Research Methods in Mathematics
Lecture 4: Rational numbers and upper bounds

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Positivity

Last time we introduced the integers. Now we can talk about positivity.

Lemma 1 *Let a be an integer. Then exactly one of the following three possibilities holds: $a = 0$; there is a natural number n so that $a = n - 0$; or there is a natural number n so that $a = 0 - n$.*

Proof □

We define an integer a to be *zero*, *positive* or *negative* according to which possibility holds. We write $a = 0$, $a > 0$ or $a < 0$ respectively. We then have the following three assertions.

✠ THREE PROPERTIES OF POSITIVITY

For any a, b in $\mathbb{Z}$,

Trichotomy:	Exactly one of the following holds: $a > 0$; $a = 0$; $a < 0$.
Closure under $+$:	If $a > 0$ and $b > 0$ then $a + b > 0$.
Closure under $\times$:	If $a > 0$ and $b > 0$ then $ab > 0$.

The rational numbers

Rational numbers fractions p/q where p and q are integers and $q \neq 0$. Collectively they are denoted $\mathbb{Q}$ (for quotient, which means ratio).

One can add rationals: $p/q + r/s = (ps + qr)/(qs)$. One can multiply them: $(p/q)(r/s) = (pr)/(qs)$. The eight properties $\triangleright$ and † still hold. So does

⊛ For every non-zero rational number x there is a rational number y so that $xy = 1$.

We denote this y (which is unique!) by x^{-1} or $1/x$.

One can define positive rational numbers x as those represented by p/q with $p > 0$ and $q > 0$. Then the three properties ✠ hold.

As with the integers, one can ask how we should actually *define* the rational numbers. The same trick as we used to define integers also works for rational numbers. We say that every rational number is represented by an ordered pair (p,q) of integers, with $q \neq 0$. We write this pair as p/q. We declare that $p/q = p'/q'$ if there exists an integer m such that $p = mp'$ and $q = mq'$; or such that $p' = mp$ and $q' = mq$. One then verifies that all the claimed properties really hold.

Inequalities

The absolute value of a rational number x is denoted by $|x|$, where this symbol is defined as follows:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Notice that $|-x| = |x|$ and that $|x-y| > 0$ unless $x = y$. Informally, one can think of $|x-y|$ as the 'distance' from x to y.

Theorem 2 *For any rational numbers x and y, we have*

$$|x+y| \leq |x| + |y|$$

and

$$|x-y| \leq |x| + |y|.$$

Proof □

Upper bounds

Suppose that R is a number system satisfying all the axioms we enumerated: $\star$, $\dagger$, $\circledast$ and ✠. The only example we know about so far is $R = \mathbb{Q}$, the rational numbers. We can write inequalities $b > a$ in this number system, and we can also write $b \geq a$ to mean that either $b > a$ or $b = a$.

Definition 3 If A is a set of numbers in R, an *upper bound* for A is a number x such that $x \geq a$ for all $a \in A$. We say A is *bounded above* if an upper bound exists. Similarly, a *lower* bound for A is a number y such that $y \leq a$ for all $a \in A$. We say A is *bounded below* if a lower bound exists.

Example 4 Let $R = \mathbb{Q}$. An upper bound for the set $A = [0,1]$ is $x = 2$; another is $x = 1$. If $A = \{1, 2, 3, \dots\}$ then A is bounded below but not above.

Definition 5 A *least upper bound* or *supremum* for A is a number x in R such that (i) x is an upper bound for A; and (ii) if x' is another upper bound for A then $x' \geq x$. If a supremum exists, it is denoted by $\sup A$.

A *greatest lower bound* or *infimum* for A is a number y in R such that (ii) y is a lower bound for A; and (ii) if y' is another lower bound for A then $y' \leq y$. If an infimum exists, it is denoted by $\inf A$.

Lemma 6 *If x and x' are both least upper bounds for A then $x = x'$. Similarly for greatest lower bounds.*

Proof □

This lemma justifies the notation $\sup A$ for 'the' supremum of A and $\inf A$ for 'the' infimum of A.

Example 7 Let $R = \mathbb{Q}$. Then

$$\sup[0,1] = \sup(0,1) = 1;$$
$$\inf[0,1] = \inf(0,1) = 0;$$
$$\sup\{-1/2, -1/3, -1/4, \dots\} = 0;$$
$$\inf\{-1/2, -1/3, -1/4, \dots\} = -1/2.$$

An 'obvious' lemma:

Lemma 8 *If $a > b$ and $b > 0$ then $a^2 > b^2$.*

Proof □

Proposition 9 *Let $R = \mathbb{Q}$. Let $A = \{a \in \mathbb{Q} : a^2 < 2\}$. Then A has an upper bound but no least upper bound.*

Proof □

Spivak reference: chapters 1–2.