{3}$ [Suggestion: Make a row of rectangles, each of area $\frac{1}{3}$, to wholly or partially cover the shaded boxes.]

34. Solve this problem found in the Rhind papyrus: “A quantity and its $\frac{1}{7}$ th added together become 19. What is the quantity?”

35. **Divvy.** The process called Divvy is like Diffy, except that the larger fraction is *divided* by the smaller. The first few rows of a sample Divvy are shown, where all of the fractions are positive.

- (a) Continue to fill in additional rows, using a calculator if you like.
 (b) Try Divvy with $\frac{2}{7}$, $\frac{4}{5}$, $\frac{3}{2}$, and $\frac{5}{6}$ in the first row.

Don't let complicated fractions put you off! Things should get better if you persist.



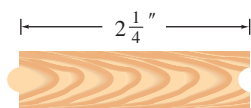
36. **(Writing)** For each fraction operation that follows, make up a realistic word problem whose solution requires the computation shown. Try to create an interesting and original situation.

- (a) $\frac{4}{5} \times \frac{7}{8}$ (b) $\frac{9}{10} \div \frac{3}{5}$

Making Connections

37. At a certain university, a student's senior thesis is acceptable if at least $\frac{3}{4}$ of the student's committee votes in its favor. What is the smallest number of favorable votes needed to accept a thesis if the committee has 3 members? 4 members? 5 members? 6 members? 7 members? 8 members?
38. A sign on a rolled-up canvas says that the canvas contains 42 square yards. The width of the canvas, which is easily measured without unrolling, is 14 feet (or $4\frac{2}{3}$ yards). What is the length of the piece of canvas, in yards?

39. Tongue-and-groove decking boards are each $2\frac{1}{4}$ inches wide. How many boards must be placed side by side to build a deck 14 feet in width?

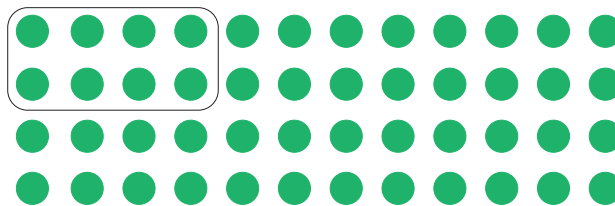


40. Six bows can be made from $1\frac{1}{2}$ yards of ribbon. How many bows can be made from $5\frac{3}{4}$ yards of ribbon?
41. Andre has 35 yards of material available to make aprons. Each apron requires $\frac{3}{4}$ yard. How many aprons can Andre make?
42. Gisela paid \$28 for a skirt that was " $\frac{1}{3}$ off." What was the original price of the skirt?
43. A soup recipe calls for $2\frac{3}{4}$ cups of chicken broth and will make enough to serve 8 people. How much broth is required if the recipe is modified to serve 6 people?
44. The label on a large 4-liter bottle claims to be the equivalent of $5\frac{1}{3}$ small bottles. What is the size of a small bottle?

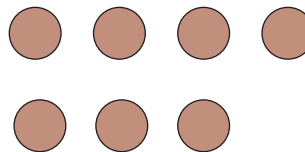
State Assessments

45. (Grade 4)
Jennie spent 15 minutes picking strawberries from the plants that are within the loop shown below. She has asked her older

brother Kyle how much time she can anticipate it will take to pick all of the strawberries in the plot.



- (a) What should Kyle tell Jennie about the total time to expect?
- (b) What should Kyle say to Jennie to explain how he found his answer.
46. **Writing** (Grade 5)
Four children want to share seven large peanut butter cookies. How many cookies does each child get if the cookies are shared equally? Explain in words and pictures how to get your answer.



47. (Massachusetts, Grade 7)
It took a ball 1 minute to roll 90 feet. What was this ball's average rate of speed, in feet per second?
- A. $\frac{2}{3}$ feet per second
- B. $1\frac{1}{2}$ feet per second
- C. 2 feet per second
- D. 3 feet per second

6.4

The Rational Number System

This section explores the properties of the rational numbers. Many properties will be familiar, since the integers have the same properties. However, we will also discover some important new properties of rational numbers that have no counterpart in the integers. This section also gives techniques for estimation and computation and presents additional examples of the application of rational numbers to the solution of practical problems.

Properties of Addition and Subtraction

To add two rational numbers, we first represent each rational number by a fraction and then add the two fractions. For example, to add the rational numbers represented by $\frac{5}{6}$ and $\frac{3}{10}$, we could use the common denominator 30; thus,

$$\frac{5}{6} + \frac{3}{10} = \frac{25}{30} + \frac{9}{30} = \frac{34}{30}.$$

We could also have used the addition formula to find that

$$\frac{5}{6} + \frac{3}{10} = \frac{5 \cdot 10 + 6 \cdot 3}{6 \cdot 10} = \frac{50 + 18}{60} = \frac{68}{60}.$$

The two answers, namely, $\frac{34}{30}$ and $\frac{68}{60}$, are different fractions. However, they are equivalent fractions,

and both represent the *same* rational number, $\frac{17}{15}$, when expressed by a fraction in simplest form.

More generally, *any* two rational numbers have a unique rational number that is their sum. That is, the rational numbers are closed under addition.

It is also straightforward to check that addition in the rational numbers is commutative and associative. For example, $\frac{5}{6} + \frac{3}{10} = \frac{3}{10} + \frac{5}{6}$ and $\frac{3}{4} + \left(\frac{-1}{3} + \frac{2}{5}\right) = \left(\frac{3}{4} + \frac{-1}{3}\right) + \frac{2}{5}$. Similarly,

0 is the additive identity. For example, $\frac{7}{9} + 0 = \frac{7}{9}$, since $0 = \frac{0}{9}$.

The rational numbers share one more property with the integers: the existence of **negatives**, or **additive inverses**. We have the following definition:

DEFINITION Negative or Additive Inverse

Let r be a rational number represented by the fraction $\frac{a}{b}$. Its **negative**, or **additive inverse**, written $-r$, is the rational number represented by the fraction $\frac{-a}{b}$.

For example, $-\frac{4}{7} = \frac{-4}{7}$. This is the additive inverse of $\frac{4}{7}$, since

$$\frac{4}{7} + \left(-\frac{4}{7}\right) = \frac{4}{7} + \frac{-4}{7} = \frac{4 + (-4)}{7} = \frac{0}{7} = 0.$$

As another example, $-\left(-\frac{3}{4}\right) = -\left(\frac{-3}{4}\right) = \frac{-(-3)}{4} = \frac{3}{4}$, which illustrates the general property

$$-\left(-\frac{a}{b}\right) = \frac{a}{b}.$$

A negative is also called an **opposite**. This term describes how a rational number and its negative are positioned on the number line: $-r$ is on the opposite side of 0 from r , at the same distance from 0.

The properties of addition on the rational numbers are listed in the following theorem:

THEOREM Properties of Addition of Rational Numbers

Let r , s , and t be rational numbers that are represented by the fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$. Then following properties hold:

Closure Property $r + s$ is a rational number; that is, $\frac{a}{b} + \frac{c}{d}$ is a fraction.

Commutative Property $r + s = s + r$; that is, $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Associative Property $(r + s) + t = r + (s + t)$; that is, $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$

Zero Is an Additive Identity $r + 0 = r$; that is, $\frac{a}{b} + 0 = \frac{a}{b}$

Existence of Additive Inverses $r + (-r) = 0$; that is, $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$, where $-\frac{a}{b} = \frac{-a}{b}$

In the integers, we discovered that subtraction was equivalent to addition of the negative. The same result holds for the rational numbers. Since the rational numbers are closed under addition and every rational number has a negative, the rational numbers are closed under subtraction.

