Advanced Calculus: MATH 410
Real Numbers

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1. Real Number System

1.1. **Introduction.** Numbers are at the heart of mathematics. By now you must be fairly familiar with them. Some basic sets of numbers are:

$$\text{natural numbers, } \mathbb{N} = \{0, 1, 2, \cdots\};$$

$$\text{integers (die Zahlen), } \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\};$$

$$\text{rational numbers (quotients), } \mathbb{Q} = \left\{\frac{p}{q} \,:\, p, q \in \mathbb{Z},\ q \neq 0\right\};$$

$$\text{real numbers, } \mathbb{R} = (-\infty, \infty);$$

$$\text{complex numbers, } \mathbb{C} = \{x + iy \,:\, x, y \in \mathbb{R}\}.$$

Each of these sets is endowed with natural algebraic operations (like 'addition' and 'multiplication') and order relations (like 'less than') by which their elements are manipulated and compared. It is fairly clear how $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are related through an increasingly richer algebraic structure. It is also fairly clear that $\mathbb{R}$ and $\mathbb{C}$ bear a similar relationship. What is less clear is the relationship between $\mathbb{Q}$ and $\mathbb{R}$. In particular, what are the properties that allow $\mathbb{R}$ and not $\mathbb{Q}$ to be identified with a 'line'? In this section we address some of these issues.

We begin by addressing the question of why the rational numbers are inadequate for mathematical analysis. Simply put, the rationals do not allow us to solve equations that we would like to solve. This was also the reason behind the introduction of the negative integers and the rational numbers. The negative integers allow us to solve equations like $x + m = n$, where m, $n \in \mathbb{N}$. The rationals allow us to solve equations like $mx = n$, where m, $n \in \mathbb{Z}$ with $m \neq 0$. However, the rationals do not allow us to solve the rather simple equation $x^2 = 2$.

Proposition 1.1. *There esists no $x \in \mathbb{Q}$ such that $x^2 = 2$.*

Proof. We argue be assuming the contrary, and showing that it leads to a contradiction. Suppose there is such an $x \in \mathbb{Q}$. Then we can write it as $x = p/q$, where p and q are nonzero integers with no common factors. Because $x^2 = 2$, we see that $p^2 = 2q^2$. Hence, 2 is a factor of p^2, which implies that 2 must also be a factor of p. We can therefore set $p = 2r$ for some nonzero integer r. Because $p^2 = 2q^2$, we see that $2^2r^2 = 2q^2$, which is the same as $q^2 = 2r^2$. Hence, 2 is a factor of q^2, which implies that 2 must also be a factor of q. It follows that 2 is a factor of both p and q, which contradicts our assumption that p and q have no common factors. Therefore no such $x \in \mathbb{Q}$ exists. $\square$

There is nothing special about 2 in our argument. The same result is obtained for equations like $x^2 = n$ where n is any positive integer that is not a perfect square. More generally, the same result is obtained for equations like $x^m = n$ where m and n are positive integers such that $n \neq k^m$ for some integer k. The problem is that there are too many "holes" like this in $\mathbb{Q}$. In this section we will see how $\mathbb{R}$ fills these holes so as to allow the solution of such equations.

1.2. **Fields.** The sets $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ endowed with their natural algebraic operations are each an example of a general algebraic structure known as a *field*.

Definition 1.1. *A* field *is a set X equipped with two distinguished binary operations, called* addition *and* multiplication*, that satisfy the addition, multiplication, and distributive axioms presented below. Taken together, these axioms constitute the so-called* field axioms*.*

Addition axioms. Addition maps any two $x, y \in X$ to their *sum* $x + y \in X$ such that:

A1: $x + y = y + x \ \forall x, y \in X$, — commutativity;
A2: $(x + y) + z = x + (y + z) \ \forall x, y, z \in X$, — associativity;
A3: $\exists 0 \in X$ such that $x + 0 = x \ \forall x \in X$, — identity;
A4: $\forall x \in X \ \exists -x \in X$, such that $x + (-x) = 0$, — inverse.

Multiplication axioms. Multiplication maps any two $x, y \in X$ to their *product* $xy \in X$ such that:

M1: $xy = yx \ \forall x, y \in X$, — commutativity;
M2: $(xy)z = x(yz) \ \forall x, y, z \in X$, — associativity;
M3: $\exists 1 \in X$ such that $1 \neq 0$ and $x1 = x \ \forall x \in X$, — identity;
M4: $\forall x \in X$ such that $x \neq 0 \ \exists x^{-1} \in X$ such that $xx^{-1} = 1$, — inverse.

Distributive axiom. Addition and multiplication are related by:

D: $x(y + z) = xy + xz \ \forall x, y, z \in X$, — distributivity.

Examples. When addition and multiplication have their usual meaning, the field axioms clearly hold in $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, but not in $\mathbb{N}$ or $\mathbb{Z}$. They also hold in $\mathbb{Z}_n \equiv \mathbb{Z}/(n\mathbb{Z})$ (the integers mod n) when n is prime. If you do not know this last example, do not worry. It is not critical in this course. You will see it in a basic algebra course.

All of the usual rules for algebraic manipulations involving addition, subtraction, multiplication, and division can be developed from the field axioms. This is not as easy as it sounds!

1.2.1. *Consequences of the Addition Axioms.* We begin by isolating the addition axioms.

Definition 1.2. *A set X equipped with a distinguished binary operation that satisfies the addition axioms is called an* Abelian group *or a* commutative group*.*

Examples. When addition has its usual meaning, the axioms for an Abelian groups clearly hold in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, but not in $\mathbb{N}$. (As defined here, $\mathbb{N}$ satisfies all these axioms but A4.) They also hold in $\mathbb{Z}_n$ for every positive integer n.

The addition axioms immediately imply the following.

Proposition 1.2. *Let X be an Abelian group.*

(a) *If $x, y, z \in X$ and $x + y = x + z$ then $y = z$.*
(b) *If $x, y \in X$ and $x + y = x$ then $y = 0$.*
(c) *If $x, y \in X$ and $x + y = 0$ then $y = -x$.*
(d) *If $x, y \in X$ then $-(x + y) = (-x) + (-y)$.*
(e) *If $x \in X$ then $-(-x) = x$.*

Proof. Exercise.

Assertion (a) states that addition enjoys a so-called cancellation law. Assertion (b) states that there is a unique additive identity of the type assumed in A3. This unique additive identity

is called *zero*. All other elements of X are said to be *nonzero*. Assertion (c) states that for every $x \in X$ there is a unique additive inverse of the type assumed in A4. This unique additive inverse is called the *negative* of x. The map defined for every $x \in X$ by $x \mapsto -x$ is called negation. Assertion (d) states that the negative of a sum is the sum of the negatives. Assertion (e) states that for every $x \in X$ the negative of the negative of x is again x.

When working with Abelian groups, it is both convenient and common to write

$$x - y\,, \qquad x + y + z\,, \qquad 2x\,, \qquad 3x\,, \qquad \cdots\,,$$

rather than

$$x + (-y)\,, \quad x + (y + z)\,, \quad x + x\,, \quad x + x + x\,, \quad \cdots\,.$$

More precisely, the symbol nx is can be defined for every group element x and every natural number n by induction. We set $0x = 0$, where the second 0 is the additive identity, and define $(n + 1)x = nx + x$ for every $n \in \mathbb{N}$. This notation satisfies the following properties.

Proposition 1.3. *Let X be an Abelian group.*

(a) *If $x \in X$ and $m, n \in \mathbb{N}$ then $(m + n)x = mx + nx$ and $(mn)x = n(mx)$.*
(b) *If $x, y \in X$ and $n \in \mathbb{N}$ then $n(x + y) = nx + ny$.*
(c) *If $x \in X$ and $n \in \mathbb{N}$ then $n(-x) = -(nx)$.*

Proof. Exercise.

Motivated by these facts, for every group element x the definition of the symbol nx can be extended to every integer n by setting $nx = (-n)(-x)$ when n is negative.

1.2.2. *Consequences of the Multiplcation Axioms.* The only connection of the multiplication axioms to addition is through the references to zero in M3 and M4. An immediate consequence of M3 is that every field has at least two elements — 0 and 1. It is also clear that the nonzero elements of a field considered with the operation of multiplication form an Abelian group.

Examples. When addition and multiplication have their usual meaning, the addition and multiplication axioms clearly hold in $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, but not in $\mathbb{N}$ or $\mathbb{Z}$. They also hold in $\mathbb{Z}_n$ when n is prime.

The multiplication axioms immediately imply the following.

Proposition 1.4. *Let X be a field.*

(a) *If $x, y, z \in X$, $x \neq 0$, and $xy = xz$ then $y = z$.*
(b) *If $x, y \in X$, $x \neq 0$, and $xy = x$ then $y = 1$.*
(c) *If $x, y \in X$, $x \neq 0$, and $xy = 1$ then $y = x^{-1}$.*
(d) *If $x, y \in X$, $x \neq 0$ and $y \neq 0$ then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.*
(e) *If $x \in X$ and $x \neq 0$ then $(x^{-1})^{-1} = x$.*

Proof. Exercise.

Assertion (a) states that multiplication enjoys a so-called cancellation law. Assertion (b) states that there is a unique multiplicative identity of the type assumed in M3. This unique multiplicative identity is called *one*. Assertion (c) states that for every nonzero $x \in X$ there is a unique multiplicative inverse of the type assumed in M4. This unique multiplicative inverse is called the *reciprocal* of x. The map defined for every nonzero $x \in X$ by $x \mapsto x^{-1}$ is called reciprocation. Assertion (d) states that the reciprocal of a product is the product of the reciprocals. Assertion (e) states that every nonzero $x \in X$ is the reciprocal of its reciprocal.

When working with fields, it is both convenient and common to write

$$x/y\,, \qquad xyz\,, \qquad x^2\,, \qquad x^3\,, \qquad \cdots\,,$$

rather than

$$xy^{-1}\,, \quad x(yz)\,, \quad xx\,, \quad xxx\,, \quad \cdots\,.$$

More precisely, the symbol x^n is can be defined for every field element x and every positive interger n by induction. We set $x^1 = x$ and define $x^{n+1} = x^n x$ for every $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the positive integers. This notation satisfies the following properties.

Proposition 1.5. *Let X be a field.*

(a) *If $x \in X$ and $m, n \in \mathbb{Z}_+$ then $x^{m+n} = x^m x^n$ and $x^{mn} = (x^m)^n$.*
(b) *If $x, y \in X$ and $n \in \mathbb{Z}_+$ then $(xy)^n = x^n y^n$.*
(c) *If $x \in X$, $x \neq 0$, and $n \in \mathbb{Z}_+$ then $x^n \neq 0$ and $(x^n)^{-1} = (x^{-1})^n$.*

Proof. Exercise.

Motivated by these facts, for every nonzero field element x the definition of the symbol x^n can be extended to every integer n by setting $x^0 = 1$, where the 1 is the multiplicative identity, and $x^n = (x^{-1})^{-n}$ when n is negative. The symbol 0^n remains undefined when n is not positive.

Exercise. Let X be a field. Extend Proposition 1.5 to $\mathbb{Z}$ by proving the following.

(a) If $x \in X$, $x \neq 0$, and $m, n \in \mathbb{Z}$ then $x^{m+n} = x^m x^n$ and $x^{mn} = (x^m)^n$.
(b) If $x, y \in X$, $x \neq 0$, $y \neq 0$, and $n \in \mathbb{Z}$ then $(xy)^n = x^n y^n$.
(c) If $x \in X$, $x \neq 0$, and $n \in \mathbb{Z}$ then $x^n \neq 0$ and $(x^n)^{-1} = (x^{-1})^n$.

1.2.3. *Consequences of the Distributive Axiom.* The distributive axiom gives the key connection between addition and multiplication. Taken together, the field axioms imply the following.

Proposition 1.6. *Let X be a field.*

(a) *If $x \in X$ then $x0 = 0$.*
(b) *If $x, y \in X$ and $xy = 0$ then $x = 0$ or $y = 0$.*
(c) *If $x, y \in X$ then $(-x)y = -(xy) = x(-y)$.*
(d) *If $x \in X$ and $x \neq 0$ then $(-x)^{-1} = -x^{-1}$.*

Proof. Exercise.

