

27 Rational Numbers

Integers such as -5 were important when solving the equation $x + 5 = 0$. In a similar way, fractions are important for solving equations like $2x = 1$. What about equations like $2x + 1 = 0$? Equations of this type require numbers like $-\frac{1}{2}$. In general, numbers of the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$ are solutions to the equation $bx = a$. The set of all such numbers is the set of **rational numbers**, denoted by \mathbb{Q} :

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

That is, the set of rational numbers consists of all fractions with their opposites. In the notation $\frac{a}{b}$ we call a the **numerator** and b the **denominator**. Note that every fraction is a rational number. Also, every integer is a rational number for if a is an integer then we can write $a = \frac{a}{1}$. Thus, $\mathbb{Z} \subset \mathbb{Q}$.

Example 27.1

Draw a Venn diagram to show the relationship between counting numbers, whole numbers, integers, and rational numbers.

Solution.

The relationship is shown in Figure 27.1 ■

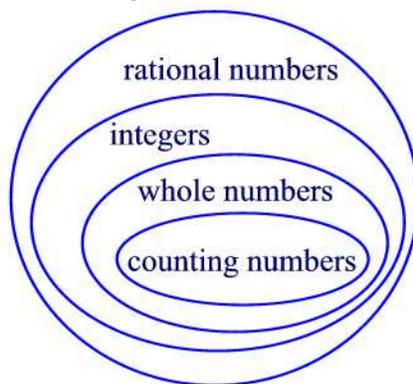


Figure 27.1

All properties that hold for fractions apply as well for rational numbers.

Equality of Rational Numbers: Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two rational numbers. Then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ (Cross-multiplication).

Example 27.2

Determine if the following pairs are equal.

- (a) $\frac{3}{-12}$ and $\frac{-36}{144}$.
 (b) $\frac{-21}{86}$ and $\frac{-51}{215}$.

Solution.

(a) Since $3(144) = (-12)(-36)$ then $\frac{3}{-12} = \frac{-36}{144}$.

(b) Since $(-21)(215) \neq (86)(-51)$ then $\frac{-21}{86} \neq \frac{-51}{215}$ ■

The Fundamental Law of Fractions: Let $\frac{a}{b}$ be any rational number and n be a nonzero integer then

$$\frac{a}{b} = \frac{an}{bn} = \frac{a \div n}{b \div n}.$$

As an important application of the Fundamental Law of Fractions we have

$$\frac{a}{-b} = \frac{(-1)a}{(-1)(-b)} = \frac{-a}{b}.$$

We also use the notation $-\frac{a}{b}$ for either $\frac{a}{-b}$ or $\frac{-a}{b}$.

Example 27.3

Write three rational numbers equal to $-\frac{2}{5}$.

Solution.

By the Fundamental Law of Fractions we have

$$-\frac{2}{5} = \frac{4}{-10} = -\frac{6}{15} = \frac{-8}{20}$$
 ■

Rational Numbers in Simplest Form: A rational number $\frac{a}{b}$ is in **simplest form** if a and b have no common factors greater than 1.

The methods of reducing fractions into simplest form apply as well with rational numbers.

Example 27.4

Find the simplest form of the rational number $\frac{294}{-84}$.

Solution.

Using the prime factorizations of 294 and 84 we find

$$\frac{294}{-84} = \frac{2 \cdot 3 \cdot 7^2}{(-2) \cdot 2 \cdot 3 \cdot 7} = \frac{7}{-2} = \frac{-7}{2}$$
 ■

Practice Problems

Problem 27.1

Show that each of the following numbers is a rational number.

- (a) -3 (b) $4\frac{1}{2}$ (c) -5.6 (d) 25%

Problem 27.2

Which of the following are equal to -3 ?

$$\frac{-3}{1}, \frac{3}{-1}, \frac{3}{1}, -\frac{3}{1}, \frac{-3}{-1}, -\frac{-3}{1}, -\frac{-3}{-1}.$$

Problem 27.3

Determine which of the following pairs of rational numbers are equal.

- (a) $\frac{-3}{5}$ and $\frac{63}{-105}$
(b) $\frac{-18}{-24}$ and $\frac{45}{60}$.

Problem 27.4

Rewrite each of the following rational numbers in simplest form.