THEOREM Subtraction Is Equivalent to Addition of the Negative

Let r and s be rational numbers represented by the fractions $\frac{a}{b}$ and $\frac{c}{d}$. Then $r - s = r + (-s)$; that is,

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right) = \frac{ad - bc}{bd}.$$

Subtraction is neither commutative nor associative, as examples such as $\frac{1}{2} - \frac{1}{4} \neq \frac{1}{4} - \frac{1}{2}$ and $1 - \left(\frac{1}{2} - \frac{1}{4}\right) \neq \left(1 - \frac{1}{2}\right) - \frac{1}{4}$ show. This means that careful attention must be given to the order of the terms and the placement of parentheses.

EXAMPLE

6.18

Subtracting Rational Numbers

Compute the following differences:

(a) $\frac{3}{4} - \frac{7}{6}$ (b) $\frac{2}{3} - \frac{-9}{8}$ (c) $\left(-2\frac{1}{4}\right) - \left(4\frac{2}{3}\right)$

Solution

(a) $\frac{3}{4} - \frac{7}{6} = \frac{3 \cdot 6}{4 \cdot 6} - \frac{4 \cdot 7}{4 \cdot 6} = \frac{18 - 28}{24} = \frac{-10}{24} = \frac{-5}{12} = -\frac{5}{12}.$

(b) $\frac{2}{3} - \frac{-9}{8} = \frac{2}{3} + \frac{9}{8} = \frac{2 \cdot 8}{3 \cdot 8} + \frac{3 \cdot 9}{3 \cdot 8} = \frac{16}{24} + \frac{27}{24} = \frac{43}{24}.$

(c) $\left(-2\frac{1}{4}\right) - \left(4\frac{2}{3}\right) = (-2 - 4) - \left(\frac{1}{4} + \frac{2}{3}\right) = -6 - \left(\frac{3}{12} + \frac{8}{12}\right) = -6\frac{11}{12}.$

Properties of Multiplication and Division

Multiplication of rational numbers includes all of the properties of multiplication on the integers. For example, let's investigate the distributive property of multiplication over addition by considering a specific case:

$$\frac{2}{5} \cdot \left(\frac{3}{4} + \frac{7}{8}\right) = \frac{2}{5} \cdot \left(\frac{6}{8} + \frac{7}{8}\right) = \frac{2}{5} \cdot \frac{13}{8} = \frac{26}{40}.$$

Add first; then multiply.

$$\frac{2}{5} \cdot \frac{3}{4} + \frac{2}{5} \cdot \frac{7}{8} = \frac{6}{20} + \frac{14}{40} = \frac{12}{40} + \frac{14}{40} = \frac{26}{40}.$$

Multiply first; then add.

These computations show that multiplication by $\frac{2}{5}$ distributes over the sum $\frac{3}{4} + \frac{7}{8}$. A similar calculation proves the general distributive property $r \cdot (s + t) = r \cdot s + r \cdot t$; that is, $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$.

An important new property of multiplication of rational numbers that is *not* true for the integers is the existence of multiplicative inverses. For example, the nonzero rational number $\frac{5}{8}$ has the multiplicative inverse $\frac{8}{5}$, since

$$\frac{5}{8} \cdot \frac{8}{5} = 1.$$

This property does not hold in the integers. For example, since there is no integer m for which $2 \cdot m = 1$, the integer 2 does not have a multiplicative inverse in the set of integers.

THEOREM Properties of Multiplication of Rational Numbers

Let r , s , and t be rational numbers represented by the fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$. Then the following properties hold:

Closure Property rs is a rational number; that is, $\frac{a}{b} \cdot \frac{c}{d}$ is a fraction.

Commutative Property $rs = sr$; that is, $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$

Associative Property $(rs)t = r(st)$; that is, $\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right)$

Distributive Property of Multiplication over Addition and Subtraction

$r(s + t) = rs + rt$ and $r(s - t) = rs - rt$; that is,
 $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$ and $\frac{a}{b} \cdot \left(\frac{c}{d} - \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} - \frac{a}{b} \cdot \frac{e}{f}$

Multiplication by 0 $0 \cdot r = 0$; that is, $0 \cdot \frac{a}{b} = 0$

One Is a Multiplicative Identity $1 \cdot r = r$; that is, $1 \cdot \frac{a}{b} = \frac{a}{b}$

Existence of Multiplicative Inverse If $r = \frac{a}{b} \neq 0$, then there is a unique rational number, namely, $r^{-1} = \frac{b}{a}$, for which $r \cdot r^{-1} = \frac{a}{b} \cdot \frac{b}{a} = 1$.

EXAMPLE 6.19

Solving an Equation with the Multiplicative Inverse

Consuela paid \$36 for a pair of shoes at a “one-fourth off” sale. What was the price of the shoes before the sale?

Solution

Let x be the original price of the shoes, in dollars. Consuela paid $\frac{3}{4}$ of this price, so $\frac{3}{4} \cdot x = 36$. To solve this equation for the unknown x , multiply both sides by the multiplicative inverse $\frac{4}{3}$ to get

$$\frac{4}{3} \cdot \frac{3}{4} \cdot x = \frac{4}{3} \cdot 36.$$

But $\frac{4}{3} \cdot \frac{3}{4} = 1$ by the multiplicative inverse property, so

$$1 \cdot x = \frac{4 \cdot 36}{3} = 48.$$

Since 1 is a multiplicative identity, it follows that $1 \cdot x = x = 48$. Therefore, the original price of the pair of shoes was \$48.

Properties of the Order Relation

The order relation has many useful properties that are not difficult to prove. The most useful are the **transitive property**, the **addition property**, the **multiplication property**, and the **trichotomy property**.

THEOREM Properties of the Order Relation on the Rational Numbers

Let r , s , and t be rational numbers represented by the fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Transitive Property If $r < s$ and $s < t$, then $r < t$; that is,

$$\frac{a}{b} < \frac{c}{d} \text{ and } \frac{c}{d} < \frac{e}{f}, \text{ then } \frac{a}{b} < \frac{e}{f}.$$

Addition Property If $r < s$, then $r + t < s + t$; that is,

$$\frac{a}{b} < \frac{c}{d}, \text{ then } \frac{a}{b} + \frac{e}{f} < \frac{c}{d} + \frac{e}{f}.$$

Multiplication Property If $r < s$ and $t > 0$, then $rt < st$; that is,

$$\frac{a}{b} < \frac{c}{d} \text{ and } \frac{e}{f} > 0, \text{ then } \frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}.$$

If $r < s$ and $t < 0$, then $rt > st$; that is,

$$\frac{a}{b} < \frac{c}{d} \text{ and } \frac{e}{f} < 0, \text{ then } \frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}.$$

Trichotomy Property Exactly one of the following holds: $r < s$, $r = s$, or $r > s$; that is,

exactly one of $\frac{a}{b} < \frac{c}{d}$, $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{b} > \frac{c}{d}$ holds.

