The individualization of rational numbers discursive routines

Aya Steiner

The individualization of rational numbers discursive routines

Aya Steiner

Haifa University, Faculty of Education, Israel; ayasteiner@gmail.com

The research presented deals with the process of learning rational numbers. Assuming that at the end of a successful learning process the formal routines taught at the school become useful for practical activities, we have documented the activities of children from different grade levels regarding school assignments and daily practical tasks. We document the participants' activities from their entrance to the 1st grade, where they have no school experience with fractions, all the way through the sixth grade, where they are expected to arrive at the fully satisfactory mastery of rational numbers. In analyzing the data, we examined how the formal routines for performing certain fractions-related school assignments changed over time and how (if at all) they converge with routines for daily practical tasks. In this paper, we present findings from two school tasks: (1) locating a fraction on the number line, and (2) naming a point on the number line.

Keywords: Fraction, rational numbers, discourse, individualization.

Introduction and theoretical background

In the research presented in this paper we investigate the development of students' thinking about rational numbers. The research is part of my doctoral dissertation under the supervision of Anna Sfard. In spite of the abundance of former research on the topic, this investigation may be expected to make a novel contribution because of two features that set it apart from former ones: (1) the study follows the process of development rather than just snapshots of children's performance with rational numbers; (2) the study is grounded in a conceptual framework markedly different from those that have been guiding the majority of other investigations.

In the last quarter of the twentieth century, learning rational numbers has been one of the topics most vigorously studied by mathematics education researchers. Numerous studies recognized the complexity of the concept and agreed that rational numbers should be characterised as a set of related but distinct constructs rather than as a homogenous single one (Behr, Lesh, Post & Silver, 1983; Kieren, 1976; Rappaport, 1962). Although this model has been the point of departure for many projects, some researchers argued that the division into interpretations of the rational number is insufficient for describing children's construction of the concept (Olive & Lobato, 2008). Psychologists took the research in a different direction and focused on the mental operations involved in constructing knowledge of non-integer quantities (Confrey & Scarano, 1995). Nevertheless, only few of the studies offered a continuous picture of the learning of rational numbers. In the Second Handbook on Mathematic Teaching and Learning, Lamon (2007) writes: "Multiplicative ideas, in particular fractions, ratio and proportion, are difficult and develop over a long period of time. Brief teaching experiments have had disappointing results. There seems to be no substitute for longitudinal research" (p. 651). She also writes that such research should consider students' intuitive and experimental knowledge as well as their formal school knowledge. In this longitudinal study I interviewed 24 children, from a wide range of ages (6-13), for two years, looking at how their execution of an activity changed over time – before, during and after the formal learning of fractions.

By adopting a theoretical perspective that views individual learning as a collective endeavor (Vygotsky, 1978), I take yet another approach to the topic. According to the basic tenet of the proposed conceptualization, known as commognitive (Sfard, 2008), mathematics is a form of communication – a discourse. The discourse on rational numbers is a mathematical discourse dealing with the mathematical object *rational number*. A mathematical object is introduced in order to account for the equivalence of many *signifiers*, which are the terms (names) and symbols used for communication. The term "three quarters" and the symbols 0.75, 3/4 and 6/8 are declared to be signifiers of the mathematical object called rational number. The leading example of mathematical signifiers featured in this article are *fractions*, whereas the objects signified by fractions are *rational numbers*. The discourse of rational numbers is identifiable by four characteristic features: its **special words** and their use, its **visual mediators** and their use, the **narratives** that are endorsed by the discourse community, and **discursive routines**, which are patterns of actions a person tends to perform in response to a *task situation,* a situation in which she feels obliged to act. Given a task-situation, the decision about what it is that needs to be done is made by the performer based on precedents - past task-situations she considers as similar enough to the present one to justify repeating at least some of the things that were done then. The participant is not always aware of what and why she chooses for repetition, or that she is even repeating anything, in the first place. The repetition-requiring elements constitute the *task* the performer feels obliged to perform. We will now say that *routine performed in a given task-situation by a given person is the task the performer saw herself performing together with the procedure she executed to perform the task.*

The origins of the historical discourse of rational numbers go back to early human attempts to expand some everyday practical activities, such as those that require comparisons of continuous quantities – length, area, etc. The development of that discourse through human history involved assigning a name (signifier) to *germinal routines*, routines of practical activities that are likely to evoke the use of fractional words. While most of these germinal routines can also be successfully dealt with without any mathematical discourse, the use of fractions allows refinement of communication: it makes it more compact, more accurate or more widely applicable. For example the activity of sharing fairly a roll of fabric among four women can be communicated as: "each woman get one *quarter* of the roll". It also involves consolidations of different germinal routines into a single one. Such consolidation was due to the use of a signifier which is applicable to several hitherto unrelated types of activities. For example, the deed of sharing a roll of fabric among four women and the deed of dividing land among four heirs can both be described as "finding one fourth of the whole".