Assertion (a) states that the product of anything with zero is zero. In particular, it shows that zero cannot have a multiplicative inverse. Hence, an element has a multiplicative inverse if and only if it is nonzero. Assertion (b) states that if a product is zero, at least one of its factors must be zero. This should be compared with (d) of Proposition 2.2. Assertions (c) and (d) state how negation, multiplication, and reciprocation relate.

The field axioms allow you to extend to any field many of the formulas that you have known for years in the context of $\mathbb{R}$ or $\mathbb{C}$. For example, you can establish the following formulas.

Proposition 1.7. *Let X be a field. Then for every $x, y \in X$ and every $n \in \mathbb{N}$ we have the* difference of powers *and* binomial *formulas*

$$x^{n+1} - y^{n+1} = (x - y)\left(x^n + x^{n-1}y + \cdots + x^{n-k}y^k + \cdots + x\,y^{n-1} + y^n\right)\,, \tag{1.1}$$

$$(x + y)^n = x^n + n\,x^{n-1}y + \cdots + \frac{n!}{(n-k)!\,k!}\,x^{n-k}y^k + \cdots + n\,x\,y^{n-1} + y^n\,. \tag{1.2}$$

Proof. Exercise.

1.3. **Ordered Sets.** The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ endowed with their natural order relation are each an example of a general structure known as an ordered set.

Definition 1.3. *An* ordered set $(X, <)$ *is a set X equipped with a distinguished binary relation "$<$", called an* order*, that satisfies the order axioms presented below.*

Order axioms. A binary relation "$<$" on a set X is called an *order* whenever:

O1: if $x, y, z \in X$ then $x < y$ and $y < z$ implies $x < z$, — transitivity;
O2: if $x, y \in X$ then exactly one of $x < y$, $x = y$, or $y < x$ is true, — trichotomy.

Examples. When "$<$" has its usual meaning of "less than", the order axioms clearly hold in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. When "$<$" has the unusual meaning of "greater than", the order axioms also clearly hold in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. We will stick with the usual meaning of "$<$" in what follows.

When working with ordered sets, it is both convenient and common to use the notation

$$x > y\,, \qquad x \le y\,, \qquad x \ge y\,,$$
$$x < y < z\,, \qquad x < y \le z\,, \qquad \cdots\,,$$

to respectively mean

$$y < x\,, \qquad x < y \text{ or } x = y\,, \qquad y < x \text{ or } x = y\,,$$
$$x < y \text{ and } y < z\,, \qquad x < y \text{ and } y \le z\,, \qquad \cdots\,.$$

1.3.1. *Bounds.* Ordered sets have associated notions of boundedness.

Definition 1.4. *Let $(X, <)$ be an ordered set. A point $x \in X$ is an* upper bound *(a* lower bound*) of a set $S \subset X$ whenever $y \le x$ ($x \le y$) for every $y \in S$. If $S \subset X$ has an upper bound (a lower bound) then S is said to be* bounded above *(*bounded below*). A set $S \subset X$ that is both bounded above and bounded below is said to be* bounded*.*

Definition 1.5. *Let $(X, <)$ be an ordered set, and let $S \subset X$ be bounded above. A point $x \in X$ is a* least upper bound *or* supremum *of S whenever:*

(i) *x is an upper bound of S;*
(ii) *if $y \in X$ is also an upper bound of S then $x \le y$.*

We similarly define a greatest lower bound *or* infimum *of S.*

If a supremum or infimum of S exists then it must be unique. The supremum of S is denoted $\sup\{S\}$ or $\sup\{z : z \in S\}$, while the infimum is denoted $\inf\{S\}$ or $\inf\{z : z \in S\}$. These notions should not be confused with those of maximum and minimum.

Definition 1.6. *Let $(X, <)$ be an ordered set, and let $S \subset X$. A point $x \in S$ is a* maximum *(*minimum*) of S whenever x is an upper (lower) bound of S.*

If a maximum or minimum of S exists then it must be unique. The maximum of S is denoted $\max\{S\}$ or $\max\{z : z \in S\}$, while the minimum is denoted $\min\{S\}$ or $\min\{z : z \in S\}$. Moreover, if a maximum (minimum) of S exists then

$$\sup\{S\} = \max\{S\} \qquad \big(\inf\{S\} = \min\{S\}\big)\,.$$

Examples. Any bounded open interval (a, b) in $\mathbb{R}$ has neither a maximum nor a minimum, yet $\sup\{(a, b)\} = b$ and $\inf\{(a, b)\} = a$. For any bounded closed interval $[a, b]$ in $\mathbb{R}$ one has $\max\{[a, b]\} = b$ and $\min\{[a, b]\} = a$. The same is true if these intervals and their endpoints are restricted to elements of $\mathbb{Q}$.

1.3.2. *Least Upper Bound Property.* What distinguishes $\mathbb{R}$ from $\mathbb{Q}$ is the following property.

Definition 1.7. *Let $(X, <)$ be an ordered set. Then X is said to have the* least upper bound property *whenever every nonempty subset of X with an upper bound has a least upper bound.*

Remark. It may seem we should also define a "greatest lower bound property", but the next proposition shows that this is unnecessary because it is exactly the same property.

Proposition 1.8. *Let $(X, <)$ be an ordered set. Let X have the* least upper bound property*. Then every nonempty subset of X with a lower bound has a greatest lower bound.*

Proof. Let $S \subset X$ be a nonempty set with a lower bound. Let $L \subset X$ be the set of all lower bounds of S. It is nonempty and bounded above by any element of S. Therefore $\sup\{L\}$ exists. It is easy to check that $\sup\{L\} = \inf\{S\}$. $\square$

Examples. When "$<$" has its usual meaning of "less than", the sets $\mathbb{N}$ and $\mathbb{Z}$ have the least upper bound property. However, as we will show in the next proposition, the set $\mathbb{Q}$ does not.

Proposition 1.9. *The set $\mathbb{Q}$ does not have the least upper bound property.*

Proof. Consider the sets

$$S = \left\{ r \in \mathbb{Q} \,:\, r > 0 \,,\, r^2 < 2 \right\} , \qquad \tilde{S} = \left\{ r \in \mathbb{Q} \,:\, r > 0 \,,\, r^2 > 2 \right\} .$$

These sets are clearly nonempty because $1 \in S$ and $2 \in \tilde{S}$. One can show that every point in $\tilde{S}$ is an upper bound for S. In order to show that S has no least upper bound, one first shows that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$. It follows (by trichotomy) that if p is a least upper bound of S then either $p \in S$ or $p \in \tilde{S}$. We will show that neither can be the case. More specifically, we will show that if $p \in S$ then p is not an upper bound of S, and that if $p \in \tilde{S}$ then p is not a least upper bound of S.

Let $p \in S$. We will construct a $q \in S$ such that $p < q$, thereby showing that p is not an upper bound of S. There are many ways to construct such a q. We are seeking a rational approximation of $\sqrt{2}$ from below that is better than p. This can be done by taking one iteration of Newton's method applied to $f(x) = 1 - 2/x^2 = 0$. Set

$$q = p - \frac{f(p)}{f'(p)} = p - \frac{p^2 - 2}{4}\, p = \frac{6 - p^2}{4}\, p \,.$$

Because $x \mapsto f(x)$ is increasing and concave over $x > 0$, a picture alone should convince you this is a suitable q. Indeed, it is clear from the above formula that $0 < p < q$. A skeptic only needs to check that $q^2 < 2$. We confirm this fact by the calculation

$$2 - q^2 = 2 - \frac{36 - 12p^2 + p^4}{16}\, p^2 = \frac{(2 - p^2)^2 (8 - p^2)}{16} > 0 \,.$$

Now let $p \in \tilde{S}$. We will construct a $q \in \tilde{S}$ such that $q < p$, thereby showing that p is not a least upper bound of S. Once again, there are many ways to construct such a q. This time we are seeking a rational approximation of $\sqrt{2}$ from above that is better than p. This can be done by taking one iteration of Newton's method and applied to $f(x) = x^2 - 2 = 0$. Set

$$q = p - \frac{f(p)}{f'(p)} = p - \frac{p^2 - 2}{2p} = \frac{p^2 + 2}{2p} \,.$$

Because $x \mapsto f(x)$ is increasing and convex over $x > 0$, a picture alone should convince you this is a suitable p. Indeed, it is clear from the above formula that $q < p$ and that $q > 0$. A skeptic only needs to check that $q^2 > 2$. We confirm this fact by the calculation

$$q^2 - 2 = \frac{p^4 + 4p^2 + 4}{4p^2} - 2 = \frac{p^4 - 4p^2 + 4}{4p^2} = \left(\frac{p^2 - 2}{2p}\right)^2 > 0\,.$$

□

Remark. An alternative construction that can be used for both cases in the above proof is

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}\,.$$

Then

$$q^2 - 2 = \frac{4p^2 + 8p + 4}{p^2 + 4p + 4} - 2 = 2\frac{p^2 - 2}{(p + 2)^2}\,.$$

While this construction yields a slicker proof, the underlying geometric picture seems less clear.

1.4. **Ordered Fields.** The sets $\mathbb{Q}$ and $\mathbb{R}$ endowed with their natural algebraic operations and order relation are each an example of a general algebraic structure known as an ordered field.

Definition 1.8. *A set X that is both a field and an ordered set is called an ordered field whenever*

OF1: *if $x, y, z \in X$ then $x < y$ implies $x + z < y + z$;*
OF2: *if $x, y \in X$ then $0 < x$ and $0 < y$ implies $0 < xy$.*

If $x > 0$ ($x < 0$, $x \geq 0$, $x \leq 0$) then we say x is positive *(*negative, nonnegative, nonpositive*). The set of all positive (negative) elements of X is denoted X_+ (X_-).*

Examples. When addition, multiplication, and "$<$" have their usual meanings, the sets $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields. In an algebra course you can learn that many other ordered fields arise in Galois theory.

1.4.1. *Consequences of the Ordered Field Axioms.* The ordered field axioms allow you to extend to any ordered field many of the rules for working with inequalities that you have known for years in the context of $\mathbb{R}$. For example, the rule that multiplying both sides of an inequality by a positive (negative) quantity will preserve (reverse) the inequality. Some of these rules are given in the following proposition.

Proposition 1.10. *Let X be an ordered field.*

(a) *If $x > 0$ then $-x < 0$, and vice versa.*
(b) *If $x > 0$ and $y < z$ then $y < x + z$ and $xy < xz$.*
(c) *If $x < 0$ and $y < z$ then $x + y < z$ and $xy > xz$.*
(d) *If $x \neq 0$ then $x^2 > 0$.*
(e) *If $0 < x < y$ and $n \in \mathbb{Z}_+$ then $0 < x^n < y^n$ and $0 < y^{-n} < x^{-n}$.*

Proof. Exercise.

The above proposition shows that X_+ satisfies the following.

P1: If $x, y \in X_+$ then $x + y \in X_+$ and $xy \in X_+$.
P2: For every $x \in X$ exactly one of $x \in X_+$, $-x \in X_+$, or $x = 0$ is true.

These so-called positivity properties alone characterize the order relation on the field X.

Proposition 1.11. *Let X be a field. Let $X_+ \subset X$ satisfy the positivity properties P1 and P2. Define the binary relation $<$ on X by*

$$x < y \qquad \textit{means} \qquad y - x \in X_+ \,. \tag{1.3}$$

Then $(X, <)$ is an ordered field.

Proof. Exercise.

Remark. Proposition 1.11 implies that we could have defined an ordered field as a field X that has a subset X_+ satisfying the positivity properties, P1 and P2. In that case the positivity properties become the positivity axioms and, upon defining the order on X by (1.3), the order axioms O1, O2, OF1, and OF2 become order properties. This is the approach taken in Fitzpatrick's book.

1.4.2. *Absolute Value Function.* There is a natural absolute value function on any ordered field.