The Density Property of Rational Numbers

Since 1 and 2 are successive integers with $1 < 2$, there is no integer between 1 and 2. But consider the rational numbers $\frac{1}{2}$ and $\frac{2}{3}$; for which,

$$\frac{1}{2} < \frac{2}{3}.$$

Alternatively, using 6 as a common denominator, we have

$$\frac{3}{6} < \frac{4}{6},$$

or with 12 as a common denominator, we have

$$\frac{6}{12} < \frac{8}{12}.$$

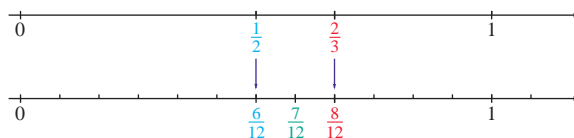
In the last form, we see that $\frac{7}{12}$ is a rational number that is between $\frac{6}{12}$ and $\frac{8}{12}$. That is,

$$\frac{1}{2} < \frac{7}{12} < \frac{2}{3},$$

as shown in Figure 6.17.

Figure 6.17

The rational number $\frac{7}{12}$ is between $\frac{1}{2}$ and $\frac{2}{3}$.



The idea used to find a rational number that is between $\frac{1}{2}$ and $\frac{2}{3}$ can be extended to show that between *any* two rational numbers there is some other rational number. This interesting fact is called the **density property** of the rational numbers.

THEOREM The Density Property of Rational Numbers

Let r and s be any two rational numbers, with $r < s$. Then there is a rational number t between r and s ; that is, $r < t < s$. Equivalently, if $\frac{a}{b} < \frac{c}{d}$, then there is a fraction $\frac{e}{f}$ such that $\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$.

EXAMPLE 6.20**Finding Rational Numbers between Two Rational Numbers**

Find a rational number between the two given rational numbers.

(a) $\frac{2}{3}$ and $\frac{3}{4}$ (b) $\frac{5}{12}$ and $\frac{3}{8}$

Solution

(a) Using $2 \cdot 3 \cdot 4 = 24$ as a common denominator, we have $\frac{2}{3} = \frac{16}{24}$ and $\frac{3}{4} = \frac{18}{24}$. Since $\frac{16}{24} < \frac{17}{24} < \frac{18}{24}$, it follows that $\frac{17}{24}$ is one answer.

(b) If we use 48 as a common denominator, we have $\frac{5}{12} = \frac{20}{48}$ and $\frac{3}{8} = \frac{18}{48}$. Thus, $\frac{19}{48}$ is one answer. Alternatively, we could use 480 as a common denominator, writing $\frac{5}{12} = \frac{200}{480}$ and $\frac{3}{8} = \frac{180}{480}$. This makes it clear that $\frac{181}{480}, \frac{182}{480}, \dots, \frac{199}{480}$ are all rational numbers between $\frac{3}{8}$ and $\frac{5}{12}$. Using a larger common denominator allows us to identify even more rational numbers between the two given rationals.

Computations with Rational Numbers

To work confidently with rational numbers, it is important to develop skills in estimation, rounding, mental arithmetic, and efficient pencil-and-paper computation.

Estimations

In many applications, the exact fractional value can be rounded off to the nearest integer value; if more precision is required, values can be rounded to the nearest half, third, or quarter.

EXAMPLE 6.21**Using Rounding of Fractions to Convert a Brownie Recipe**

Krishna's recipe, shown in the box, makes two dozen brownies. He'll need five dozen for the Math Day picnic, so he wants to adjust the quantities of his recipe. How should this be done?

Solution**Brownie Recipe
(2 dozen)**

4 squares chocolate
 $\frac{3}{4}$ cup butter
 2 cups sugar
 3 eggs
 1 teaspoon vanilla
 1 cup flour
 $1\frac{1}{4}$ cups chopped walnuts

Since Krishna needs $2\frac{1}{2}$ times the number of brownies given by his recipe, he will multiply the quantities by $\frac{5}{2}$. For example, $\frac{5}{2} \times 4 = 10$, so he will use 10 squares of chocolate. Similarly, he will use $\frac{5}{2} \times 2 = 5$ cups of sugar, $2\frac{1}{2}$ teaspoons of vanilla, and $2\frac{1}{2}$ cups of flour. However, $\frac{5}{2} \times \frac{3}{4} = \frac{15}{8} = 1\frac{7}{8}$, so Krishna will use "just short" of 2 cups of butter. Similarly, $\frac{5}{2} \times \frac{5}{4} = \frac{25}{8} = 3\frac{1}{8}$, so Krishna will use about 3 cups of chopped walnuts. Finally, $\frac{5}{2} \times 3 = \frac{15}{2} = 7\frac{1}{2}$, so he will use either 7 or 8 eggs.

By taking advantage of the properties, formulas, and algorithms associated with the various operations, it is often possible to simplify the computational process. Some useful strategies are demonstrated in the next example.

Computational Strategies for Rational Number Arithmetic

$$\begin{array}{ll} \text{(a)} 53 - 29\frac{3}{5} & \text{(b)} \left(2\frac{1}{8} - 4\frac{2}{3}\right) + 7\frac{7}{8} \\ \text{(c)} \frac{7}{15} \times 90 & \text{(d)} \frac{3}{8} \times 14 + \frac{3}{4} \times 25 \\ \text{(e)} 4\frac{1}{6} \times 18 & \text{(f)} \frac{5}{8} \times \left(\frac{7}{10} \times \frac{24}{49}\right) \end{array}$$

(a) Adding $\frac{2}{5}$ to each term gives $53 - 29\frac{3}{5} = 53\frac{2}{5} - 30 = 23\frac{2}{5}$.

(b) $\left(2\frac{1}{8} - 4\frac{2}{3}\right) + 7\frac{7}{8} = \left(2 + 7 + \frac{1}{8} + \frac{7}{8}\right) - 4\frac{2}{3}$
 $= 10 - 4\frac{2}{3} = 10\frac{1}{3} - 5 = 5\frac{1}{3}$.

(c) $\frac{7}{15} \times 90 = 7 \times \frac{90}{15} = 7 \times 6 = 42$.

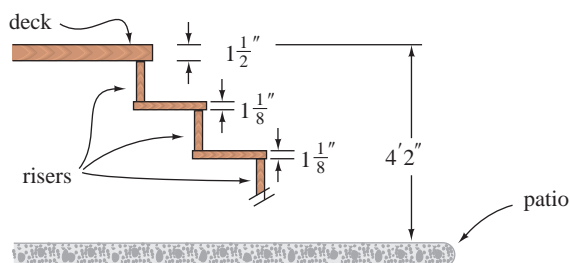
(d) $\frac{3}{8} \times 14 + \frac{3}{4} \times 25 = \frac{3}{8} \times 14 + \frac{3}{8} \times 50$
 $= \frac{3}{8} \times (14 + 50) = \frac{3}{8} \times 64$
 $= 3 \times \frac{64}{8} = 3 \times 8 = 24$.

(e) $4\frac{1}{6} \times 18 = \left(4 + \frac{1}{6}\right) \times 18 = 4 \times 18 + \frac{1}{6} \times 18 = 72 + 3 = 75$.

(f) $\frac{5}{8} \times \left(\frac{7}{10} \times \frac{24}{49}\right) = \frac{5}{10} \times \frac{7}{49} \times \frac{24}{8} = \frac{1}{2} \times \frac{1}{7} \times 3 = \frac{3}{14}$.

Designing Wooden Stairs

A deck is 4' 2" above the surface of a patio. How many steps and risers should there be in a stairway that connects the deck to the patio? The decking is $1\frac{1}{2}$ " thick, the stair treads are $1\frac{1}{8}$ " thick, and the steps should each rise the same distance, from one to the next. Calculate the vertical dimension of each riser.



Solution**Understand the Problem**

We have been given some important dimensions, including the thickness of the treads and the deck. We have *not* been told what dimension to cut the risers. This, we see from the figure, depends on the number of steps we choose, with a lower rise corresponding to a greater number of steps. We must choose a number of steps that feels natural to walk on. We must also be sure that each step rises the same amount.