Today, it is through the process of learning that children gradually become participants of this historically established discourse. The way it happens is bound to deviate from the historical trajectory because the formal discourse has already been established and the child is not required to name signifiers on her own initiative or to formulate discursive patterns. Hence, learning rational numbers is the process of *individualization* of the formal discourse of rational numbers. It is the process at the end of which the learner fluently participates in the discourse, according to her needs. Individualization begins with the learner's exposure to new words and symbols embedded in such everyday expressions as "half an hour" or "quarter to six", and proceeds with the formal learning of rational numbers in school. Since rational number is not a physical object, it cannot be displayed in class. Instead, the child is introduced to the discourse of rational numbers as a formal language: she is presented with the signifiers within its typical discursive context, and the formal routines. According to the commognitive approach there is no other way to begin the process of individualizing a new routine than by adopting it as a *ritual*. The adoption of a routine begins with an imitation of an expert's moves (e.g., father, mother, and teacher) which is motivated by the child's social needs, meaning participation is ritual. Being unaware of the practical application of the outcome, the child would not recognize the outcome of such routine as the exclusive aim of the performance. Later, the child's ritual routine undergoes de-ritualization; there is a change of focus in the performance of the routine, from the procedure being performed to the desired result. Eventually it becomes *exploration*, that is, a routine whose focus is on the outcome and whose success is evaluated by answering the question of whether a new endorsed narrative has been produced (Sfard & Lavie, 2005).

This paper focuses on the individualization of two formal routines for manipulating rational numbers performed by Ada and Noa, two fourth graders who are new participants in the formal discourse of rational numbers. Through presenting the girls with the same two school-like tasks repeatedly, five times over two years, we describe the de-ritualization of these formal routines. Ada and Noa were randomly chosen. The way the girls acted appeared rather standard; what they did was similar to what was done by most other participants.

Research questions and method

The purpose of the study is to describe the process of individualization of formal discourse routines of rational numbers taught in school. In the part of the study presented we will describe one case of the individualization of two school routines: “locate a fraction on the number line (as a point)” and “name a point on the number line”. Hence we focus here on the questions:

1. How do these routines change throughout the school learning process?
2. Are they applicable in reproducing a point on the axis according to verbal instructions?

In order to answer these questions, I repeatedly interviewed 12 pairs of children from grades one to six (two pairs from each grade level) over two years. In Israeli schools, fractions are introduced in the third trimester of the third grade. The curriculum of grades 4, 5 and 6 was designed to gradually introduce fractions so that by the end of 6th grade, students would have satisfactory mastery of the fraction and its different interpretations. This is an ongoing study, in which interviews are conducted by introducing a predesigned assignments. The children are then engaged in the activity in order to accomplish the task. As a preliminary step to this project, we composed a battery of 28 assignments of two kinds: (i) school tasks; (ii) assignments which are meant to spur a performance of a germinal routine that occasions the use of rational numbers (such as finding parts and fair sharing of both discrete and continues quantities). This second group of assignments are meant to check the applicability of the formal routines. As we noted earlier in this paper, our aim is to characterize the way in which Ada and Noa individualized the formal routines of locating a fraction and naming a point on the number line (routines of lasting and naming, for short). We do so by looking at five consecutive interviews in which the girls repeatedly performed the respective school assignments and one practical activity. We labeled the interview as presented in Table 1.

Table 1. The five rounds of interviews (ITV) with Ada and Noa

Labeled	ITV1	ITV2	ITV3	ITV4	ITV5
Date	4.2015	12.2015	5.2016	11.2016	5.2017
Grade	4^{th}	5^{th}	5^{th}	6^{th} g	6^{th}

Ritual and exploration were defined by considering the task of the routine. We are dealing with a ritual if the routine is oriented exclusively at the process, that is, the task at hand – the set of elements of the precedent performances the performer regards as requiring repetition – is related to the process in its entirety. In other words, these repetition-requiring elements are the specific operations implemented in the precedent situation, not just their outcome. We are dealing with an exploration if only the outcome of precedent performances counts as important.