Definition 1.9. *Let X be an ordered field. The* absolute value *function on X is defined by*

$$|x| = \begin{cases} x & \textit{if } x > 0 \\ 0 & \textit{if } x = 0 \\ -x & \textit{if } x < 0 \end{cases} .$$

Some of its properites are given in the following proposition. They should all look very familiar to you. However, your goal now is to understand how they follow from Definition 1.9 and the ordered field axioms.

Proposition 1.12. *Let X be an ordered field. Then for every $x, y \in X$*

(a) $|x| \geq 0$,
(b) $|x| = 0$ *if and only if* $x = 0$,
(c) $|x + y| \leq |x| + |y|$,
(d) $|xy| = |x|\,|y|$,
(e) $\big||x| - |y|\big| \leq |x - y|$.

Proof. Exercise.

With the absolute value function we can define the *distance* between points $x, y \in X$ by $d(x, y) = |x - y|$. This distance function satisfies the following proposition.

Proposition 1.13. *Let X be an ordered field. Let $d(x, y) = |x - y|$ where $|\cdot|$ is given by Definition 1.9. Then for every $x, y, z \in X$*

(a) $d(x, y) \geq 0$,
(b) $d(x, y) = 0$ *if and only if* $x = y$,
(c) $d(x, y) = d(y, x)$,
(d) $d(x, x) = d(x, y) + d(y, z)$.

Proof. Exercise.

We can also characterize bounded sets with the absolute value function.

Proposition 1.14. *Let X be an ordered field. Then $S \subset X$ is bounded if and only if there exists an $m \in X_+$ such that*

$$x \in S \quad \Longrightarrow \quad |x| \leq m \,.$$

Proof. Exercise.

1.5. **Real Numbers.** We now state without proof the main theorem of this section.

Theorem 1.1. *There exists a unique (up to an isomorphism) ordered field with the least upper bound property that contains $\mathbb{Q}$ (up to an isomorphism) as a subfield.*

Proof. Proofs of this theorem are quite long and technical. You can find a proof of all but the uniqueness in the book by W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1976, and another in the book by T. Tao, *Analysis I*, Hindustan Book Agency, 2006 (available through the American Mathematical Society). The proof in Rudin is based upon a construction due to Dedekind in which the real numbers are built from subsets of the rationals called Dedekind cuts. The proof in Tao is based upon a construction due to Cantor in which the real numbers are built from equivalence classes of Cauchy sequences within the rationals. Both Dedekind and Cantor published their constructions in 1872. □

Definition 1.10. *The* real numbers *are defined to be the unique ordered field with the least upper bound property whose existence is guaranteed by Theorem 1.1. This field is denoted $\mathbb{R}$.*

Remark. The least upper bound property that sets $\mathbb{R}$ apart from $\mathbb{Q}$. As we will see, it is why $\mathbb{R}$ can be identified with a line.

1.5.1. *Intervals.* Intervals are special subsets of $\mathbb{R}$ that will play a leading role in our study. They are denoted with the so-called interval notation. The empty set $\emptyset$ is considered to be an interval. For every $a \in \mathbb{R}$ we define $[a, a] = \{a\}$. For every $a, b \in \mathbb{R}$ such that $a < b$ we define

$$(a, b) = \big\{x \in \mathbb{R} : a < x < b\big\}, \qquad [a, b) = \big\{x \in \mathbb{R} : a \le x < b\big\},$$
$$(a, b] = \big\{x \in \mathbb{R} : a < x \le b\big\}, \qquad [a, b] = \big\{x \in \mathbb{R} : a \le x \le b\big\}.$$

For every $a, b \in \mathbb{R}$ we define

$$(a, \infty) = \big\{x \in \mathbb{R} : a < x\big\}, \qquad (-\infty, b) = \big\{x \in \mathbb{R} : x < b\big\},$$
$$[a, \infty) = \big\{x \in \mathbb{R} : a \le x\big\}, \qquad (-\infty, b] = \big\{x \in \mathbb{R} : x \le b\big\}.$$

Finally, we define $(-\infty, \infty) = \mathbb{R}$. Here have not defined the symbols ∞ and $-\infty$ outside of the context of an unbounded interval. Unless it is stated explicitly otherwise, when we write (a, b), $[a, b)$, or $(a, b]$ it is implied that $a < b$, while when we write $[a, b]$ it is implied that $a \le b$.

Exercise. Prove that for every $a \in \mathbb{R}$ and $r \in \mathbb{R}_+$ we have

$$\{x \in \mathbb{R} : |x - a| < r\} = (a - r, a + r), \qquad \{x \in \mathbb{R} : |x - a| \le r\} = [a - r, a + r].$$

1.5.2. *Consequences of the Real Number Axioms.* The following important properties relate the reals $\mathbb{R}$ with the positive integers $\mathbb{Z}_+$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$.

Proposition 1.15. *The following hold.*

- *If $x, y \in \mathbb{R}$ and $x > 0$ then there exists $n \in \mathbb{Z}_+$ such that*
$$nx > y.$$
- *If $x \in \mathbb{R}$ then there exists a unique $m \in \mathbb{Z}$ such that*
$$m \in (x - 1, x] \quad \big(\text{or equivalently } x \in [m, m + 1)\big).$$
- *If $x, y \in \mathbb{R}$ and $x < y$ then there exists a $q \in \mathbb{Q}$ such that*
$$x < q < y.$$

Remark. The first assertion above is called the *Archimedean property* of $\mathbb{R}$, the second is a statement about the uniform distribution of the integers, while the third asserts that $\mathbb{Q}$ is *dense* in $\mathbb{R}$ — i.e. that between any two reals lies a rational.

Proof. Suppose the first assertion is false. Then the set $S = \{nx : n \in \mathbb{N}\}$ is bounded above by y. By the least upper bound property S has a supremum. Let $z = \sup\{S\}$. Because $x > 0$ one has that $z - x < z$. Hence, $z - x$ is not an upper bound for S because $z = \sup\{S\}$. This implies there exists some $n \in \mathbb{N}$ such that $z - x < nx$. But then $z < (n+1)x$, which contradicts the fact z is an upper bound of S. Therefore the first assertion holds.

To prove the second assertion, by the first assertion there exists $k, l \in \mathbb{Z}_+$ such that $-x < k$ and $x < l$. It follows that $-k < x < l$. Because

$$x \in (-k, l) \subset \bigcup_{m=-k}^{l-1} [m.m+1)\,,$$

there exists some $m \in \mathbb{Z}$ such that

$$-k \leq m < l \qquad \text{and} \qquad m \leq x < m+1\,.$$

It then follows that $m \in (x-1, x]$. To prove uniqueness, suppose that $m, n \in \mathbb{Z}$ and $m, n \in (x-1, x]$. Without loss of generality we may suppose that $m \leq n$ — otherwise simply exchange m and n. Because $x - 1 < m \leq n \leq x$, we see that $n - m \in \mathbb{Z}$ satisfies

$$0 \leq n - m < x - (x-1) = 1\,.$$

It follows that $n - m = 0$, thereby establishing the uniqueness in the second assertion.

To prove the third assertion, because $y - x > 0$, by the first assertion there exists $n \in \mathbb{Z}_+$ such that $n(y-x) > 1$. Then by the second assertion there exists a unique $m \in (nx, nx+1]$. Combining these facts yields

$$nx < m \leq nx + 1 < nx + n(y-x) = ny\,.$$

Because n is positive, we conclude that

$$x < \frac{m}{n} < y\,.$$

Therefore the third assertion holds with $q = m/n$. □

1.5.3. *Rational Powers.* Recall that we showed that $y^2 = 2$ had no solution within $\mathbb{Q}$. One of the most important consequences of the fact $\mathbb{R}$ has least upper bound property is the existence of solutions to such equations.

Proposition 1.16. *For every $x \in \mathbb{R}_+$ and every $n \in \mathbb{Z}_+$ there exists a unique $y \in \mathbb{R}_+$ such that $y^n = x$.*

Proof. The uniqueness of such a y is clear because if $y < z$ then $y^n < z^n$. So we only have to show such a y exists. Consider the sets

$$S = \left\{r \in \mathbb{R}_+ : r^n < x\right\}, \qquad \tilde{S} = \left\{r \in \mathbb{R}_+ : r^n > x\right\}.$$

The set S is nonempty because $s = x/(1+x) < 1$ implies $s^n < s < x$, whereby $s \in S$. The set $\tilde{S}$ is nonempty because $1 + x \in \tilde{S}$. One can show that every point in $\tilde{S}$ is an upper bound for S. Let $y = \sup\{S\}$. Then (by trichotomy) either $y \in S$, $y \in \tilde{S}$, or $y^n = x$. We will show that the first two cases cannot occur, which will thereby prove the theorem. More specifically, we

will show that no point in S is an upper bound of S, and that no point in $\tilde{S}$ is the least upper bound of S.

Let $p \in S$. We can construct a $q \in S$ such that $p < q$, thereby showing that p is not an upper bound of S. This can be done by taking one iteration of Newton's method applied to $f(r) = 1 - x/r^n = 0$. The details are left as an exercise.

Now suppose $p \in \tilde{S}$. We can construct a $q \in \tilde{S}$ such that $q < p$, thereby showing that p is not a least upper bound of S. This can be done by taking one iteration of Newton's method and applied to $f(r) = r^n - x = 0$. The details are left as an exercise. $\square$

The number y asserted in Proposition 1.16 is written $x^{\frac{1}{n}}$. You can then show that $(x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$ for every $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$. By setting $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$, we can therefore define x^p for every $x \in \mathbb{R}_+$ and $p \in \mathbb{Q}$. One can then show the following.

Proposition 1.17. *Let $x, y \in \mathbb{R}_+$ and $p, q \in \mathbb{Q}$. Then $x^p > 0$ and*

(a) $x^{p+q} = x^p x^q$,
(b) $(xy)^p = x^p y^p$,
(c) $x^{pq} = (x^p)^q$, *and* $(x^p)^{-1} = (x^{-1})^p$.

Proof. Exercise.

1.6. **Extended Real Numbers.** It is often convenient to extend the real numbers $\mathbb{R}$ by appending two elements designated $-\infty$ and ∞. This enlarged set is called the extended real numbers and is denoted by $\mathbb{R}_{\text{ex}}$.

The order $<$ on $\mathbb{R}$ is extended to $\mathbb{R}_{\text{ex}}$ by defining

$$-\infty < x < \infty \quad \text{for every } x \in \mathbb{R}\,.$$

The ordered set $(\mathbb{R}_{\text{ex}}, <)$ has the property that ∞ $(-\infty)$ is an upper (lower) bound for every subset of $\mathbb{R}_{\text{ex}}$. It also has the least upper bound property. Indeed, the supremum of any $S \subset \mathbb{R}_{\text{ex}}$ is given by

$$\sup\{S\} = \begin{cases} \infty & \text{if } S \cap \mathbb{R} \text{ has no upper bound in } \mathbb{R} \text{ or } \infty \in S\,, \\ -\infty & \text{if } S = \{-\infty\} \text{ or } S = \emptyset\,, \\ \sup\{S \cap \mathbb{R}\} & \text{otherwise}\,, \end{cases}$$

where $\emptyset$ denotes the empty set. In particular, every $S \subset \mathbb{R}$ that has no upper bound in $\mathbb{R}$ (and therefore no supremum in $\mathbb{R}$) has $\sup\{S\} = \infty$ in $\mathbb{R}_{\text{ex}}$. Similar statements hold for lower bounds and infimums.