Devise a Plan

If we know what vertical rise from step to step is customary, we can first get a reasonable estimate of the number of steps to use. Once it is agreed what number of steps to incorporate into the design, we can calculate the height of each riser. The riser meeting the deck must be adjusted to account for the decking being thicker than the stair tread.

Carry Out the Plan

A brief survey of existing stairways shows that most steps rise 5" to 7" from one to the next. Since $4' 2'' = 50''$, $\frac{50}{5} = 10$, and $\frac{50}{7} = 7\frac{1}{7}$, then 8, or perhaps 9, steps will work well, including the step onto the deck itself. Let's choose 8. Then $\frac{50''}{8} = 6\frac{2''}{8} = 6\frac{1''}{4}$; that is, each combination of riser plus tread is to be $6\frac{1''}{4}$. Since the treads are $1\frac{1''}{8}$ thick, the first seven risers require the boards to be cut $6\frac{1''}{4} - 1\frac{1''}{8} = 5\frac{1''}{8}$ wide. Since the decking is $1\frac{1''}{2}$ thick, the uppermost riser is $6\frac{1''}{4} - 1\frac{1''}{2} = 4\frac{3''}{4}$ wide.

Look Back

If we are concerned that the steps will take up too much room on the patio, we might use a steeper, 7-step design. Each riser plus tread would then rise $7\frac{1''}{7}$. Experienced carpenters would think of this value as a "hair" more than $7\frac{1''}{8}$ and cut the lower six risers from a board 8" wide, for a total rise of $7\frac{1''}{8}$. The last riser is cut to make any final small adjustment.

6.4**Problem Set****Understanding Concepts**

1. Explain what properties of addition of rational numbers can be used to make this sum very easy to compute:

$$\left(3\frac{1}{5} + 2\frac{2}{5}\right) + 8\frac{1}{5}$$

2. What properties can you use to make these computations easy?

(a) $\frac{2}{5} + \left(\frac{3}{5} + \frac{2}{3}\right)$

(b) $\frac{1}{4} + \left(\frac{2}{5} + \frac{3}{4}\right)$

(c) $\frac{2}{3} \cdot \frac{1}{8} + \frac{2}{3} \cdot \frac{7}{8}$

(d) $\frac{3}{4} \cdot \left(\frac{5}{9} \cdot \frac{4}{3}\right)$

3. Find the negatives (i.e., the additive inverses) of the given rational numbers. Show each number and its negative on the number line.

(a) $\frac{4}{5}$

(b) $\frac{-3}{2}$

(c) $\frac{8}{-3}$

(d) $\frac{4}{2}$

4. Compute the given sums of rational numbers. Explain what properties of rational numbers you find useful. Express your answers in simplest form.

(a) $\frac{1}{6} + \frac{2}{-3}$

(b) $\frac{-4}{5} + \frac{3}{2}$

(c) $\frac{9}{4} + \frac{-7}{8}$

(d) $\frac{3}{4} + \frac{-5}{8} + \frac{7}{-12}$

5. Compute the given differences of rational numbers. Explain what properties you find useful. Express your answers in simplest form.

(a) $\frac{2}{5} - \frac{3}{4}$

(b) $\frac{-6}{7} - \frac{4}{7}$

(c) $\frac{3}{8} - \frac{1}{12}$

6. Compute the given differences of rational numbers. Explain what properties you find useful. Express your answers in simplest form.

(a) $3\frac{2}{5} - \frac{7}{10}$

(b) $2\frac{1}{3} - 5\frac{3}{4}$

(c) $-4\frac{2}{3} - \frac{-19}{6}$

7. Calculate the given products of rational numbers. Explain what properties of rational numbers you find useful. Express your answers in simplest form.

(a) $\frac{3}{5} \cdot \frac{7}{8} \cdot \left(\frac{5}{3}\right)$

(b) $\frac{-2}{7} \cdot \frac{3}{4}$

(c) $\frac{-4}{3} \cdot \frac{6}{-16}$

8. Calculate the given products of rational numbers. Explain what properties of rational numbers you find useful. Express your answers in simplest form.

(a) $3\frac{1}{8} \cdot 2\frac{1}{5} \cdot (40)$

(b) $\frac{14}{15} \cdot \frac{60}{7}$

(c) $\left(\frac{4}{11} \cdot \frac{22}{7}\right) \cdot \left(\frac{-3}{8}\right)$

9. Find the reciprocals (i.e., the multiplicative inverses) of the given rational numbers. Show each number and its reciprocal on the number line.

(a) $\frac{3}{2}$

(b) $\frac{4}{-9}$

(c) $\frac{-4}{-11}$

10. Find the reciprocals (i.e., the multiplicative inverses) of the given rational numbers. Show each number and its reciprocal on the number line.

(a) 5

(b) -2

(c) $2\frac{1}{2}$

11. Use the properties of the operations of rational number arithmetic to perform these calculations. Express your answers in simplest form.

(a) $\frac{2}{3} \cdot \frac{4}{7} + \frac{2}{3} \cdot \frac{3}{7}$

(b) $\frac{4}{5} \cdot \frac{2}{3} - \frac{3}{10} \cdot \frac{2}{3}$

12. Use the properties of the operations of rational number arithmetic to perform these calculations. Express your answer in simplest form.

(a) $\frac{4}{7} \cdot \frac{3}{2} - \frac{4}{7} \cdot \frac{6}{4}$

(b) $\left(\frac{4}{7} \cdot \frac{2}{5}\right) \div \frac{2}{7}$

13. Justify each step in the proof of the distributive property of multiplication over addition that follows. In this case, the two addends have a common denominator, but the operation is always possible for any rational addends by using a common denominator for the two fractions.

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{d}\right) = \frac{a}{b} \cdot \frac{c+e}{d}$$

(a) Why?

$$= \frac{a \cdot (c+e)}{b \cdot d}$$

(b) Why?

$$= \frac{a \cdot c + a \cdot e}{b \cdot d}$$

(c) Why?

$$= \frac{a \cdot c}{b \cdot d} + \frac{a \cdot e}{b \cdot d}$$

(d) Why?

$$= \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{d}$$

(e) Why?

14. If $\frac{a}{b} \cdot \frac{4}{7} = \frac{2}{3}$, what is $\frac{a}{b}$? Carefully explain how you obtain your answer. What properties do you use?

15. Solve each equation for the rational number x . Show your steps and explain what property justifies each step.

(a) $4x + 3 = 0$

(b) $x + \frac{3}{4} = \frac{7}{8}$

(c) $\frac{2}{3}x + \frac{4}{5} = 0$

(d) $3\left(x + \frac{1}{8}\right) = -\frac{2}{3}$

16. Arrange each group of rational numbers in increasing order. Show, at least approximately, the numbers on the number line.