- (a) $\frac{5}{-7}$ (b) $\frac{21}{-35}$ (c) $\frac{-8}{-20}$ (d) $\frac{-144}{180}$

Problem 27.5

How many different rational numbers are given in the following list?

$$\frac{2}{5}, 3, \frac{-4}{-10}, \frac{39}{13}, \frac{7}{4}$$

Problem 27.6

Find the value of x to make the statement a true one.

- (a) $\frac{-7}{25} = \frac{x}{500}$ (b) $\frac{18}{3} = \frac{-5}{x}$

Problem 27.7

Find the prime factorizations of the numerator and the denominator and use them to express the fraction $\frac{247}{-77}$ in simplest form.

Problem 27.8

- (a) If $\frac{a}{b} = \frac{a}{c}$, what must be true?
(b) If $\frac{a}{c} = \frac{b}{c}$, what must be true?

Addition of Rational Numbers

The definition of adding fractions extends to rational numbers.

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Example 27.5

Find each of the following sums.

(a) $\frac{2}{-3} + \frac{1}{5}$ (b) $\frac{-2}{5} + \frac{4}{-7}$ (c) $\frac{3}{7} + \frac{-5}{7}$

Solution.

(a) $\frac{2}{-3} + \frac{1}{5} = \frac{2 \cdot (-5)}{(-3) \cdot (-5)} + \frac{1 \cdot 3}{5 \cdot 3} = \frac{-10}{15} + \frac{3}{15} = \frac{(-10)+3}{15} = \frac{-7}{15}$
(b) $\frac{-2}{5} + \frac{4}{-7} = \frac{(-2) \cdot 7}{5 \cdot 7} + \frac{4 \cdot (-5)}{(-5) \cdot (-7)} = \frac{-14}{35} + \frac{-20}{35} = \frac{(-14)+(-20)}{35} = \frac{-34}{35}$
(c) $\frac{3}{7} + \frac{-5}{7} = \frac{3+(-5)}{7} = \frac{-2}{7}$ ■

Rational numbers have the following properties for addition.

Theorem 27.1

Closure: $\frac{a}{b} + \frac{c}{d}$ is a unique rational number.

Commutative: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Associative: $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$

Identity Element: $\frac{a}{b} + 0 = \frac{a}{b}$

Additive inverse: $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$

Example 27.6

Find the additive inverse for each of the following:

(a) $\frac{3}{5}$ (b) $\frac{-5}{11}$ (c) $\frac{2}{-3}$ (d) $-\frac{2}{5}$

Solution.

(a) $-\frac{3}{5} = \frac{-3}{5} = \frac{3}{-5}$

(b) $\frac{5}{11}$

(c) $\frac{2}{3}$

(d) $\frac{2}{5}$ ■

Subtraction of Rational Numbers

Subtraction of rational numbers like subtraction of fractions can be defined in terms of addition as follows.

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right).$$

Using the above result we obtain the following:

$$\begin{aligned}\frac{a}{b} - \frac{c}{d} &= \frac{a}{b} + \left(-\frac{c}{d}\right) \\ &= \frac{a}{b} + \frac{-c}{d} \\ &= \frac{ad+b(-c)}{bd} \\ &= \frac{ad-bc}{bd}\end{aligned}$$

Example 27.7

Compute $\frac{103}{24} - \frac{-35}{16}$.

Solution.

$$\begin{aligned}\frac{103}{24} - \frac{-35}{16} &= \frac{206}{48} - \frac{-105}{48} \\ &= \frac{206 - (-105)}{48} \\ &= \frac{206 + 105}{48} = \frac{311}{48} \blacksquare\end{aligned}$$

Practice Problems

Problem 27.9

Use number line model to illustrate each of the following sums.

(a) $\frac{3}{4} + \frac{-2}{4}$ (b) $\frac{-3}{4} + \frac{2}{4}$ (c) $\frac{-3}{4} + \frac{-1}{4}$

Problem 27.10

Perform the following additions. Express your answer in simplest form.

(a) $\frac{6}{8} - \frac{-25}{100}$ (b) $\frac{-57}{100} + \frac{13}{10}$

Problem 27.11

Perform the following subtractions. Express your answer in simplest form.

(a) $\frac{137}{214} - \frac{-1}{3}$ (b) $\frac{-23}{100} - \frac{198}{1000}$

Problem 27.12

Compute the following differences.