The operations of addition and multiplication on R cannot be extended so as to make $\mathbb{R}_{\text{ex}}$ into a field. It is however natural to extend addition by defining for every $x \in \mathbb{R}$

$$x + \infty = \infty + x = \infty\,, \qquad x - \infty = -\infty + x = -\infty\,,$$

and by defining

$$\infty + \infty = \infty\,, \qquad -\infty - \infty = -\infty\,,$$

while leaving $\infty - \infty$ and $-\infty + \infty$ undefined. In particular, $-\infty$ and ∞ do not have additive inverses. Similarly, it is natural to extend multiplication by defining for every nonzero $x \in \mathbb{R}$

$$x\,\infty = \infty\,x = \begin{cases} \infty & \text{if } x > 0\,, \\ -\infty & \text{if } x < 0\,, \end{cases} \qquad x\,(-\infty) = (-\infty)\,x = \begin{cases} -\infty & \text{if } x > 0\,, \\ \infty & \text{if } x < 0\,, \end{cases}$$

and by defining

$$\infty\,\infty = (-\infty)\,(-\infty) = \infty\,, \qquad\qquad \infty\,(-\infty) = (-\infty)\,\infty = -\infty\,,$$

while leaving $0\,\infty$, $\infty\,0$, $0\,(-\infty)$, and $(-\infty)\,0$ undefined. In particular, $-\infty$ and ∞ do not have multiplicative inverses.

Interval notation extends naturally to $\mathbb{R}_{\text{ex}}$. Given any $a \in \mathbb{R}_{\text{ex}}$ we define $[a, a] = \{a\}$. Given any a, $b \in \mathbb{R}_{\text{ex}}$ such that $a < b$ we define the sets

$$\begin{aligned} (a,b) &= \big\{x \in \mathbb{R}_{\text{ex}} \,:\, a < x < b\big\}\,, \qquad & [a,b) &= \big\{x \in \mathbb{R}_{\text{ex}} \,:\, a \le x < b\big\}\,, \\ (a,b] &= \big\{x \in \mathbb{R}_{\text{ex}} \,:\, a < x \le b\big\}\,, \qquad & [a,b] &= \big\{x \in \mathbb{R}_{\text{ex}} \,:\, a \le x \le b\big\}\,. \end{aligned} \tag{1.4}$$

When these sets are contained within $\mathbb{R}$ the notation coincides with the interval notation we introduced earlier. We therefore call these sets *intervals* too. The new intervals are the ones that contain either $-\infty$ or ∞ — namely, ones that have the form $[-\infty, b)$, $[-\infty, b]$, $(a, \infty]$, or $[a, \infty]$ for some a, $b \in \mathbb{R}_{\text{ex}}$. In particular, one has $\mathbb{R}_{\text{ex}} = [-\infty, \infty]$.

We will often use the following characterization of intervals.

Proposition 1.18. (Interval Characterization Theorem.) *A set $S \subset \mathbb{R}_{\text{ex}}$ is an interval if and only if it has the property that*

$$\forall x, y \in \mathbb{R} \qquad x, y \in S \quad \textit{and} \quad x < y \qquad \Longrightarrow \qquad (x, y) \subset S\,. \tag{1.5}$$

Proof. ($\Longrightarrow$) It is clear from (1.4) that if S is an interval then it has property (1.5). In particular, the empty set and every singleton set (a set with only a single point in it) have property (1.5).

($\Longleftarrow$) The empty set and every singleton set is an interval. So we only have to consider sets that contain at least two points.

Let $S \subset \mathbb{R}_{\text{ex}}$ contain at least two points and have property (1.5). Because $\mathbb{R}_{\text{ex}}$ has the least upper bound property, while every subset of $\mathbb{R}_{\text{ex}}$ is bounded, we can set $a = \inf\{S\}$ and $b = \sup\{S\}$. You should be able to argue that $a < b$ because S has at least two points in it.

First, we show that $(a, b) \subset S$. Let $z \in (a, b)$. We claim that there exists $x \in (a, z)$ and $y \in (z, b)$ such that x, $y \in S$. (Otherwise z is either a lower or upper bound for S, which contradicts either $a = \inf\{S\}$ or $a = \sup\{S\}$.) Hence, property (1.5) implies that $z \in (x, y) \subset S$. Therefore, $(a, b) \subset S$.

Next, we claim that if $x < a$ or $x > b$ then $x \notin S$ because that would contradict either $a = \inf\{S\}$ or $a = \sup\{S\}$. When this fact is combined with the fact that $(a, b) \subset S$ it follows that S is an interval with

$$S = \begin{cases} (a,b) & \text{if } a \notin S \text{ and } b \notin S\,, \\ [a,b) & \text{if } a \in S \text{ and } b \notin S\,, \\ (a,b] & \text{if } a \notin S \text{ and } b \in S\,, \\ [a,b] & \text{if } a \in S \text{ and } b \in S\,, \end{cases}$$

where the sets (a, b), $[a, b)$, $(a, b]$, and $[a, b]$ are defined by (1.4). $\square$

2. Sequences of Real Numbers

2.1. **Sequences and Subsequences.** Sequences play a central role in analysis. We introduce them here in the context of an arbitrary set X before specializing to sets of real numbers.

Definition 2.1. *A* sequence *in a set X is a map from $\mathbb{N}$ into X, often denoted $\{x_k\}$ or $\{x_k\}_{k\in\mathbb{N}}$, where $k \mapsto x_k$ maps the index $k \in \mathbb{N}$ to the point $x_k \in X$.*

Remark. Any countable ordered set may be used as the index set instead of $\mathbb{N}$.

When X is an ordered set, sequences that either preserve or reverse order are special.

Definition 2.2. *Let $(X, <)$ be an ordered set. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in X is called*

$$\begin{aligned} \text{increasing} \quad & \textit{whenever } x_l > x_k \textit{ for every } k, l \in \mathbb{N} \textit{ with } l > k\,, \\ \text{nondecreasing} \quad & \textit{whenever } x_l \geq x_k \textit{ for every } k, l \in \mathbb{N} \textit{ with } l > k\,, \\ \text{decreasing} \quad & \textit{whenever } x_l < x_k \textit{ for every } k, l \in \mathbb{N} \textit{ with } l > k\,, \\ \text{nonincreasing} \quad & \textit{whenever } x_l \leq x_k \textit{ for every } k, l \in \mathbb{N} \textit{ with } l > k\,. \end{aligned}$$

It is called monotonic *if it is either nondecreasing or nonincreasing.*

When dealing with sequences, it is convenient to introduce the concepts of *eventually* and *frequently*.

Definition 2.3. *Let $\mathcal{A}(x)$ be any assertion about any $x \in X$. (For example, $\mathcal{A}(x)$ could be the assertion "$x \in S$" for a given $S \subset X$.) Let $\{x_k\}$ be a sequence in X. Then one says:*

$$\text{“}\mathcal{A}(x_k) \text{ eventually } \textit{as } k \to \infty\text{” when } \exists m \in \mathbb{N} \textit{ such that } \forall k \geq m\ \mathcal{A}(x_k)\,;$$

$$\text{“}\mathcal{A}(x_k) \text{ frequently } \textit{as } k \to \infty\text{” when } \forall m \in \mathbb{N}\ \exists k \geq m \textit{ such that } \mathcal{A}(x_k)\,.$$

When there is no possible confusion as to the index set, one says simply "$\mathcal{A}(x_k)$ eventually" or "$\mathcal{A}(x_k)$ frequently", dropping the "as $k \to \infty$".

Exercise. Show that $2^{-k} < .001$ eventually.

Exercise. Let $\{x_k\}$ be a sequence in X. Let $\mathcal{A}(x)$ be any assertion about any $x \in X$ and let $\sim \mathcal{A}(x)$ be its negation. Show that the negation of "$\mathcal{A}(x_k)$ eventually" is "$\sim \mathcal{A}(x_k)$ frequently".

Exercise. Show that $\cos(k) > .5$ frequently, but not eventually.

Another useful concept is that of a *subsequence*.

Definition 2.4. *A* subsequence *$\{x_{n_k}\}_{k\in\mathbb{N}}$ of a sequence $\{x_k\}_{k\in\mathbb{N}}$ in a set X is a map from $\mathbb{N}$ into X given by $k \mapsto x_{n_k}$, where $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence in $\mathbb{N}$.*

Example. If $\{x_k\}$ is a sequence in a set X, then $\{x_{2k}\}$ is the subsequence with indices that are even, while $\{x_{k^3}\}$ is the subsequence with indices that are cubes.

Exercise. Consider the sequence $\{2^k\}$. Write out the first three terms (i.e. $k = 0, 1, 2$) in the subsequences $\{2^{3k}\}$ and $\{2^{2k+1}\}$.

In an ordered set, subsequences of monotonic sequences are again monotonic.

Proposition 2.1. *Let $(X, <)$ be an ordered set. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in X that is increasing (nondecreasing, decreasing, nonincreasing). Then every subsequence of $\{x_k\}_{k\in\mathbb{N}}$ is also increasing (nondecreasing, decreasing, nonincreasing).*

Proof. Exercise.

You should test your understanding of the concepts in this section by proving the following.

Proposition 2.2. *Let X be a set. Let $\mathcal{A}(x)$ be any assertion about any $x \in X$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then $\mathcal{A}(x_k)$ frequently as $k \to \infty$ if and only if there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\mathcal{A}(x_{n_k})$ eventually as $k \to \infty$.*

Proof. Exercise.

2.2. **Convergence and Divergence.** The most important concept related to sequences is that of convergence. Here we see it in the context of real sequences.

Definition 2.5. *A sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is said to* converge *or is said to be* convergent *whenever there exists a point $a \in \mathbb{R}$ such that for every $\epsilon > 0$ one has that*

$$|a_k - a| < \epsilon \text{ eventually as } k \to \infty\,. \tag{2.1}$$

This is denoted as

$$a_k \to a \text{ as } k \to \infty\,,$$

or as

$$\lim_{k\to\infty} a_k = a\,.$$

One then says that the sequence converges to a. A sequence that does not converge is said to diverge *or is said to be* divergent.

An immediate consequence of this definition and ideas from the previous section is the following proposition.

Proposition 2.3. *A sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ diverges if and only if for every $a \in \mathbb{R}$ there exists an $\epsilon_a > 0$ such that*

$$|a_k - a| \geq \epsilon_a \text{ frequently as } k \to \infty\,.$$

Proof. Exercise.

Definition 2.5 does not assert that there is a unique number a that satisfies (2.1). The following proposition establishes this and more.

Proposition 2.4. *If a sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ converges, there is a unique point in $\mathbb{R}$ to which it converges. Moreover, the set $\{a_k\} \subset \mathbb{R}$ is bounded.*

Proof. Here we prove only the boundedness assertion. The proof of the uniqueness assertion is left as an exercise.

Let $\{a_k\}_{k\in\mathbb{N}}$ converge to $a \in \mathbb{R}$. Then $\exists m \in \mathbb{N}$ such that $\forall k \geq m$ $|a_k - a| < 1$. In particular, for every $k \geq m$ one has that $a - 1 < a_k < a + 1$. Then for every $k \in \mathbb{N}$ we have

$$|a_k| < 1 + \max\left\{|a_0|, |a_1|, \ldots, |a_{m-1}|, |a|\right\}.$$

The sequence $\{a_k\}_{k\in\mathbb{N}}$ is therefore bounded. □

Definition 2.6. *The unique point to which a convergent sequence in $\mathbb{R}$ converges is called the* limit *of the sequence.*

An important characterization of the limit of a convergent sequence is given by the following.

Proposition 2.5. *Let $\{b_k\}_{k\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$. Let $b \in \mathbb{R}$. Then*

$$\lim_{k\to\infty} b_k = b\,,$$

if and only if

$$a < b < c \qquad \Longrightarrow \qquad a < b_k < c \quad \textit{eventually}\,.$$

Proof. Exercise.

It is fairly easy to check that subsequences of a convergent sequence are also convergent, and have the same limit.

Proposition 2.6. *If a sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ converges to a limit $a \in \mathbb{R}$ then every subsequence of $\{a_k\}_{k\in\mathbb{N}}$ also converges to a.*

Proof. Exercise.

The main theorem regarding algebraic operations, order, and limits is the following.