(a) $\frac{4}{5}, -\frac{1}{5}, \frac{2}{5}$

(b) $-\frac{3}{7}, \frac{4}{7}, -\frac{5}{7}$

(c) $\frac{3}{8}, \frac{1}{2}, \frac{3}{4}$

(d) $-\frac{7}{12}, -\frac{2}{3}, -\frac{3}{4}$

17. Verify these inequalities:

(a) $\frac{-4}{5} < \frac{-3}{4}$

(b) $\frac{1}{10} > -\frac{1}{4}$

(c) $\frac{-19}{60} > \frac{-1}{3}$

18. Solve the inequalities that follow. Show all your steps.

(a) $x + \frac{2}{3} > -\frac{1}{3}$

(b) $x - \left(-\frac{3}{4}\right) < \frac{1}{4}$

(c) $\frac{3}{4}x < -\frac{1}{2}$

(d) $-\frac{2}{5}x + \frac{1}{5} > -1$

19. Find a rational number that is between the two given rational numbers.

(a) $\frac{4}{9}$ and $\frac{6}{11}$

(b) $\frac{1}{9}$ and $\frac{1}{10}$

(c) $\frac{14}{23}$ and $\frac{7}{12}$

(d) $\frac{141}{568}$ and $\frac{183}{737}$

20. Find three rational numbers between $\frac{1}{4}$ and $\frac{2}{5}$.

21. For each given rational number, choose the best estimate from the list provided.

- (a) $\frac{104}{391}$ is approximately $\frac{1}{3}$, $\frac{1}{4}$, or $\frac{1}{2}$.
 (b) $\frac{217}{340}$ is approximately $\frac{1}{3}$, $\frac{1}{2}$, or $\frac{2}{3}$.
 (c) $\frac{-193}{211}$ is approximately $-\frac{1}{2}$, -1 , 1 , or $\frac{1}{2}$.
 (d) $\frac{453}{307}$ is approximately $\frac{3}{4}$, 1 , $1\frac{1}{3}$, or $1\frac{1}{2}$.

22. Use estimations to choose the best approximation of the following expressions:

- (a) $3\frac{19}{40} + 5\frac{11}{19}$ is approximately 8 , $8\frac{1}{2}$, 9 , or $9\frac{1}{2}$.
 (b) $2\frac{6}{19} + 5\frac{1}{3} - 4\frac{7}{20}$ is approximately 3 , $3\frac{1}{3}$, $3\frac{1}{4}$, or 4 .
 (c) $17\frac{8}{9} \div 5\frac{10}{11}$ is approximately 2 , 3 , $3\frac{1}{2}$, or 4 .

23. (Writing) First do the given calculations mentally. Then use words and equations to explain your strategy.

- (a) $\frac{1}{2} + \frac{1}{4} + \frac{3}{4}$ (b) $\frac{5}{2} \cdot \left(\frac{2}{5} - \frac{2}{10}\right)$
 (c) $\frac{3}{4} \cdot \frac{12}{15}$ (d) $\frac{2}{9} \div \frac{1}{3}$

24. (Writing) First do the given calculations mentally. Then use words and equations to explain your strategy.

- (a) $2\frac{2}{3} \times 15$ (b) $3\frac{1}{5} - 1\frac{1}{4} + 7\frac{4}{5}$
 (c) $6\frac{1}{8} - 8\frac{1}{4}$ (d) $\frac{2}{3} \cdot \frac{7}{4} - \frac{2}{3} \cdot \frac{1}{4}$

25. (a) Hal owns $3\frac{1}{2}$ acres and just purchased an adjacent plot of $\frac{3}{4}$ acres. The answer is $4\frac{1}{4}$. What is the question?

- (b) Janet lives $1\frac{3}{4}$ miles from school. On the way to school, she stopped to walk the rest of the way with Brian, who lives $\frac{1}{2}$ mile from school. The answer is $1\frac{1}{4}$. What is the question?

- (c) A family room is $5\frac{1}{2}$ yards wide and 6 yards long. The answer is 33 square yards. What is the question?

- (d) Clea made $3\frac{1}{2}$ gallons of ginger ale, which she intends to bottle in “fifths” (i.e., bottles that contain a fifth of a gallon). The answer is 17 full bottles and $\frac{1}{2}$ of another bottle. What is the question?

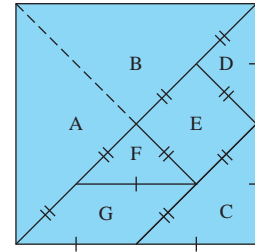
26. Invent interesting word problems that lead you to the given expression.

- (a) $\frac{3}{4} + \frac{1}{2}$ (b) $4\frac{1}{2} \div \frac{3}{4}$ (c) $7\frac{2}{3} \times \frac{1}{4}$

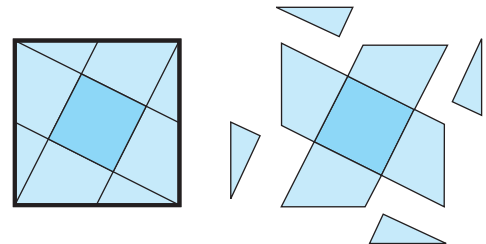
Into the Classroom

The area model of fractions is a useful visualization of the meaning of fractions. The unit of area is a simple shape that is partitioned into b congruent pieces and then a of the pieces are shaded to give a pictorial representation of the fraction $\frac{a}{b}$. A challenging problem is to reverse the model: Shade part of a figure of unit area and determine the fraction that describes this area. The following two problems can initiate a lively class discussion.

27. A unit square is partitioned into the seven tangram pieces A, B, C, D, E, F, and G. What fraction gives the area of each piece?



28. Start with a square piece of paper and join each corner to the midpoint of the opposite side, as shown in the left-hand figure. Shade the slanted square that is formed inside the original square. What fraction of the large square have you shaded? It will help to cut the triangular pieces as shown on the right and reattach them to form some new squares.



Responding to Students

29. When asked to evaluate the sum $\frac{1}{3} + \frac{3}{5}$, a student claimed that the answer, when simplified, is $\frac{1}{2}$. How do you suspect the student arrived at this answer? Describe what you might do to help this student's understanding.
30. When asked to evaluate the difference $\frac{9}{11} - \frac{3}{22}$, a student gave the answer $\frac{165}{242}$. What would you suggest to the student?

Thinking Critically

In 1858, the Scotsman H. A. Rhind purchased an Egyptian papyrus copied by the scribe A'h-mose (or Ahmes) in 1650 B.C. from an earlier document written in about 1850 B.C. Problems 31–34 are adapted from the Rhind papyrus.

31. “A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, amount to 33.” That is, in modern notation, $\frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x + x = 33$. What is the quantity?

32. “Divide 100 loaves among five men in such a way that the share received shall be in arithmetic progression and that one-seventh of the sum of the largest three shares shall be equal to the sum of the smallest two.” (*Suggestion:* Denote the shares as $s, s + d, s + 2d, s + 3d$, and $s + 4d$.)

33. The ancient Egyptians measured the steepness of a slope by the fraction $\frac{x}{y}$, where x is the number of hands of horizontal “run” and y is the number of cubits of vertical “rise.” Seven hands form a cubit. Problem 56 of the Rhind papyrus asks for the steepness of the face of a pyramid 250 cubits high and having a square base 360 cubits on a side. The papyrus gives the answer $5\frac{1}{25}$. Show why this answer is correct.

34. (a) The Rhind papyrus is about 6 yards long and $\frac{1}{3}$ yard wide.

What is its area in square yards?

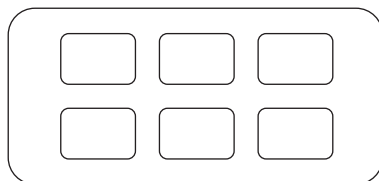
- (b) The Moscow papyrus, another source of mathematics of ancient Egypt, is about the same length as the Rhind papyrus, but has $\frac{1}{4}$ the area. What is the width of the Moscow papyrus?