Table 2. Features of routine performances in a task-situation

aspect of routine R	*extremal types*		*examples of analysis-guiding questions about the available performances*
	Ritual	Exploration	
Agentivity	The performer, in a given type of task-situation, is always executing the procedure as it was in the precedent. She is not making independent decisions.	The procedure implemented by the performer required making independent decisions; in some performances, the performer was task-setter.	From one performance to another, was there any increase in the number of independent decisions made by the performer?
Boundedness	Different steps of the performance do not depend on one another, even if they should.	Outcome of one step in the performance feeds into the next one.	Does each step in the performance that should depend on the outcome of the former does, indeed, utilize that outcome?
Objectification	The talk is about processes; The focal signifiers, if used, are either stand-alone or appear as adjectives or adverbs.	The focal signifiers are used as a noun, a name of an independently existing object.	Was there an increase in the performer's use of the focal signifiers as nouns?
Flexibility	The performer, over time, uses the same single procedure in similar task-situations.	In a given type of task-situation, the performer uses a range of procedures, with their choice depending on the parameters of the situation (such as the numbers that were given)	Was there an increase in the number of different procedures P's performs in reaction to what can count as the same task-situation?
Substantiation	The performer is either unable to give any substantiation of the correctness of her performance or simply repeats the performance claiming its correctness.	The performer argues for the correctness of her previous performance by employing a different procedure and showing that she gets the same outcome	Is P now less dependent on other people's judgement in gauging the appropriateness of the execution?
Applicability	The routine is performed only if task-situation is reproduced in almost all its details.	No restrictions	Was there an increase in the use of the procedure in new contexts?

The analyst's question is how she can diagnose a degree of ritualization of a routine or a change in it by considering the records of specific performances. According to the commognitive approach, there are features of performances that the analyst can actually see and can take as indicative of a degree of ritualization (or de-ritualization). The features of Table 2 indicate how much the performance is directed toward a specific procedure and how much toward the specific outcome (Lavie, Steiner & Sfard, 2018). Among those features are: i) vi the procedure is functional; iii) Agentivity: the degree of decision making while operating; iv) Substantiation: the performance arguments for the correction of the outcome; v) Objectification: the transition to referring to the rational number as an independent object in the mediating discourse; vi)

Applicability: the performer finds the routine useful in situations where there is no significant similarity between the two task situations.

Here are some principles that the analyst must keep in mind while performing such diagnosis:

1. Most of the features appearing in Table 2 cannot be diagnosed directly on the basis of any specific performance. Instead, one needs to look at a whole series of performances for what counts for an expert as the same task-situation.
2. The properties of performances are not independent from one another. Sometimes, when you decide on one of them, the other almost automatically turns thru as well.
3. Diagnosing the features is an interpretive activity, in which any claim is tentative and subject to change; the change in interpretation may come at any time as a result from broadening the context and considering additional performances.

Because of space limitations, we will present in this article illustration only for some of the properties listed in Table 2. We reiterate that although the examples we chose are typical, in that they represented phenomena we saw in the rest of Noa and Ada's data and also in those of other participants, we will not present here finding but rather an illustration of the analytic method.

Illustrations of the analytic method

We will present representative examples showing how different features of routines summarized in Table 2 were identified in our data. The examples presented here are taken from the five consecutive interviews in which the girls repeatedly performed the following two fourth grade textbook assignments:

Could you draw the number-line with the numbers 0, 1, 2, 3, 4 on it?
Is it possible that $\frac{5}{12}$ is on the number-line that you have drawn?
Can you show where?

Figure 1. Text with Assignment 1

Pluto the Wolf runs from his house (marked with 0) to the bowl with food (marked with 1). What part of the way did he already make, approximately?

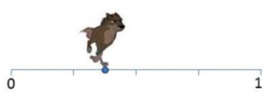

Figure 2. Text with assignment 2

Flexibility

The rise in the flexibility of a routine means that there is now more than one way to perform the task. Stating that a routine preformed in a task-situation became more flexible means that there is now more than one optional procedure to perform the task. The routine become more flexible through the adaption of the procedure to typical features of the current task. Looking at Table 3, which present the procedures performed by Ada and Noa in the activity over Assignment 1, it might seems at first glance that the same procedure of dividing into twelfth and then reaching to the fifth twelfth is executed over and over again. However, despite the similarity between the procedures implemented by the girls, we argue that there is a process of streamlining - the adaptation of the procedure of the formal routine that they have learned to the specific assignment with which they are coping. Ada shifted from dividing the interval [0,1] into twelfth to another method of division that was easier for her to implement. Then she made yet another alternation when she divided only the interval [0,0.5].