Proposition 2.7. *Let $\{a_k\}$ and $\{b_k\}$ be convergent sequences in $\mathbb{R}$ with $a_k \to a$ and $b_k \to b$ as $k \to \infty$. Then*

$$\begin{aligned}
&\textit{(i)} && (a_k + b_k) \to (a + b)\,,\\
&\textit{(ii)} && -a_k \to -a\,,\\
&\textit{(iii)} && a_k b_k \to ab\,,\\
&\textit{(iv)} && 1/a_k \to 1/a \quad \textit{provided no division by zero occurs}\,.
\end{aligned}$$

Moreover, if $a_k \le b_k$ frequently then $a \le b$. (Equivalently, if $a < b$ then $a_k < b_k$ eventually.)

Proof. Exercise.

When working with real sequences, it is useful to distinguish two of the many ways in which a sequence might diverge — namely, those when the sequence "approaches" either ∞ or $-\infty$.

Definition 2.7. *A sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is said to* diverge *to ∞ (to $-\infty$) if for every $b \in \mathbb{R}$ one has that*

$$a_k > b \ \textit{eventually} \ \ (a_k < b \ \textit{eventually}) \ \ \textit{as } k \to \infty.$$

This is denoted as

$$a_k \to \infty \ \ (a_k \to -\infty) \ \ \textit{as } k \to \infty\,,$$

or as

$$\lim_{k\to\infty} a_k = \infty \quad \left(\lim_{k\to\infty} a_k = -\infty\right)\,.$$

One then says that the sequence approaches ∞ (approaches $-\infty$).

2.3. **Monotonic Sequences.** For monotonic sequences the least upper bound property can be employed to show the existence of limits.

Proposition 2.8. (Monotonic Sequence Theorem) *Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ that is nondecreasing (nonincreasing). Then it converges if and only if it is bounded above (bounded below). Moreover, if it converges then*

$$\lim_{k\to\infty} a_k = \sup\{a_k\} \quad \left(\lim_{k\to\infty} a_k = \inf\{a_k\}\right) ,$$

while if it diverges then

$$\lim_{k\to\infty} a_k = \infty \quad \left(\lim_{k\to\infty} a_k = -\infty\right) .$$

Proof. We give the proof for the nondecreasing case only; the nonincreasing case goes similarly. The proof of the assertion in the case of divergence in is left as an exercise.

($\Rightarrow$) This follows from Proposition 2.4, which states that every convergent sequence is bounded.

($\Leftarrow$) Because $\{a_k\}$ is bounded above, we can set $a = \sup\{a_k\,;\, k \in \mathbb{N}\}$ by the least upper bound property. We claim that $a_k \to a$ as $k \to \infty$. Let $\epsilon > 0$ be arbitrary. There exists some $m_\epsilon \in \mathbb{N}$ such that $0 \le a - a_{m_\epsilon} < \epsilon$. (For if not, it would mean that $a_k \le a - \epsilon$ for every $k \in \mathbb{N}$, which would contradict the definition of a.) But then for every $k > m_\epsilon$ one has that $0 \le a - a_k \le a - a_{m_\epsilon} < \epsilon$, which establishes the claim. □

For monotonic sequences it is enough to know what happens to a single subsequence.

Proposition 2.9. *Let $\{a_k\}$ be a monotonic sequence in $\mathbb{R}$. Then $\{a_k\}$ is convergent if and only if it has a convergent subsequence.*

Proof. Exercise.

The Monotonic Sequence Theorem has the following consequence, often attributed to Cantor.

Proposition 2.10. (Nested Interval Theorem.) *Let $\{[a_k, b_k]\}_{k\in\mathbb{N}}$ be a sequence of closed, bounded intervals in $\mathbb{R}$ that is nested in the sense that*

$$[a_{k+1}, b_{k+1}] \subset [a_k, b_k] \quad \textit{for every } k \in \mathbb{N} .$$

Then the sequences $\{a_k\}$ and $\{b_k\}$ converge and

$$\bigcap_{k\in\mathbb{N}} [a_k, b_k] = [a, b] , \quad \textit{where} \quad a = \lim_{k\to\infty} a_k , \quad b = \lim_{k\to\infty} b_k , \quad \textit{with } a \le b .$$

Proof. Because the sequence of intervals $\{[a_k, b_k]\}$ is nested, it follows that the sequence $\{a_k\}$ is nondecreasing while the sequence $\{b_k\}$ is nonincreasing. Hence, for every $m, n \in \mathbb{N}$ we have $a_m \le a_n \le b_n$ when $m \le n$ and $a_m \le b_m \le b_n$ when $m \ge n$. This implies the sequence $\{a_k\}$ is bounded above by every b_n, while the sequence $\{b_k\}$ is bounded below by every a_n. The Monotone Sequence Theorem then implies that the sequences $\{a_k\}$ and $\{b_k\}$ converge with

$$a = \lim_{k\to\infty} a_k = \sup_k\{a_k\} \le b_n , \qquad b = \lim_{k\to\infty} b_k = \inf_k\{b_k\} \ge a_n .$$

Because $a_k \le b_k$ for every $k \in \mathbb{N}$ it follows that $a \le b$.

If $x \in [a, b]$ then because $[a, b] \subset [a_k, b_k]$ for every $k \in \mathbb{N}$, it follows that $x \in \cap_k [a_k, b_k]$. If $x < a$ then because $a_k \to a$, it follows that $x < a_k$ eventually, whereby $x \notin \cap_k [a_k, b_k]$. Similarly, if $b < x$ then because $b_k \to b$, it follows that $b_k < x$ eventually, whereby $x \notin \cap_k [a_k, b_k]$. □

Remark. This theorem shows the above intersection of nested intervals is always nonempty. In particular, when $a = b$ this intersection consists of a single point. To better appreciate significance of this result, you should do the following exercise.

Exercise. Show that a nested sequence of closed, bounded intervals in $\mathbb{Q}$ can have an empty intersection.

2.4. **Limit Superior and Limit Inferior.** The power of the Monotonic Sequence Theorem (Proposition 2.8) lies in the fact that from every real sequence $\{a_k\}_{k\in\mathbb{N}}$ that is bounded above (bounded below), we can construct a nonincreasing (nondecreasing) sequence from its "tails". Specifically, we construct the sequence $\{\overline{a}_k\}$ ($\{\underline{a}_k\}$) with elements defined by

$$\overline{a}_k = \sup\big\{a_l \,:\, l \geq k\big\} \quad \big(\underline{a}_k = \inf\big\{a_l \,:\, l \geq k\big\}\big)\ .$$

This sequence is clearly nonincreasing (nondecreasing). The convergence of such sequences is characterized by the Monotonic Sequence Theorem, which motivates the following definition.

Definition 2.8. *For every sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$, define its* limit superior *and* limit inferior *by*

$$\limsup_{k\to\infty} a_k \equiv \begin{cases} \lim\limits_{k\to\infty} \overline{a}_k & \textit{if } \sup\{a_k\} < \infty\,, \\ \infty & \textit{otherwise}\,; \end{cases}$$

$$\liminf_{k\to\infty} a_k \equiv \begin{cases} \lim\limits_{k\to\infty} \underline{a}_k & \textit{if } \inf\{a_k\} > -\infty\,, \\ -\infty & \textit{otherwise}\,. \end{cases}$$

These are called simply the "$\limsup$*" and "*$\liminf$*" for short.*

Remark. By the Monotonic Sequence Theorem (Proposition 2.8) we have that

$$\limsup_{k\to\infty} a_k \equiv \begin{cases} \inf\{\overline{a}_k\} & \text{if } \sup\{a_k\} < \infty\,, \\ \infty & \text{otherwise}\,; \end{cases}$$

$$\liminf_{k\to\infty} a_k \equiv \begin{cases} \sup\{\underline{a}_k\} & \text{if } \inf\{a_k\} > -\infty\,, \\ -\infty & \text{otherwise}\,. \end{cases}$$

Example. Consider the sequence $\{a_k\}$ given by

$$a_k = (-1)^k\,\frac{k+1}{k} \quad \text{for } k \in \mathbb{Z}_+\,.$$

The first eight terms of the sequences $\{\overline{a}_k\}$, $\{a_k\}$, and $\{\underline{a}_k\}$ are

$\overline{a}_k$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{9}{8}$	$\frac{9}{8}$
a_k	-2	$\frac{3}{2}$	$-\frac{4}{3}$	$\frac{5}{4}$	$-\frac{6}{5}$	$\frac{7}{6}$	$-\frac{8}{7}$	$\frac{9}{8}$
$\underline{a}_k$	-2	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{6}{5}$	$-\frac{6}{5}$	$-\frac{8}{7}$	$-\frac{8}{7}$	$-\frac{10}{9}$

Notice that $\{a_k\}$ diverges while $\{\overline{a}_k\}$ and $\{\underline{a}_k\}$ are both monotonic and converge to 1 and -1 respectively. Therefore

$$\limsup_{k\to\infty} a_k = 1\,, \qquad \liminf_{k\to\infty} a_k = -1\,.$$

Remark. Notice that, unlike the limit, the lim sup and lim inf are defined for every real sequence, taking values in $\mathbb{R}_{\text{ex}}$, and that in general

$$-\infty \le \liminf_{k\to\infty} a_k \le \limsup_{k\to\infty} a_k \le \infty\,.$$

Example. The sequence $\{(-1)^k k\}$ is neither bounded above nor bounded below. Therefore

$$\limsup_{k\to\infty}(-1)^k k = \infty\,, \qquad \liminf_{k\to\infty}(-1)^k k = -\infty\,.$$

The key to learning how to use lim sup and lim inf is an understanding of the following characterizations. These should be compared with the characterization of the limit of a convergent sequence given by Proposition 2.5

Proposition 2.11. *Let $\{b_k\}$ be a sequence in $\mathbb{R}$. Let $b \in \mathbb{R}$. Then*

$$\limsup_{k\to\infty} b_k = b \qquad \Big(\liminf_{k\to\infty} b_k = b\Big)\,, \tag{2.2}$$

if and only if

$$b < c \quad \Longrightarrow \quad b_k < c \quad \textit{eventually (frequently)}\,, \tag{2.3}$$

and

$$a < b \quad \Longrightarrow \quad a < b_k \quad \textit{frequently (eventually)}\,. \tag{2.4}$$

Proof. We give the proof of the lim sup assertion. The lim inf assertion is proved similarly.
($\Rightarrow$) Suppose that (2.2) holds. Let $a < b$ and $c > b$. Then because

$$b = \limsup_{k\to\infty} b_k = \lim_{k\to\infty} \overline{b}_k\,,$$

where $\overline{b}_k = \sup\{b_l \,:\, l \ge k\}$, it follows that

$$a < \overline{b}_k < c \qquad \text{eventually}\,.$$

Because $b_k \le \overline{b}_k$ for every k, we see directly that $b_k < c$ eventually, whereby (2.3) holds. Moreover, $a < \overline{b}_k$ implies that for some $l \ge k$ one has $a < b_l$. (Otherwise a would be an upper bound for the set $\{b_l \,:\, l \ge k\}$, which contradicts the fact b is the least upper bound of this set.) Hence, $a < b_k$ frequently, whereby (2.4) holds.
($\Leftarrow$) Suppose that (2.3) and (2.4) hold. Let $a < b$ and $c > b$ be arbitrary. Then (2.3) implies that

$$\overline{b}_k = \sup\{b_l \,:\, l \ge k\} \le c \quad \text{eventually}\,,$$

while (2.4) implies that

$$a < \overline{b}_k = \sup\{b_l \,:\, l \ge k\} \quad \text{eventually}\,.$$

(If for some k one had $\overline{b}_k \le a$ then $b_l \le a$ for every $l \ge k$, which contradicts (2.4).) Thus, we see

$$a \le \inf\{\overline{b}_k\} \le c\,.$$

But $a < b$ and $c > b$ were arbitrary, so that

$$\limsup_{k\to\infty} b_k = \inf\{\overline{b}_k\} = b\,,$$

whereby (2.2) holds. $\square$

Convergent sequences may be characterized in terms of their lim sup and lim inf as follows.

Proposition 2.12. *Let $\{a_k\}$ be a sequence in $\mathbb{R}$. Then $\{a_k\}$ converges if and only if*

$$-\infty < \liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k < \infty\,,$$

in which case

$$\lim_{k\to\infty} a_k = \liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k\,.$$

Proof. Exercise. Hint: Use Propositions 2.5 and 2.11.