35. (Writing) Three-quarters of the pigeons occupy two-thirds of the pigeonholes in a pigeon house (one pigeon per hole), and the other one-quarter of the pigeons are flying around. If all the pigeons were in the pigeonholes, just five holes would be empty. How many pigeons and how many holes are there? Give an answer based on the following diagram, adding labels to the diagram and carefully describing your reasoning in words:



36. (Writing) A bag contains red and blue marbles. One-sixth of the marbles are blue, and there are 20 more red marbles than blue marbles. How many red marbles and how many blue marbles are there?

- (a) Give an answer based on the following set diagram, adding labels and describing how to use the diagram to obtain an answer:



- (b) Give an answer based on finding and solving equations.

37. (a) Let x, y , and z be arbitrary rational numbers. Show that the sums of the three entries in every row, column, and diagonal in the Magic Square are the same.

| | | |
|-------------|-------------|-------------|
| $x - z$ | $x - y + z$ | $x + y$ |
| $x + y + z$ | x | $x - y - z$ |
| $x - y$ | $x + y - z$ | $x + z$ |

- (b) Let $x = \frac{1}{2}$, $y = \frac{1}{3}$, and $z = \frac{1}{4}$. Find the corresponding Magic Square.
- (c) Here is a partial Magic Square that corresponds to the form shown in part (a). Find x, y , and z , and then fill in the remaining entries of the square.

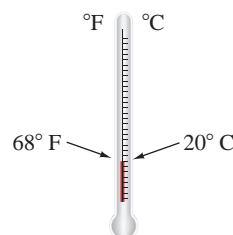
| | | |
|-----------------|--|---|
| $-\frac{5}{12}$ | | 1 |
| | | |
| $-\frac{1}{3}$ | | |

(Hint: $2x = (x + y) + (x - y)$.)

Making Connections

38. The Fahrenheit and Celsius temperature scales are related by the formula $F = \frac{9}{5}C + 32$. For example, a temperature of 20° Celsius corresponds to 68° Fahrenheit, since










$$\frac{9}{5} \cdot 20 + 32 = 36 + 32 = 68.$$



- (a) Derive a formula for C as it depends on F . Deduce the equivalent formula $C = \frac{5}{9}(F - 32)$ relating the two temperature scales.
- (b) Fill in the missing entries in this table:

| | | | | | | | |
|------------------|-------------|-------------|-----------|------------|------------|-------------|-------------|
| $^\circ\text{C}$ | -40° | | 0° | 10° | 20° | | |
| $^\circ\text{F}$ | | -13° | | | 68° | 104° | 212° |

- (c) Electronic signboards frequently give the temperature in both Fahrenheit and Celsius degrees. What is the temperature if both readings are the same (i.e., $F = C$)? The negative of one another (i.e., $F = -C$)? Answer the latter question with an exact rational number and the approximate integer that would be seen on the signboard.
39. Earth revolves around the sun once in 365 days, 5 hours, 48 minutes, and 46 seconds. Since this *solar* year is more than 365 days long, it is important to have 366-day-long *leap* years to keep the seasons in the same months of the calendar year.
- (a) The solar year is very nearly 365 days and 6 hours, or $365\frac{6}{24} = 365\frac{1}{4}$ days, long. Explain why having the years divisible by 4 as leap years is a good rule to decide when leap years should occur.
- (b) Show that a solar year is $365\frac{20,926}{24 \cdot 60 \cdot 60}$ days long.
- (c) Estimate the value given in part (b) with $365\frac{20,952}{24 \cdot 60 \cdot 60}$, and then show that this number can also be written as $365\frac{1}{4} - \frac{1}{100} + \frac{1}{400}$.
- (d) What rule for choosing leap years is suggested by the approximation $365\frac{1}{4} - \frac{1}{100} + \frac{1}{400}$ to the solar year?
- (Hint: 1900 was not a leap year, but 2000 was a leap year.)
40. Fractions have a prominent place in music, where the time value of a note is given as a fraction of a whole note. The note values and their corresponding fractions are shown in this table:

| Name | Whole | Half | Quarter | Eighth | Sixteenth |
|------------|---|--|--|--|--|
| Note |  |  or  |  or  |  or  |  or  |
| Time Value | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ |

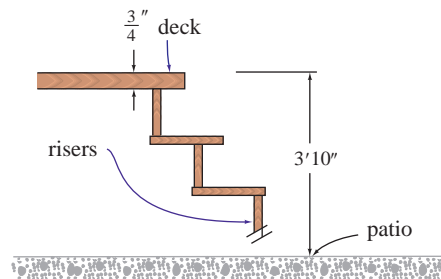
The time signature of the music can be interpreted as a fraction that gives the duration of the measure. Here are two examples:

$$\begin{array}{cc} \text{Musical notation: } \text{3/4} & \text{Musical notation: } \text{6/8} \\ \frac{3}{4} = \frac{1}{4} + \frac{1}{2} & \frac{6}{8} = \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{1}{4} \end{array}$$

In each measure shown, fill in the upper number of the time signature by adding the note values shown in the measure.

- (a) 
- (b) 
- (c) 

41. (Writing) Design a stairway that connects the patio to the deck. (See Example 6.23.)

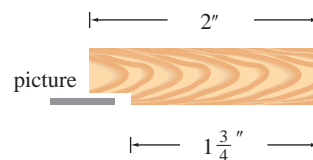


Assume that the tread of each stair is $\frac{1}{2}$ inch thick.

42. This recipe makes six dozen cookies. Adjust the quantities to have a recipe for four dozen cookies.

| | |
|-----------------------|--------------------------|
| 1 cup shortening | $1\frac{1}{2}$ cup sugar |
| 1 tsp baking soda | 3 eggs |
| 3 cups unsifted flour | $\frac{1}{2}$ tsp salt |
| 9 oz mincemeat | |

43. Anja has a box of photographic print paper. Each sheet measures 8 by 10 inches. She could easily cut a sheet into four 4-by-5-inch rectangles. However, to save money, she wants to get six prints, all the same size, from each 8-by-10-inch sheet. Describe how she can cut the paper.
44. Krystoff has an 8' piece of picture frame molding, shown in cross section as follows:

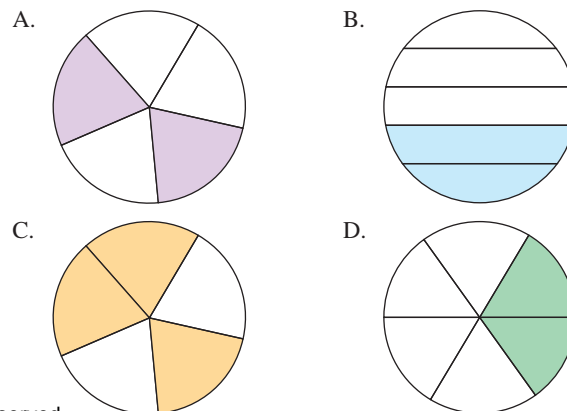


Is this 8' piece sufficient to frame a 16"-by-20" picture? Allow for saw cuts and some extra space to ensure that the picture fits easily into the frame.

State Assessments

45. (Grade 5)

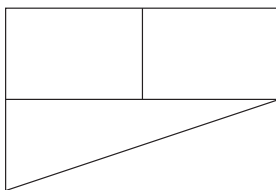
Which figure shows $\frac{2}{5}$ has been shaded?



46. (Writing) (Grade 5)

Becky cut a rectangular dish of brownies into four pieces as shown below. Not all of the pieces are the same shape, but she knows that the pieces are the same size.

Explain how Becky knows this.