(a) $\frac{2}{3} - \frac{-9}{8}$ (b) $(-2\frac{1}{4}) - 4\frac{2}{3}$

Multiplication of Rational Numbers

The multiplication of fractions is extended to rational numbers. That is, if $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Multiplication of rational numbers has properties analogous to the properties of multiplication of fractions. These properties are summarized in the following theorem.

Theorem 27.2

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then we have the following:

Closure: The product of two rational numbers is a unique rational number.

Commutativity: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.

Associativity: $\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$.

Identity: $\frac{a}{b} \cdot 1 = \frac{a}{b} = 1 \cdot \frac{a}{b}$.

Inverse: $\frac{a}{b} \cdot \frac{b}{a} = 1$. We call $\frac{b}{a}$ the **reciprocal** of $\frac{a}{b}$ or the **multiplicative inverse** of $\frac{a}{b}$.

Distributivity: $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$.

Example 27.8

Perform each of the following multiplications. Express your answer in simplest form.

(a) $\frac{-5}{6} \cdot \frac{7}{3}$ (b) $\frac{-3}{10} \cdot \frac{-25}{27}$

Solution.

(a) We have

$$\frac{-5}{6} \cdot \frac{7}{3} = \frac{(-5) \cdot 7}{6 \cdot 3} = \frac{-35}{18}.$$

(b)

$$\frac{-3}{10} \cdot \frac{-25}{27} = \frac{-1}{2} \cdot \frac{-5}{9} = \frac{(-1)(-5)}{2(9)} = \frac{5}{18} \blacksquare$$

Example 27.9

Use the properties of multiplication of rational numbers to compute the following.

- (a) $-\frac{3}{5} \cdot \left(\frac{11}{17} \cdot \frac{5}{3}\right)$
 (b) $\frac{2}{3} \cdot \left(\frac{3}{2} + \frac{5}{7}\right)$
 (c) $\frac{5}{9} \cdot \frac{2}{7} + \frac{2}{7} \cdot \frac{4}{9}$

Solution.

$$(a) -\frac{3}{5} \cdot \left(\frac{11}{17} \cdot \frac{5}{3}\right) = \frac{-3}{5} \cdot \frac{11 \cdot 5}{17 \cdot 3} = \frac{-3}{5} \cdot \frac{55}{51} = \frac{-1}{1} \cdot \frac{11}{17} = \frac{-11}{17}$$

$$(b) \frac{2}{3} \cdot \left(\frac{3}{2} + \frac{5}{7}\right) = \frac{2}{3} \cdot \frac{31}{14} = \frac{1}{3} \cdot \frac{31}{7} = \frac{31}{21}$$

$$(c) \frac{5}{9} \cdot \frac{2}{7} + \frac{2}{7} \cdot \frac{4}{9} = \frac{5}{9} \cdot \frac{2}{7} + \frac{4}{9} \cdot \frac{2}{7} = \left(\frac{5}{9} + \frac{4}{9}\right) \cdot \frac{2}{7} = \frac{2}{7} \blacksquare$$

Division of Rational Numbers

We define the division of rational numbers as an extension of the division of fractions. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers with $\frac{c}{d} \neq 0$. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Using words, to find $\frac{a}{b} \div \frac{c}{d}$ multiply $\frac{a}{b}$ by the reciprocal of $\frac{c}{d}$.
 By the above definition one gets the following two results.

$$\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$$

and

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}.$$

Remark 27.1

After inverting, it is often simplest to "cancel" before doing the multiplication. Cancelling is dividing one factor of the numerator and one factor of the denominator by the same number. For example: $\frac{2}{9} \div \frac{3}{12} = \frac{2}{9} \times \frac{12}{3} = \frac{2 \times 12}{9 \times 3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}$.

Remark 27.2

Exponents and their properties are extended to rational numbers in a natural way. For example, if a is any rational number and n is a positive integer then

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}} \quad \text{and} \quad a^{-n} = \frac{1}{a^n}$$

Example 27.10

Compute the following and express the answers in simplest form.

$$(a) \frac{-7}{4} \div \frac{2}{3} \quad (b) \frac{13}{17} \cdot \frac{-4}{9} \quad (c) \frac{-18}{23} \cdot \frac{-6}{23}$$

Solution.