When adding and comparing sequences, lim sup and lim inf generally behave as follows.

Proposition 2.13. *Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Then*

$$\limsup_{k\to\infty}(a_k + b_k) \le \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,,$$
$$\liminf_{k\to\infty}(a_k + b_k) \ge \liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k\,,$$

whenever the sum on the right-hand side is defined. Moreover, if $a_k \le b_k$ eventually then

$$\limsup_{k\to\infty} a_k \le \limsup_{k\to\infty} b_k\,, \qquad \liminf_{k\to\infty} a_k \le \liminf_{k\to\infty} b_k\,.$$

Proof. Exercise.

When multiplying sequences, lim sup and lim inf generally behave as follows.

Proposition 2.14. *Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. If $\{a_k\}$ is convergent with*

$$\lim_{k\to\infty} a_k = a > 0\,,$$

then

$$\limsup_{k\to\infty} a_k b_k = a \limsup_{k\to\infty} b_k\,, \qquad \liminf_{k\to\infty} a_k b_k = a \liminf_{k\to\infty} b_k\,.$$

Proof. Exercise.

When they converge, the lim sup and lim inf of a sequence are actually limits of some of its subsequences.

Proposition 2.15. *Let $\{a_k\}$ be a sequence in $\mathbb{R}$. If $\{a_{n_k}\}$ is any subsequence of $\{a_k\}$ then*

$$\liminf_{k\to\infty} a_k \le \liminf_{k\to\infty} a_{n_k} \le \limsup_{k\to\infty} a_{n_k} \le \limsup_{k\to\infty} a_k\,.$$

Moreover, there exist subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ such that

$$\lim_{k\to\infty} a_{n_k} = \limsup_{k\to\infty} a_k\,, \qquad \lim_{k\to\infty} a_{m_k} = \liminf_{k\to\infty} a_k\,.$$

Proof. Exercise. Hint: Use Propositions 2.5 and 2.11.

In particular, the following famous theorem of Bolzano and Weierstrass is a consequence.

Proposition 2.16. (Bolzano-Weierstrass Theorem) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $\{b_k\}$ be a bounded sequence in $\mathbb{R}$. This implies that there exists $[a, c] \subset \mathbb{R}$ such that $\{b_k\} \subset [a, c]$. Then

$$-\infty < a \le \liminf_{k\to\infty} b_k \le \limsup_{k\to\infty} b_k \le c < \infty\,.$$

The result then follows by Proposition 2.15. $\square$

2.5. **Cauchy Criterion.** When a sequence is monotonic, just knowing that it is bounded tells you that it is convergent. When a sequence is not monotonic, determining whether it is convergent or divergent is generally much harder. For example, to establish convergence directly from Definition 2.5 you must first know the limit of the sequence. Cauchy introduced a criterion for convergence that does not require knowledge of the limit.

Definition 2.9. *A sequence* $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ *is said to be* Cauchy *whenever for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}$ *such that*

$$k, l \geq N_\epsilon \quad \Longrightarrow \quad |a_k - a_l| < \epsilon\,. \tag{2.5}$$

In other words, a sequence is Cauchy if for every $\epsilon > 0$ one can find a tail of the sequence such that any two terms in the tail are within ϵ of each other. Roughly speaking, a Cauchy sequence is one whose terms generally get closer together.

The main result of this section is the so-called Cauchy criterion for convergence — namely, that a sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy. The easier half of this criterion is established by the following.

Proposition 2.17. *A convergent sequence in* $\mathbb{R}$ *is Cauchy.*

Proof. Let $\{a_k\}$ be a convergent sequence in $\mathbb{R}$ with limit a. Let $\epsilon > 0$. Then by the definition of convergence there exists $N_\epsilon \in \mathbb{N}$ such that

$$k \geq N_\epsilon \quad \Longrightarrow \quad |a_k - a| < \frac{\epsilon}{2}\,.$$

It follows from the triangle inequality that if $k, l \geq N_\epsilon$ then

$$|a_k - a_l| = \big|(a_k - a) + (a - a_l)\big| \leq |a_k - a| + |a_l - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

Hence, the sequence $\{a_k\}$ is Cauchy. □

We now take the first step toward establishing the harder half of the Cauchy criterion.

Proposition 2.18. *A Cauchy sequence in* $\mathbb{R}$ *is bounded.*

Proof. The proof is very similar to the proof that a convergent sequence is bounded. It is left as an exercise. □

We are now ready to establish the Cauchy Criterion.

Proposition 2.19. (Cauchy Criterion) *A sequence in* $\mathbb{R}$ *is convergent if and only if it is Cauchy.*

Proof. Proposition 2.17 established that convergent sequences are Cauchy. We only need to establish the other direction.

Let $\{a_k\}$ be a Cauchy sequence in $\mathbb{R}$. By Proposition 2.18 the sequence $\{a_k\}$ is bounded. By the Bolzano-Weierstrass Theorem (Proposition 2.16) it has a convergent subsequence $\{a_{n_k}\}$. Let a be the limit of this convergent subsequence. We will use the fact $\{a_k\}$ is a Cauchy sequence to show that it converges to a.

Let $\epsilon > 0$. Because the subsequence $\{a_{n_k}\}$ converges to a while the sequence $\{a_k\}$ is Cauchy, there exists an $N_\epsilon \in \mathbb{N}$ such that

$$k \geq N_\epsilon \quad \Longrightarrow \quad |a_{n_k} - a| < \frac{\epsilon}{2}\,,$$

and

$$k, l \geq N_\epsilon \quad \Longrightarrow \quad |a_k - a_l| < \frac{\epsilon}{2}\,.$$

Because $k \geq N_\epsilon$ implies that $n_k \geq N_\epsilon$, the line above implies that

$$k \geq N_\epsilon \quad \Longrightarrow \quad |a_k - a_{n_k}| < \frac{\epsilon}{2}\,.$$

It follows from the triangle inequality that if $k \geq N_\epsilon$ then

$$|a_k - a| \leq |a_k - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

The sequence $\{a_k\}$ therefore converges to a. $\square$

3. Series of Real Numbers

3.1. **Infinite Series.** Any finite set of real numbers can be summed. Here we study one way to make sense of the sum of an infinite sequence of real numbers.

Definition 3.1. *Given any real sequence* $\{a_k\}_{k=0}^{\infty}$*, for every* $m, n \in \mathbb{N}$ *with* $m \leq n$ *define the* sigma notation*:*

$$\sum_{k=m}^{n} a_k \equiv a_m + a_{m+1} + \cdots + a_{n-1} + a_n \,.$$

Associate with the sequence of terms $\{a_k\}$ *the so-called sequence of* partial sums $\{s_n\}$ *defined by*

$$s_n \equiv \sum_{k=0}^{n} a_k \,.$$

It is convenient to encode $\{s_n\}$ *with the formal infinite series*

$$\sum_{k=0}^{\infty} a_k \,.$$

If the sequence $\{s_n\}$ *converges to a limit* s *then we say that the series* converges*, and that* s *is the* sum *of the series. In that case we write*

$$\sum_{k=0}^{\infty} a_k = s \,.$$

If the sequence $\{s_n\}$ *diverges then we say that the series* diverges*.*

Remark. It is clear that changing, adding, or removing a finite number of terms in a series does not affect whether the series converges or diverges, but if it converges, the sum would almost always be affected. For example,

$$\sum_{k=0}^{\infty} a_k \quad \text{converges} \qquad \Longleftrightarrow \qquad \sum_{k=5}^{\infty} a_k \quad \text{converges}\,,$$

but when they do converge the sums will generally differ — namely,

$$\text{in general} \quad \sum_{k=0}^{\infty} a_k \neq \sum_{k=5}^{\infty} a_k \,.$$

More specifically, these sums will be equal if and only if $a_0 + a_1 + a_2 + a_3 + a_4 = 0$.

Example. Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \,.$$

The nth partial sum is given by

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2} \right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n} \right) - \frac{1}{n+1} = 1 - \frac{1}{n+1} \,. \end{aligned}$$

One then sees that the series is convergent with

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} s_n = 1 \,.$$

The previous example shows a series that can be put into so-called *telescoping form.*

Definition 3.2. *A formal inifinite series with terms $\{a_k\}_{k=m}^{\infty}$ is said to be in* telescoping form *if $a_k = c_{k-1} - c_k$ for some sequence $\{c_k\}_{k=m-1}^{\infty}$, so that the series is expressed as*

$$\sum_{k=m}^{\infty} (c_{k-1} - c_k) \,. \tag{3.1}$$

If a series can be put into telescoping form with a sequence $\{c_k\}$ that is known explicitly then the convergence or divergence of the series can be easily determined. Moreover, if it converges then its sum can be easily determined. This is because the sequence $\{c_k\}$ is simply related to the sequence of partial sums $\{s_n\}$. Indeed, for every $n \geq m$ one sees that

$$\begin{aligned} s_n &= \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c_{k-1} - c_k) \\ &= (c_{m-1} - c_m) + (c_m - c_{m+1}) + \cdots + (c_{n-2} - c_{n-1}) + (c_{n-1} - c_n) \\ &= c_{m-1} + (-c_m + c_m) + \cdots + (-c_{n-1} + c_{n-1}) - c_n \\ &= c_{m-1} - c_n \,. \end{aligned}$$

It follows immediately that the sequence $\{s_n\}$ converges if and only if the sequence $\{c_k\}$ converges, and that when these sequences converge one has that

$$\lim_{n\to\infty} s_n = c_{m-1} - \lim_{k\to\infty} c_k \,.$$

Hence, the following proposition holds.

Proposition 3.1. *Let $\{c_k\}_{k=m-1}^{\infty}$ be a real sequence. Then*

$$\sum_{k=m}^{\infty} (c_{k-1} - c_k) \quad \textit{converges} \qquad \Longleftrightarrow \qquad \{c_k\}_{k=m-1}^{\infty} \quad \textit{converges} \,.$$

Moreover, when these are convergent one has

$$\sum_{k=m}^{\infty} (c_{k-1} - c_k) = c_{m-1} - \lim_{k\to\infty} c_k \,.$$

Remark. The above considerations show that if a series is in the telescoping form (3.1) then there is a $c \in \mathbb{R}$ such that $c_k = c - s_k$ for every $k \geq m$, where $\{s_k\}_{k=m}^{\infty}$ is the sequence of partial sums. This means that finding an explicit telescoping form for a series is equivalent to finding an explicit expression for its partial sums. It should be clear that one can do this in only the rarest of cases.

Given a general infinite series, it is usually impossible to evaluate the limit of the sequence of partial sums. However, one can commonly determine whether a series is convergent or divergent without explicitly evaluating this limit. The following proposition gives the simplest test for divergence.

Proposition 3.2. (Divergence Test) *Let $\{a_k\}$ be a real sequence.*

If the series $\sum_{k=0}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k = 0$.

Equivalently, if $\lim_{k\to\infty} a_k \neq 0$ then the series $\sum_{k=0}^{\infty} a_k$ diverges.

Proof. The proof is based on the fact that the kth term in a formal infinite series can be expressed as $a_k = s_k - s_{k-1}$, where $s_{-1} = 0$ and $\{s_k\}_{k\in\mathbb{N}}$ is the sequence of partial sums. If the series converges then one knows that

$$\lim_{k\to\infty} s_k = s\,, \qquad \lim_{k\to\infty} s_{k-1} = s\,,$$

where s is the sum of the series. It thereby follows that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = s - s = 0\,. \quad \square$$

Remark. You can easily find examples of a series whose terms converge to zero, yet the series is divergent. One such example is the *harmonic series*:

$$\sum_{k=1}^{\infty} \frac{1}{k}\,.$$

Clearly $1/k \to 0$ as $k \to \infty$. However, we will soon show that this series diverges.

3.2. **Geometric Series.** An important example is that of geometric series.