Table 3. The girls' procedures in implementing Assignment 1

	ITV1 4.2015	ITV2 12.2015	ITV3 5.2016	ITV4 11.2016
Noa	(1) Divide all the intervals into halves (2) Divide [0,1] into twelfths (3) Counting five segments from zero	(1) Divide [0,1] into half (2) Divide [0,0.5] and [0.5,1] into sixths (3) Count five segments from zero	(1) Divide [0,1] into two halves (2) Divide [0,0.5] and [0.5,1] into halves (3) Divide all new segments into three equal parts (4) Count five segments from zero	(1) Divide [0,1] into half (2) Divide [0,0.5] and [0.5,1] into halves (3) Divide all new segments into three (4) Count one segment from half
Ada	(1) Divide [0,1] into twelfth (2) Count five segments from zero	(1) Divide [0,1] into half (2) Divide [0,0.5] and [0.5,1] into sixth (3) Count five segments from zero	(1) Divide [0,1] into half (2) Divide [0,0.5] into sixth (3) Count five segments from zero	(1) Divide [0,1] into half (2) Divide [0,0.5] into sixth (3) Count one segment from half

Another sign of flexibility can be found in the excerpt taken from ITV3 where the equivalence of different procedures is explicitly recognized by Ada.

Ada: Ok, I did it between one to zero because I know it is not yet a whole and then I divided it into half, and then I sevises both halves to sixth, I did not complete the division of that one' and here is the five.

Noa: I am dividing it to half to half and then divide into three.

Ada: But it is the method, like, it is less important, the how we divide it.

Objectification

For the discourse of rational numbers to be considered sufficiently developed, it is not enough that the discussant is acquainted with the different signifiers and realizations of rational numbers. It is also important that she objectifies rational numbers, that is, becomes able to see all of the relevant signifiers as signifying full-fledged objects in their own right. Hence, the increasing tendency of using the signifiers as nouns is yet another indication for the progress of de-ritualization. Throughout the repeated performances of Assignment 2 we could see changes in the linguistic use of the signifier fourth in the girls mediating discourse. The transformation from referring to the location of the wolf as: "It is in the middle of the second quarter" in ITV1, to utterance like this: "It is right in the middle between forth and two forth" in ITV3 indicates that the girls have objectified the signifier fourth. In the repeated performance of assignment 1 Ada have objectified the signifier five over twelve. Along ITV1,2 and 3, Ada did not related to the fraction five over twelve as a number, but rather as a set of twelfths with five entities. As can be seen in the following excerpt taken from ITV2, Five is the number of twelfths that we need to count from zero:

Ada: I divided it into twelfth and then I checked where the five was, because it is five over twelve.

The objectification of five over twelve in the discourse of Ada is expressed in the following excerpt taken from TLV4 as Ada referred to it as a name of the location that is needed to be found.

Ada: I divided the zero to one into twelfth and then I just found the five over twelve.

Substantiability

When focused only on the procedure, the performer would substantiate her actions by simply describing the procedure she executed. Hence, a step forward in the process of de-ritualization would be substantiation of the outcome by showing that an alternative procedure would yield the same outcome. We found that over time the routines of the two task-situations became more substantiate. One of the signs of substantiation can be found in the change in Noa's explanation of her choice to divide the interval unto eight equal parts in performing Assignment 2 (Table 2).

Table 4. Signs of substantiation in Noa's explanation

ITV1	because four times two equals eight ... do not know how to explain it, so it's eighths
ITV3	And that four times two, that's eight.
ITV4	Here it was divided into **quarters**, I deliberately emphasized the **quadrants** and that's **half**, oh and it was here, it was exactly **half** between **quarter** and **two quarters**. So I had to divide it, to multiply ... multiply the **denominator** by two to get it to the middle. Because **half** ... that's ... for that you have to double by two. So I multiplied the fo ... four times two, it's equal to eight, and I divided it so that each part is eighth.