$$(a) \frac{-7}{4} \div \frac{2}{3} = \frac{-7}{4} \cdot \frac{3}{2} = \frac{(-7)(3)}{(4)(2)} = \frac{-21}{8}.$$

$$(b) \frac{13}{17} \cdot \frac{-4}{9} = \frac{13}{17} \cdot \frac{9}{-4} = \frac{13 \cdot 9}{17 \cdot (-4)} = \frac{-117}{68}.$$

$$(c) \frac{-18}{23} \cdot \frac{-6}{23} = \frac{-18}{23} \cdot \frac{23}{-6} = \frac{3}{1} \cdot \frac{1}{1} = 3 \blacksquare$$

Practice Problems**Problem 27.13**

Multiply the following rational numbers. Write your answers in simplest form.

$$(a) \frac{3}{5} \cdot \frac{-10}{21} \quad (b) \frac{-6}{11} \cdot \frac{-33}{18} \quad (c) \frac{5}{12} \cdot \frac{48}{-15} \cdot \frac{-9}{8}$$

Problem 27.14

Find the following quotients. Write your answers in simplest form.

$$(a) \frac{-8}{9} \div \frac{2}{9} \quad (b) \frac{12}{15} \div \frac{-4}{3} \quad (c) \frac{-13}{24} \div \frac{-39}{-48}$$

Problem 27.15

State the property that justifies each statement.

$$(a) \left(\frac{5}{7} \cdot \frac{7}{8}\right) \cdot \frac{-8}{3} = \frac{5}{7} \cdot \left(\frac{7}{8} \cdot \frac{-8}{3}\right)$$

$$(b) \frac{1}{4} \left(\frac{8}{3} + \frac{-5}{4}\right) = \frac{1}{4} \cdot \frac{8}{3} + \frac{1}{4} \cdot \frac{-5}{4}$$

Problem 27.16

Compute the following and write your answers in simplest form.

$$(a) \frac{-40}{27} \div \frac{-10}{9} \quad (b) \frac{21}{25} \div \frac{-3}{5} \quad (c) \frac{-10}{9} \div \frac{-9}{8}$$

Problem 27.17

Find the reciprocals of the following rational numbers.

$$(a) \frac{4}{-9} \quad (b) 0 \quad (c) \frac{-3}{2} \quad (d) \frac{-4}{-9}$$

Problem 27.18

Compute: $\left(\frac{-4}{7} \cdot \frac{2}{-5}\right) \div \frac{2}{-7}$.

Problem 27.19

If $\frac{a}{b} \cdot \frac{-4}{7} = \frac{2}{3}$ what is $\frac{a}{b}$?

Problem 27.20

Compute $-4\frac{1}{2} \times -5\frac{2}{3}$

Problem 27.21

Compute $-17\frac{8}{9} \div 5\frac{10}{11}$

Problem 27.22

Compute each of the following:

(a) $-\left(\frac{3}{4}\right)^2$ (b) $\left(-\frac{3}{4}\right)^2$ (c) $\left(\frac{3}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^7$

Comparing and Ordering Rational Numbers

In this section we extend the notion of "less than" to the set of all rationals. We describe two equivalent ways for viewing the meaning of less than: a number line approach and an addition (or algebraic) approach. In what follows, $\frac{a}{b}$ and $\frac{c}{d}$ denote any two rationals.

Number-Line Approach

We say that $\frac{a}{b}$ is **less than** $\frac{c}{d}$, and we write $\frac{a}{b} < \frac{c}{d}$, if the point representing $\frac{a}{b}$ on the number-line is to the left of $\frac{c}{d}$. For example, Figure 27.2 shows that $\frac{1}{2} < \frac{2}{3}$.



Figure 27.2

Example 27.11

Use the number line approach to order the pair of numbers $\frac{3}{7}$ and $\frac{5}{2}$.

Solution.

When the two numbers have unlike denominators then we find the least common denominator and then we order the numbers. Thus, $\frac{3}{7} = \frac{6}{14}$ and $\frac{5}{2} = \frac{35}{14}$.

Hence, on a number line, $\frac{3}{7}$ is to the left of $\frac{5}{2}$. ■

Addition Approach

As in the case of ordering integers, we say that $\frac{a}{b} < \frac{c}{d}$ if there is a unique fraction $\frac{e}{f}$ such that $\frac{a}{b} + \frac{e}{f} = \frac{c}{d}$.

Example 27.12

Use the addition approach to show that $\frac{-3}{7} < \frac{5}{2}$.

Solution.

Since $\frac{5}{2} = \frac{-3}{7} + \frac{41}{14}$ then $\frac{-3}{7} < \frac{5}{2}$. ■

Notions similar to less than are included in the following table.