Definition 3.3. *A formal infinite series of the form*

$$\sum_{k=0}^{\infty} ar^k$$

for some nonzero a and some $r \in \mathbb{R}$ is called a geometric *series.*

The convergence or divergence of a geometric series is easy to determine because it is one of those rare series where one can find an explicit experssion for its partial sums. For every $n \in \mathbb{N}$ let s_n denote the partial sum given by

$$s_n = \sum_{k=0}^{n} ar^k\,.$$

It is clear that if $r = 1$ then $s_n = (n+1)a$ and the series will diverge. So suppose that $r \neq 1$. One checks that

$$s_n - rs_n = \sum_{k=0}^{n} ar^k - \sum_{k=0}^{n} ar^{k+1} = \sum_{k=0}^{n} ar^k - \sum_{k=1}^{n+1} ar^k = a - ar^{n+1}\,,$$

whereby the partial sum s_n is found to be

$$s_n = \frac{a - ar^{n+1}}{1-r}\,.$$

By letting n tend to ∞ in this expression we find that

$$\sum_{k=0}^{\infty} ar^k = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1\,, \\ \text{diverges} & \text{otherwise}\,. \end{cases}$$

Remark. The fact the geometric series diverges when $|r| \geq 1$ can also be seen easily from the Divergence Test. Indeed, in that case you see that

$$\lim_{k\to\infty} ar^k = \begin{cases} a & \text{(and hence is nonzero) if } r = 1\,, \\ \text{diverges} & \text{(and hence is nonzero) if } |r| \geq 1 \text{ and } r \neq 1\,, \end{cases}$$

whereby the Divergence Test (Proposition 3.2) shows that the geometric series diverges. Of course, the Divergence Test does not show the geometric series converges when $|r| < 1$.

Exercise. Consider a formal infinite series of the form

$$\sum_{k=1}^{\infty} kr^k$$

for some $r \in \mathbb{R}$. Find all the values of r for which this series converges and evaluate the sum. (Hint: Find an explicit expression for the partial sums and evaluate the limit. The explicit expression may be derived from the analogous expression for a geometric series.)

3.3. **Series with Nonnegative Terms.** If the terms of an infinite series are nonnegative then the associated sequence of partial sums will be nondecreasing. Hence, the least upper bound property can be employed in the guise of the Monotonic Sequence Theorem (Proposition 2.8) to show the convergence or divergence of the series. Specifically, one has the following proposition, which lies at the heart of most proofs about the convergence or divergence of series with nonnegative terms.

Proposition 3.3. (Series with Nonnegative Terms Theorem) *Let $\{a_k\}_{k=m}^{\infty}$ be a nonnegative sequence. Then*

$$\sum_{k=m}^{\infty} a_k \quad \textit{converges} \qquad \Longleftrightarrow \qquad \{s_k\}_{k=m}^{\infty} \quad \textit{is bounded above}\,,$$

where $\{s_k\}_{k=m}^{\infty}$ is the sequence of partial sums associated with the formal infinite series.

Proof. One first shows that the sequence $\{s_k\}_{k=m}^{\infty}$ is nondecreasing. One then applies the Monotonic Sequence Theorem. The details are left as an exercise. $\square$

One way to establish whether or not a sequence of partial sums is bounded above is to compare it with a sequence for which the answer is known. This is often done with one of the following comparison tests.

Proposition 3.4. (Comparison Tests for Series with Nonnegative Terms) *Let $\{a_k\}$ and $\{b_k\}$ be nonnegative sequences that satisfy one of the following comparison conditions:*
(i) the direct comparison

$$\exists M \in \mathbb{R}_+ \quad \textit{such that} \quad a_k \le M\, b_k \quad \textit{eventually};$$

(ii) the limit comparison (if each b_k is positive)

$$\limsup_{k\to\infty} \frac{a_k}{b_k} < \infty\,;$$

(iii) the ratio comparison (if each a_k and b_k is positive)

$$\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k} < \infty \quad \textit{eventually}.$$

Then

$$\sum_{k=0}^{\infty} b_k \quad \textit{converges} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges}, \tag{3.2}$$

$$\left(\sum_{k=0}^{\infty} a_k \quad \textit{diverges} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} b_k \quad \textit{diverges}. \right)$$

Proof. First, condition (i) implies (3.2) because it and the fact that $\sum b_k$ converges yields the upper bound

$$\sum_{k=0}^{n} a_k \le M \sum_{k=0}^{n} b_k \le M \sum_{k=0}^{\infty} b_k < \infty\,.$$

Proposition 3.3 therefore implies that $\sum a_k$ converges. Next, condition (ii) implies condition (i) (and hence (3.2)) upon observing that

$$\limsup_{k\to\infty} \frac{a_k}{b_k} < M < \infty \quad \Longrightarrow \quad a_k \le M b_k \quad \text{eventually}\,.$$

Finally, condition (iii) implies condition (ii) (and hence (3.2)) upon observing that

$$\begin{aligned} \frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k} < \infty \quad \text{eventually} \quad &\Longrightarrow \quad \frac{a_{k+1}}{b_{k+1}} \le \frac{a_k}{b_k} \quad \text{eventually} \\ &\Longrightarrow \quad \left\{ \frac{a_k}{b_k} \right\} \quad \text{is nonincreasing eventually} \\ &\Longrightarrow \quad \lim_{k\to\infty} \frac{a_k}{b_k} < \infty\,. \quad \square \end{aligned}$$

Exercise. The proof argues that the direct comparison test applies whenever the limit comparison test applies, and that the limit comparison test applies whenever the ratio comparison test applies. Can you find an examples where (a) the direct comparison test applies but the limit comparison test fails, or (b) the limit comparison test applies but the ratio comparison test fails?

Example. We can apply the direct comparison test to show the harmonic series diverges. Consider the comparison

$$1 \le 1\,, \quad \tfrac{1}{2} \le \tfrac{1}{2}\,, \quad \tfrac{1}{4} \le \tfrac{1}{3}\,, \quad \tfrac{1}{4} \le \tfrac{1}{4}\,, \quad \tfrac{1}{8} \le \tfrac{1}{5}\,, \quad \tfrac{1}{8} \le \tfrac{1}{6}\,, \quad \tfrac{1}{8} \le \tfrac{1}{7}\,, \quad \tfrac{1}{8} \le \tfrac{1}{8}\,, \quad \cdots .$$

Summing both sides, one sees that

$$1 + \frac{n}{2} \le \sum_{k=1}^{2^n} \frac{1}{k}\,.$$

The partial sums clearly diverge.

3.4. **Series with Nonincreasing Positive Terms.** The harmonic series is a special case of the so-called p-series, which is formally given by

$$\sum_{k=1}^{\infty} \frac{1}{k^p}\,. \tag{3.3}$$

Because the terms of this series are nonincreasing and positive, the following two convergence tests can be applied. Proposition 3.3 plays a central role in their proofs.

Proposition 3.5. (Cauchy 2^k Test) *Let $\{a_k\}$ be a nonincreasing, positive sequence. Then*

$$\sum_{k=1}^{\infty} a_k \quad \textit{converges} \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty} 2^k a_{2^k} \quad \textit{converges}\,.$$

Proof. The result is a consequence of the direct comparisons

$$a_{2^{j+1}} \le a_k \le a_{2^j} \quad \text{for } 2^j \le k < 2^{j+1}\,,$$

which yield the bounds

$$\sum_{j=0}^{n-1} 2^{j-1} a_{2^{j+1}} \le \sum_{k=1}^{2^n-1} a_k \le \sum_{j=0}^{n-1} 2^j a_{2^j} \qquad \text{for every } n \in \mathbb{Z}_+\,.$$

The details are left as an exercise. □

Example. Because $\{1/k^p\}$ is a nonincreasing, positive sequence, Proposition 3.5 implies that the p-series (3.3) converges or diverges as the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k .$$

But this is a geometric series that clearly converges for $p > 1$ and diverges for $p \le 1$.

Remark. The proof of the Cauchy 2^k test outlined above extends the argument by which we showed the harmonic series diverges. Indeed, the harmonic series is just the p-series for $p = 1$.

Remark. The Cauchy 2^k test can be generalized to subsequences of $\{a_k\}$ of the form $\{a_{n_k}\}$ where there exist constants $\underline{m}$ and $\overline{m}$ such that

$$0 < \underline{m} \le \frac{n_{k+1} - n_k}{n_k - n_{k-1}} \le \overline{m} < \infty\,.$$

For example, one can choose $n_k = m^k$ for some $m \in \mathbb{N}$ with $m > 1$. This satisfies

$$\frac{n_{k+1} - n_k}{n_k - n_{k-1}} = \frac{m-1}{1-m^{-1}} = m\,,$$

whereby $\underline{m} = \overline{m} = m$. This leads to a "$m^k$ test" The statement and proof is left to you.

The second convergence test of this section requires the use of integrals — in fact, the use of *improper* integrals. These will be developed rigorously later in the course. However, here we will assume you have some familiarity with them from your elementary calculus courses.

Proposition 3.6. (Integral Test) *Let f be a nonincreasing, positive, locally integrable (continuous, for example) function over $[0, \infty)$. Then*

$$\sum_{k=0}^{\infty} f(k) \quad \textit{converges} \quad \Longleftrightarrow \quad \int_0^{\infty} f(x)\,\mathrm{d}x \quad \textit{converges}\,,$$

where the integral is understood in the sense of an improper integral.

Proof. The key fact we need from integration theory is that the improper integral

$$\int_0^{\infty} f(x)\,\mathrm{d}x \quad \text{converges}$$

whenever the sequence $\{S_n\}$ converges, where each S_n is defined by

$$S_n = \int_0^{n} f(x)\,\mathrm{d}x\,.$$

Because $\{S_n\}$ is an increasing sequence, showing convergence reduces to showing it is bounded above.

The result will then be a consequence of the fact that

$$S_n = \sum_{k=1}^{n} \int_{k-1}^{k} f(x)\,\mathrm{d}x \quad \text{for every } n \in \mathbb{Z}_+\,,$$

and the direct comparisons

$$f(k) \le \int_{k-1}^{k} f(x)\,\mathrm{d}x \le f(k-1)\,, \quad \text{for every } k \in \mathbb{Z}_+\,.$$

These facts should be clear to you based on your knowledge of definite integrals from elementary calculus. If not, a picture should help clarify things. We will establish them rigorously later. Here we will assume they are true and complete the proof.

By summing the above direct comparisons, one obtains

$$\sum_{k=1}^{n} f(k) \le S_n \le \sum_{k=1}^{n} f(k-1) = \sum_{k=0}^{n-1} f(k)\,.$$

The remaining details are left as an exercise. □

Example. Because $\{1/k^p\}$ is a nonincreasing, positive sequence, Proposition 3.6 implies that the p-series (3.3) converges or diverges as the improper integral

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x\,.$$

But for $p \neq 1$ one can easily check that

$$S_k = \int_1^{k} \frac{1}{x^p}\,\mathrm{d}x = \frac{1}{p-1}\left(1 - \frac{1}{k^{p-1}}\right),$$

while for $p = 1$ one has

$$S_k = \int_1^k \frac{1}{x}\,\mathrm{d}x = \log(k)\,.$$

(Here $\log(\,\cdot\,)$ denotes the natural logarithm.) One then sees that the sequence $\{S_k\}$ converges for $p > 1$ and diverges for $p \le 1$. The same is therefore true for the p-series.

3.5. **Alternating Series.** Until now we have only studied convergence tests for nonnegative series. The underlying tool has been the Monotonic Sequence Theorem (Proposition 2.8), which was used to prove Proposition 3.3. Here we use the Monotonic Sequence Theorem to obtain the following characterization of convergence for a special class of series with alternating sign.