In ITV1 Noa is having hard time substantiate her actions. In ITV3 she is more confident and describes the calculation that led her to eighth, but she does not use words that are typical to the discourse of rational numbers as we would expect. In ITV4 she is describing the calculation again but this time she is using new words that are typical to the discourse of rational numbers that she didn’t use before, like half quarter and denominator. In addition Noa is using another procedure that of division that her partner Ada was using in TLV4 to in order to substation her action.

Applicability of the two formal routines

We speak about routine’s applicability while considering the range of task-situations for which its performances so far are likely to constitute precedents. In order to learn of the applicability of the two formal routines we presented the girls with the following activity that is supposed to evoke the use of the two formal routines. Each girl is given a card. Ada got the card in Figure 3 and had to instruct Noa over the phone to draw the point on her card (Figure 4) in the same location.

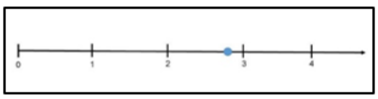

Figure 3: The card given to the instructor

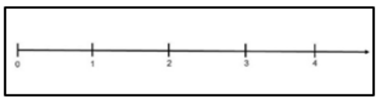

Figure 4: The instructed participants' card

Using her demonstrated ability to name the point with a fractional number (23/4) would have helped Ada in this assignment, and would have been useful for Noa too, since she could translate a fraction into a number line location, as she had already shown during the execution of assignment 1. And yet, none of these routines were applied. We take this fact as an evidence of their ritualized character. Noa and Ada were asked to perform the telephone assignment over and over again in every session, but although their procedure underwent several refinements as is shown above, it did not involve point-naming and point-reading. It was only 18 months after ITR1 took place that the girls evoked that routine in that context, as shown in excerpt from ITR5:

Ada: Ok, divide between 2 and 3 into quarters (Looks at the page and marks with a finger the division of the interval between 2 and 3)

Noa: Ok... three quarters? (Looks at her page)

Ada: Yes (Looks at her page and smiles)

Both: (Laugh)

Noa: (Divides the interval between 2 and 3 into four more-or-less equal parts)

Ada: Perfect. So two and three quarters, approximately (Looks at her page)

Here, it was the first time Ada was preforming the formal routine for naming the point to be copied with a fractional number. Later, Noa used the point-reading formal routine for reproducing the desired location on the line. We can summarize by saying that in ITR5 we could finally recognize that the formal routines were applicable in a non-school-like assignment.

Concluding remarks

In the above, we have described the analytic method by which we operate in this study. As time-consuming and laborious a task it is to conduct this kind of analysis we feel that it is worthwhile to undertake this effort. Underlying the decision to study the notion of routine is our belief that conceptualizing learning as a process of routinization may generate new insights.

Conceptualizing the learning of rational number as de-ritualization of the formal school routines, which is reflected primarily in the shift of focus from the implementation of a certain procedure to achieving a desired outcome, allows us to look at familiar phenomena that are considered challenging in the world of education in a new and insightful way. We know from previous research that the process of de-ritualization is gradual and slow and only too often will not be completed in school. Such analytic method may provide answers to outstanding questions in the research of learning rational numbers. Questions like the ones listed in the Second Handbook on Mathematics Teaching and Learning (Lamon, 2007) may not get holistic and comprehensive responses. They are questions like: How does one measure rational number sense? How does a child come to understand a rational number as a single quantity as oppose to regarding it as a pair of numbers? What are the benchmarks by which to judge that children's knowledge is moving in a desirable direction? And how can we assess depth of understanding of the rational numbers?

References

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). Chicago, IL: Northwestern University Press.

Confrey, J., & Scarano, G. H. (1995). Splitting reexamined: Results from a three year longitudinal study of children in grades three to five. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the Seventeenth Annual Meeting for the Psychology of Mathematics Education-NA* (vol. 1, pp. 421-426). Columbus, OH: ERIC.

Kieren, T. E. (1976). On the Mathematical, cognitive and instructional. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop.* Columbus, OH: ERIC.

Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323-371). Hillsdale, NJ: Erlbaum Associates.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Publishing.

Lavie, I., Steiner, A., & Sfard, A. (2018). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, doi.org/10.1007/s10649-018-9817-4.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom. The case of function. *International Journal of Educational Research*, *51*, 10-27.

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In K. Heid & G. Blume (Eds.) *Research on technology and the teaching and learning of mathematics* (pp. 1-53). Greenwich, CT: Information Age Publishing.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction*, *23*, 237-309.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.