47. (Grade 7)

In Hank's garden, $\frac{3}{4}$ of his tomato plants have blossoms,

and of these $\frac{2}{3}$ have green tomatoes. How many of

Hank's tomato plants have green tomatoes?

- A. More than $\frac{2}{3}$ B. Between $\frac{2}{3}$ and $\frac{3}{4}$
C. Less than $\frac{1}{2}$ D. Exactly $\frac{1}{2}$

The Chapter in Relation to Future Teachers

If elementary school teachers were asked to list topics in the mathematics curriculum that they felt were the most challenging, both for themselves and their students, fractions will often head their lists. Unlike the integers, which are usually associated with the familiar idea of counting, fractions play a much wider range of roles. Here is a list of three roles highlighted in this chapter:

- *Fractions are an extension of measurement.* The unit is subdivided, and the measurement of a quantity (distance, area, time, etc.) is given by counting both the number of whole units and the number of subdivisions of a unit.
- *Fractions represent the rational numbers.* Indeed, any given rational number can be represented by any of infinitely many equivalent fractions. Moreover, the rational numbers *are* a system of numbers. That is, like the integers, the rational numbers can be identified with a point or distance on the number line, their arithmetic operations obey certain algebraic properties, and the result of an arithmetic operation $+$, \times , $-$, or \div is computed by an accompanying formula.
- *Fractions are operators.* That is, a fraction is an action as opposed to an object, much in the way a verb is opposed to a noun. For example, at a storewide $\frac{1}{4}$ -off sale, we would determine the price of any item by multiplying its current price by $\frac{3}{4}$. Now $\frac{3}{4}$ has become an operator.

These meanings of fractions have been illustrated throughout the chapter by physical models and visualizations: paper folding, colored regions, fraction strips, fraction circles, rectangular arrays, and the like. It is important to understand how the representations and models are of value as instructional aids.

The **Common Core** makes it clear why instruction must focus on conveying understanding:

Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut.

Our work with fractions and rational numbers is not yet complete. In Chapter 7, we will explore which decimal numbers represent rational numbers, see how fractions are related to ratios and proportional reasoning, and learn that a percent is a fraction with 100 as its denominator.



Connecting the
Standards for
Mathematical
Practice and the
Standards for
Mathematical
Content.

Chapter 6 Summary

Section 6.1 The Basic Concepts of Fractions and Rational Numbers

Page Reference

CONCEPTS

- **Fraction:** A number of the form $\frac{a}{b}$ that conveys a quantity obtained by splitting a unit into b equal parts, and then taking a of those parts.

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| | |
|--|---|
| <ul style="list-style-type: none"> • Colored regions, or area model: The unit is a geometric figure such as a polygon or disc, sectioned into b parts of equal area, of which a parts are colored to represent the fraction $\frac{a}{b}$. | 266 |
| <ul style="list-style-type: none"> • Fraction strip: The unit is a rectangle partitioned by vertical lines that partition it into b identical subrectangles, of which a are shaded to represent $\frac{a}{b}$. | 266 |
| <ul style="list-style-type: none"> • Fraction circle: The unit is a circle partitioned into b congruent sectors, so that shading a of the sectors gives a representation of the fraction $\frac{a}{b}$. | 266 |
| <ul style="list-style-type: none"> • Set model: The unit is a set of b elements, so that a subset of a elements represents the fraction $\frac{a}{b}$. | 267 |
| <ul style="list-style-type: none"> • Number line: Each unit interval of the integer number line is partitioned into b segments of the same length, so that the fraction $\frac{a}{b}$ is the length of a jump by a of these segments. Moreover, the position of the point at the distance of this jump from 0 is also a representation of $\frac{a}{b}$. | 267 |
| <ul style="list-style-type: none"> • Strategies for ordering fractions: Look for a common denominator. | 273 |
| DEFINITIONS <ul style="list-style-type: none"> • A fraction is a pair of integers a and b, $b \neq 0$, written a/b or $\frac{a}{b}$. • In the expression $\frac{a}{b}$, a is the numerator and b is the denominator. • Equivalent fractions are fractions that express the same quantity. • A fraction $\frac{a}{b}$ is in simplest form if a and b have no common divisor larger than 1 and b is positive. • Fractions with the same denominator are said to have a common denominator. • The least common denominator is the least common multiple of the denominators of two (or more) fractions. • A rational number is any number that can be represented by a fraction or any of its equivalent fractions. | 265 265 268 270 271 271 272 |
| THEOREM <ul style="list-style-type: none"> • Cross-product property: The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. | 269 |
| PROPERTY <ul style="list-style-type: none"> • Fundamental law of fractions: Let $\frac{a}{b}$ be a fraction. Then $\frac{a}{b} = \frac{an}{bn}$, for any integer $n \neq 0$. | 268 |
| NOTATION <ul style="list-style-type: none"> • Set of rational numbers, Q: The set of all numbers that can be expressed as the fraction a/b where a and b are integers and b is not zero. | 272 |

| Section 6.2 Addition and Subtraction of Fractions | Page Reference |
|--|---|
| <p>DEFINITIONS</p> <ul style="list-style-type: none"> • Addition of fractions: For fractions $\frac{a}{b}$ and $\frac{c}{d}$ with the common denominator b, let $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. For fractions $\frac{a}{b}$ and $\frac{c}{d}$ with unlike denominators b and d, use a common denominator such as bd so that $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$. • Subtraction of fractions, $\frac{a}{b} - \frac{c}{d}$: Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. Then $\frac{a}{b} - \frac{c}{d} = \frac{e}{f}$ if and only if $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$. This implies the formula $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. • A mixed number is the sum of a natural number and a positive fraction. For example, $2\frac{3}{4}$ is a mixed number. • A proper fraction is a fraction a/b for which $0 \leq a < b$. | <p>282</p> <p>287</p> <p>284</p> <p>285</p> |
| Section 6.3 Multiplication and Division of Fractions | Page Reference |
| <p>CONCEPTS</p> <ul style="list-style-type: none"> • Multiplication: Extending the area model of multiplication motivates the definition of multiplication of fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. • Division: Given fractions $\frac{a}{b}$ and $\frac{c}{d} \neq 0$, the division $\frac{a}{b} \div \frac{c}{d}$ represents: <ul style="list-style-type: none"> (a) the fractional number of times $\frac{c}{d}$ can be removed from $\frac{a}{b}$ (repeated-subtraction, or grouping, model) (b) the fractional amount per unit when the quantity $\frac{a}{b}$ is partitioned into $\frac{c}{d}$ parts (sharing, or partitive, model) (c) the multiplicative factor $x = \frac{m}{n}$ for which $\frac{a}{b} = x \cdot \frac{c}{d}$ (multiplicative model) | <p>291</p> <p>296</p> |
| <p>DEFINITIONS</p> <ul style="list-style-type: none"> • Multiplication of fractions, $\frac{a}{b} \times \frac{c}{d}$: Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. Then their product is given by $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. • Division of fractions, $\frac{a}{b} \div \frac{c}{d}$: Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions, where $\frac{c}{d}$ is not zero. Then $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$ if and only if $\frac{a}{b} = \frac{m}{n} \cdot \frac{c}{d}$. • The reciprocal, or multiplication inverse, of a nonzero fraction $\frac{c}{d}$ is the fraction $\frac{d}{c}$, so that $\frac{c}{d} \cdot \frac{d}{c} = 1$. | <p>294</p> <p>297</p> <p>300</p> |

FORMULA

- **Invert-and-multiply algorithm for division:** The division $\frac{a}{b} \div \frac{c}{d}$ is equivalent to the product $\frac{a}{b} \cdot \frac{d}{c}$ obtained by multiplying $\frac{a}{b}$ by the reciprocal of the divisor $\frac{c}{d}$.
That is, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

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Section 6.4 The Rational Number System**Page Reference****CONCEPT**

- **Rational numbers:** The system of numbers represented by fractions, so that the arithmetic operations (addition, subtraction, multiplication, division), properties, and order relations (less than and greater than) correspond to the same operations, properties, and order relations for fractions. A rational number is represented by any choice of the equivalent fractions corresponding to the rational numbers.