Inequality Symbol	Meaning
$<$	less than
$>$	greater than
\leq	less than or equal
\geq	greater than or equal

The following rules are valid for any of the inequality listed in the above table.

Rules for Inequalities

• **Trichotomy Law:** For any rationals $\frac{a}{b}$ and $\frac{c}{d}$ exactly one of the following is true:

$$\frac{a}{b} < \frac{c}{d}, \frac{a}{b} > \frac{c}{d}, \frac{a}{b} = \frac{c}{d}.$$

• **Transitivity:** For any rationals $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ if $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$ then $\frac{a}{b} < \frac{e}{f}$.

• **Addition Property:** For any rationals $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} + \frac{e}{f} < \frac{c}{d} + \frac{e}{f}$.

• **Multiplication Property:** For any rationals $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}$ if $\frac{e}{f} > 0$ and $\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}$ if $\frac{e}{f} < 0$.

• **Density Property:** For any rationals $\frac{a}{b}$ and $\frac{c}{d}$, if $\frac{a}{b} < \frac{c}{d}$ then

$$\frac{a}{b} < \frac{1}{2} \left(\frac{a}{b} + \frac{c}{d} \right) < \frac{c}{d}.$$

Practice Problems

Problem 27.23

True or false: (a) $\frac{-2}{3} < \frac{-3}{7}$ (b) $\frac{15}{-9} > \frac{-13}{4}$.

Problem 27.24

Show that $\frac{-3}{4} < \frac{-1}{4}$ using the addition approach.

Problem 27.25

Show that $\frac{-3}{4} < \frac{-1}{4}$ by using a number line.

Problem 27.26

Put the appropriate symbol, $<$, $=$, $>$ between each pair of numbers to make a true statement.

(a) $-\frac{5}{6}$ _____ $-\frac{11}{12}$

(b) $-\frac{1}{3}$ _____ $\frac{5}{4}$

(c) $\frac{-12}{15}$ _____ $\frac{36}{-45}$

(d) $-\frac{3}{12}$ _____ $\frac{-4}{20}$

Problem 27.27

Find three rational numbers between $\frac{1}{4}$ and $\frac{2}{5}$.

Problem 27.28

The properties of rational numbers are used to solve inequalities. For example,

$$\begin{aligned} x + \frac{3}{5} &< \frac{-7}{10} \\ x + \frac{3}{5} + \left(-\frac{3}{5}\right) &< \frac{-7}{10} + \left(-\frac{3}{5}\right) \\ x &< -\frac{13}{10} \end{aligned}$$

Solve the inequality

$$-\frac{2}{5}x + \frac{1}{5} > -1.$$

Problem 27.29

Solve each of the following inequalities.

(a) $x - \frac{6}{5} < \frac{-12}{7}$

(b) $\frac{2}{5}x < -\frac{7}{8}$

(c) $\frac{-3}{7}x > \frac{8}{5}$

Problem 27.30

Verify the following inequalities.

(a) $\frac{-4}{5} < \frac{-3}{4}$ (b) $\frac{1}{10} < \frac{1}{4}$ (c) $\frac{19}{-60} > \frac{-1}{3}$

Problem 27.31

Use the number-line approach to arrange the following rational numbers in increasing order:

(a) $\frac{4}{5}, -\frac{1}{5}, \frac{2}{5}$
(b) $\frac{-7}{12}, \frac{-2}{3}, \frac{3}{-4}$

Problem 27.32

Find a rational number between $\frac{5}{12}$ and $\frac{3}{8}$.

Problem 27.33

Complete the following, and name the property that is used as a justification.

(a) If $\frac{-2}{3} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{5}$ then $\frac{-2}{3}$ _____ $\frac{7}{5}$.

(b) If $\frac{-3}{5} < \frac{-6}{11}$ then $(\frac{-3}{5}) \cdot (\frac{2}{3})$ _____ $(\frac{-6}{11}) \cdot (\frac{2}{3})$

(c) If $\frac{-4}{7} < \frac{7}{4}$ then $\frac{-4}{7} + \frac{5}{8} < \frac{7}{4} +$ _____

(d) If $\frac{-3}{4} > \frac{11}{3}$ then $(\frac{-3}{4}) \cdot (\frac{-5}{7})$ _____ $(\frac{11}{3}) \cdot (\frac{-5}{7})$

(e) There is a rational number _____ any two unequal rational numbers.