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DEFINITION

- The **negative**, or **additive inverse**, of rational number $\frac{a}{b}$ is written $-\frac{a}{b}$ and is the rational number corresponding to the fraction $\frac{(-a)}{b}$.

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THEOREMS

- **Properties of addition of rational numbers:** Given three rational numbers r , s , and t , addition is closed ($r + s$ is a rational number), commutative ($rs = sr$), associative ($r + (s + t) = (r + s) + t$), 0 is the additive identity ($r + 0 = 0 + r = r$), and $-r$ is the additive inverse ($r + (-r) = (-r) + r = 0$, where $-r = -\frac{a}{b}$ if $r = \frac{a}{b}$).
- **Subtraction is equivalent to the addition of the negative:**
$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right) = \frac{ad - bc}{bd}.$$
- **Properties of multiplication of rational numbers:** Given three rational numbers r , s , and t , multiplication is closed ($r \cdot s$ is a rational number), commutative ($r \cdot s = s \cdot r$), associative ($r \cdot (s \cdot t) = (r \cdot s) \cdot t$), and distributive over addition ($r \cdot (s + t) = r \cdot s + r \cdot t$), has the multiplicative identity 1 ($r \cdot 1 = 1 \cdot r = r$), and has the multiplication by 0 property ($r \cdot 0 = 0 \cdot r = 0$). Moreover, each nonzero rational number $r = \frac{a}{b} \neq 0$ has the multiplicative inverse r^{-1} given by its reciprocal $\frac{b}{a}$, so that $r \cdot r^{-1} = \frac{a}{b} \cdot \frac{b}{a} = 1$.
- **Properties of the order relation on rational numbers:** Given three rational numbers r , s , and t , the order relation $<$ is transitive (if $r < s$ and $s < t$, then $r < t$), has the addition property (if $r < s$, then $r + t < s + t$), has the multiplication by a positive property (if $r < s$ and $t > 0$, then $r \cdot t < s \cdot t$), and satisfies the law of trichotomy (exactly one of the relations $r < s$, $r = s$, $r > s$ holds).
- **Density property:** Given two different rational numbers r , s , with $r < s$, there is a rational number t between r and s . Indeed, there are infinitely many rational numbers t for which $r < t < s$.

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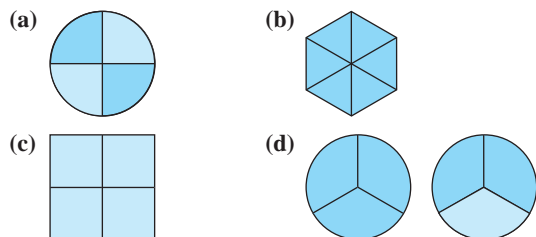
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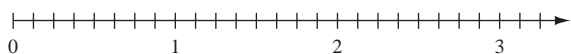
Chapter Review Exercises

Section 6.1

1. What fraction is represented by the darker blue shading in each of the colored-region models shown? In (a) and (d), the unit is the region inside one circle. In (b) and (c), the units are the regions inside the hexagon and square, respectively.



2. Illustrate $\frac{2}{3}$
- on the number line,
 - with a fraction strip,
 - with a colored-region model, and
 - with the set model.
3. Give three different fractions, each equivalent to $-\frac{3}{4}$.
4. Label the points on the number line that correspond to these rational numbers:
- $\frac{3}{4}$
 - $\frac{12}{8}$
 - 1
 - $2\frac{3}{8}$



5. Express each rational number by a fraction in simplest form.

- $\frac{27}{81}$
- $\frac{100}{825}$
- $\frac{378}{72}$
- $\frac{3^5 \cdot 7^2 \cdot 11^3}{3^2 \cdot 7^3 \cdot 11^2}$

6. Order these fractions from smallest to largest:

$$\frac{1}{2}, \frac{13}{27}, \frac{25}{49}, \frac{13}{30}, \frac{26}{49}$$

7. Find the least common denominator of each set of fractions.

- $\frac{4}{9}, \frac{5}{12}$
- $\frac{7}{18}, \frac{5}{6}, \frac{1}{3}$

Section 6.2

8. Illustrate $\frac{3}{4} + \frac{7}{8}$ on the number line.
9. Illustrate $\frac{3}{4} - \frac{1}{3}$ with fraction strips.
10. Compute these sums and differences:

- $\frac{3}{8} + \frac{1}{4}$
- $\frac{2}{9} + \frac{-5}{12}$

$$(c) \frac{4}{5} - \frac{2}{3}$$

$$(d) 5\frac{1}{4} - 1\frac{5}{6}$$

11. Perform these calculations:

$$(a) \frac{1}{3} + \frac{5}{8} - \frac{5}{6}$$

$$(b) \left(\frac{2}{3} - \frac{5}{4}\right) \div \frac{3}{4}$$

$$(c) \frac{4}{7} \cdot \left(\frac{35}{4} + \frac{-42}{12}\right)$$

$$(d) \frac{123}{369} \div \frac{1}{3}$$

Section 6.3

12. Illustrate these products by labeling and coloring appropriate rectangular regions:

$$(a) 3 \times \frac{1}{3}$$

$$(b) \frac{2}{3} \times 4$$

$$(c) \frac{5}{6} \times \frac{3}{2}$$

13. Gina, Hank, and Igor want to share $2\frac{1}{4}$ pizzas equally. Draw an appropriate diagram to show how much pizza each should be given. What conceptual model of division should you use?

14. Each quart of soup calls for $\frac{2}{3}$ of a cup of pinto beans. How many quarts of soup can be made with 3 cups of beans? Draw an appropriate diagram to find your answer. What conceptual model of division are you using?

15. On a map, it is $7\frac{1}{8}$ miles from Arlington to Banks. If the scale of the map is $1\frac{1}{4}$ inches per mile, how far is it between the two towns?

Section 6.4

16. Perform these calculations, expressing your answers in simplest form:

$$(a) \frac{-3}{4} + \frac{5}{8}$$

$$(b) \frac{4}{5} - \frac{-7}{10}$$

$$(c) \left(\frac{3}{8} \cdot \frac{-4}{27}\right) \div \frac{1}{9}$$

$$(d) \frac{2}{5} \cdot \left(\frac{3}{4} - \frac{5}{2}\right)$$

17. Solve each equation. Give the rational number x as a fraction in simplest form.

$$(a) 3x + 5 = 11$$

$$(b) x + \frac{2}{3} = \frac{1}{2}$$

$$(c) \frac{3}{5}x + \frac{1}{2} = \frac{2}{3}$$

$$(d) -\frac{4}{3}x + 1 = \frac{1}{4}$$

18. Solve the given equations and inequalities for all possible rational numbers x . Show all of your steps.

$$(a) 2x + 3 > 0$$

$$(b) \frac{5}{4}x > -\frac{1}{3}$$

$$(c) \frac{1}{2} < 4x + \frac{5}{6}$$

19. Find two rational numbers between $\frac{5}{6}$ and $\frac{10}{11}$.