Proposition 3.7. (Alternating Series Test) *Let $\{a_k\}$ be a positive, nonincreasing sequence in $\mathbb{R}$. Then*

$$\sum_{k=0}^{\infty}(-1)^k a_k \quad \textit{converges} \qquad \Longleftrightarrow \qquad \lim_{k\to\infty} a_k = 0\,.$$

Proof. The direction $\Longrightarrow$ is just Proposition 3.2 (Divergence Test). To prove the other direction, let

$$s_n = \sum_{k=0}^{n}(-1)^k a_k\,.$$

First, the picture is that

$$\{s_{2k}\}_{k\in\mathbb{N}} \quad \text{is nonincreasing}\,, \quad \text{while} \quad \{s_{2k+1}\}_{k\in\mathbb{N}} \quad \text{is nondecreasing}\,,$$

and that

$$s_{2k} > s_{2j+1} \quad \text{for every } j,k \in \mathbb{N}\,.$$

Indeed, the first two assertions follow because

$$\begin{aligned} s_{2k+2} - s_{2k} &= \quad a_{2k+2} - a_{2k+1} \le 0\,, \\ s_{2k+3} - s_{2k+1} &= -a_{2k+3} + a_{2k+2} \ \ge 0\,. \end{aligned}$$

Next, because $s_{2k} > s_{2k+1}$ for every $k \in \mathbb{N}$, for any $j \le k$ one has

$$s_{2k} > s_{2k+1} \ge s_{2j+1}\,, \qquad s_{2j} \ge s_{2k} > s_{2k+1}\,.$$

The result follows by exchanging j and k in the last inequality. The monotonic subsequences $\{s_{2k}\}$ and $\{s_{2k+1}\}$ are thereby bounded below and above respectively. By the Monotonic Sequence Theorem they therefore converge. Let

$$\overline{s} = \lim_{k\to\infty} s_{2k}\,, \qquad \underline{s} = \lim_{k\to\infty} s_{2k+1}\,.$$

Then

$$\overline{s} - \underline{s} = \lim_{k\to\infty}\big(s_{2k} - s_{2k+1}\big) = \lim_{k\to\infty} a_{2k+1} = 0\,,$$

whereby $\overline{s} = \underline{s}$. The last step is to show that this fact implies that $\{s_k\}$ converges. This is left as an exercise. $\square$

Examples.

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^p} \quad \text{converges for } p > 0\,, \qquad \sum_{k=2}^{\infty}\frac{(-1)^k}{\log(k)} \quad \text{converges}\,.$$

3.6. **Absolute Convergence.** The Monotonic Sequence Theorem has been the tool underlying all the convergence tests we have studied so far. We now use the Cauchy criterion to establish a test that does not require the series to be nonnegative.

Proposition 3.8. (Absolute Convergence Test) *Let $\{a_k\}$ be a real sequence. Then*

$$\sum_{k=0}^{\infty} |a_k| \quad \textit{converges} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges}\,.$$

Proof. Let $\{p_n\}$ and $\{q_n\}$ be the sequences of partial sums given by

$$p_n = \sum_{k=0}^{n} |a_k|\,, \qquad q_n = \sum_{k=0}^{n} a_k\,.$$

By hypotheses $\{p_n\}$ is convergent, and thereby Cauchy. The idea of the proof is to show that $\{q_n\}$ is Cauchy, and thereby convergent.

The key to doing so is the fact that for every $m, n \in \mathbb{N}$ one has the inequality

$$|q_n - q_m| \leq |p_n - p_m|\,.$$

This is trivially true when $m = n$. When $n > m$ the triangle inequality yields

$$|q_n - q_m| = \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| = |p_n - p_m|\,.$$

The case $n < m$ goes similarly.

Let $\epsilon > 0$. Because $\{p_n\}$ is Cauchy there exists an $N_\epsilon \in \mathbb{N}$ such that

$$m, n \geq N_\epsilon \quad \Longrightarrow |p_n - p_m| < \epsilon\,.$$

Because $|q_n - q_m| \leq |p_n - p_m|$, one immediately sees that

$$m, n \geq N_\epsilon \quad \Longrightarrow |q_n - q_m| < \epsilon\,.$$

Hence, $\{q_n\}$ is Cauchy, and thereby convergent. □

Proposition 3.8 motivates the following definition.

Definition 3.4. *If $\{a_k\}$ is a real sequence such that*

$$\sum_{k=0}^{\infty} |a_k| \quad \textit{converges}\,,$$

then one says

$$\sum_{k=0}^{\infty} a_k \qquad \begin{array}{c} \text{converges absolutely}\,, \\ \textit{or} \\ \textit{is}\ \text{absolutely convergent}\,. \end{array}$$

Example. Consider the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}\,.$$

This converges for $p > 0$ by the alternating series test, but it converges absolutely only for $p > 1$. This example shows that not every convergent series is absolutely convergent. In other words, absolute convergence is a stronger property than convergence.

When the definition of absolute convergence is combined with the Comparison Tests for series with nonnegative terms (Proposition 3.4), we get an array of new comparison tests for absolute convergence that can be applied to general series.

Proposition 3.9. (Absolute Comparison Tests) *Let $\{a_k\}$ and $\{b_k\}$ be real sequences that satisfy one of the following comparison conditions:*
(i) the direct comparison

$$\exists M \in \mathbb{R}_+ \quad \textit{such that} \quad |a_k| \le M\, b_k \quad \textit{eventually};$$

(ii) the limit comparison (if each b_k is positive)

$$\limsup_{k\to\infty} \frac{|a_k|}{b_k} < \infty\,;$$

(iii) the ratio comparison (if each $|a_k|$ and b_k is positive)

$$\frac{|a_{k+1}|}{|a_k|} \le \frac{b_{k+1}}{b_k} < \infty \quad \textit{eventually}.$$

Then

$$\sum_{k=0}^{\infty} b_k \quad \textit{converges} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges absolutely}. \tag{3.4}$$

Proof. Exercise.

Example. Because $|\cos(kx)| \le 1$ for every $x \in \mathbb{R}$ and $k \in \mathbb{Z}_+$, direct comparison with the p-series shows that the series

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^p} \quad \text{converges absolutely for } p > 1\,.$$

3.7. **Root and Ratio Tests.** The root and ratio tests both draw their conclusions about the convergence of a series based on absolute comparisons with a geometric series.

Proposition 3.10. (Root Test) *Let $\{a_k\}$ be a real sequence. Let*

$$\rho = \limsup_{k\to\infty} \sqrt[k]{|a_k|}\,.$$

Then

$$\rho < 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges absolutely},$$

$$\rho > 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{diverges}.$$

If $\rho = 1$ the series may either converge or diverge.

Proof. The convergence conclusion when $\rho < 1$ follows by a direct comparison of the series with a convergent geometric series. Specifically, by Proposition 2.11 one has that

$$\rho < r < 1 \quad \Longrightarrow \quad \limsup_{k\to\infty} \sqrt[k]{|a_k|} < r \quad \Longrightarrow \quad |a_k| < r^k \quad \text{eventually}\,.$$

The absolute convergence follows from the direct comparison test of Proposition 3.9.

The divergence conclusion when $\rho > 1$ follows by showing that $\limsup |a_k| > 0$. Specifically,

$$1 < r < \rho \quad \Longrightarrow \quad \limsup_{k\to\infty} \sqrt[k]{|a_k|} > r \quad \Longrightarrow \quad |a_k| > r^k \quad \text{frequently}\,.$$

This implies there exists a subsequence $\{a_{n_k}\}_k$ of $\{a_k\}_k$ such that

$$|a_{n_k}| > r^{n_k} \quad \text{eventually}\,.$$

Then $\limsup |a_k| \geq \limsup |a_{n_k}| \geq \lim r^{n_k} = \infty$. But $\limsup |a_k| > 0$ implies the sequence $\{a_k\}$ does not converge to zero, which by the Divergence Test (Proposition 3.2) implies the associated series diverges.

We leave as an exercise them problem of finding examples of both a convergent and a divergent series with $\rho = 1$. □

Proposition 3.11. (Ratio Test) *Let $\{a_k\}$ be a nonzero real sequence. Then*

$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges absolutely}\,,$$

$$\frac{|a_{k+1}|}{|a_k|} \geq 1 \quad \textit{eventually} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{diverges}\,.$$

Proof. As with the proof of the root test, the convergence conclusion follows by a direct comparison of the series with a convergent geometric series, while the divergence conclusion follows by showing that $\limsup |a_k| > 0$. Specifically, by Proposition 2.11 one has that

$$\rho < r < 1 \quad \Longrightarrow \quad \limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < r \quad \Longrightarrow \quad \frac{|a_{k+1}|}{|a_k|} < r \quad \text{eventually}\,.$$

An induction argument can then be used to show that for some $m \in \mathbb{N}$ one has

$$|a_k| \leq |a_m|\, r^{k-m} \quad \text{for every } k \geq m\,.$$

Because the geometric series

$$\sum_{k=m}^{\infty} \frac{|a_m|}{r^m}\, r^k \quad \text{converges}\,,$$

the comparison theorem implies

$$\sum_{k=0}^{\infty} a_k \quad \text{converges}\,.$$

The proof of the divergence assertion is left as an exercise. □

Remark. Some books give the divergence criterion of the ratio test as

$$\liminf_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} > 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \text{diverges}\,.$$

This is clearly weaker than the one we give.

Remark. The version of the ratio test most commonly found in elementary calculus texts is the following, which makes a much stronger hypothesis than our version.

Proposition 3.12. (Elementary Ratio Test) *Let $\{a_k\}$ be a nonzero real sequence such that*

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \quad \textit{exists}\,. \tag{3.5}$$

Then

$$\rho < 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges absolutely}\,,$$

$$\rho > 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k \quad \textit{diverges}\,.$$

If $\rho = 1$ the series may either converge or diverge.

Hypothesis (3.5) above requires the existence of a limit, whereas there is no such requirement in Proposition 3.11. (Recall that the lim sup exists in $\mathbb{R}_{\text{ex}}$ for every sequence, even ones that diverge.)

Remark. The root test is sometimes harder to apply, but as the following indicates, its convergence assertion can be sharper.

Proposition 3.13. *Let $\{a_k\}$ be a positive sequence. Then*

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \le \liminf_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}\,. \tag{3.6}$$

Proof. Exercise. (The middle inequality is obvious, so just prove the other two.)

Exercise. Find a series for which all the inequalities in (3.6) are strict.

Remark. Because both the root and ratio tests draw their conclusion about the convergence of a series based on comparison with a geometric series, they should only be used when such a comparison makes sense. For example, these tests can be used to assert the absolute convergence of series like

$$\sum_{k=1}^{\infty} k^4 2^{-k}\,, \qquad \sum_{l=0}^{\infty} e^{-l^2} 4^l\,, \qquad \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!}(-3)^m\,,$$

but will not yield any information about the convergence of series like

$$\sum_{k=2}^{\infty} \frac{\log(k)}{k^2}\,, \qquad \sum_{l=0}^{\infty} \left(\frac{3l+2}{l^4+2}\right)^{\frac{1}{2}}\,, \qquad \sum_{m=2}^{\infty} \frac{(-1)^m}{m(\log(m))^2}\,.$$

Example. Find the least upper bound and greatest lower bound of the set

$$S = \left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(3n)!}{n!}\frac{(2n)!}{(4n)!} x^n \quad \text{converges} \right\}\,.$$

This can be easily done employing the ratio test. Indeed, because

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(3n+3)!}{(n+1)!}\dfrac{(2n+2)!}{(4n+4)!}|x|^{n+1}}{\dfrac{(3n)!}{n!}\dfrac{(2n)!}{(4n)!}|x|^n} = \frac{(3n+3)(3n+2)(3n+1)(2n+2)(2n+1)}{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)}|x|\,.$$

one finds that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{3^3 \times 2^2}{4^4}|x| = \frac{3^3}{4^3}|x|\,.$$

The elementary ratio test, Proposition 3.12, therefore implies that the series converges when $|x| < (4/3)^3$ and diverges when $|x| > (4/3)^3$. The least upper bound of S is thereby $(4/3)^3$ while the greatest lower bound is $-(4/3)^3$.

3.8. **Dirichlet Test.** We now apply the Cauchy criterion to establish a test that, like the Alternating Series Test, can be applied to series that do not converge absolutely.

Proposition 3.14. (Dirichlet Test) *Let $\{a_k\}_{k\in\mathbb{N}}$ be a positive, nonincreasing sequence in $\mathbb{R}$ such that*

$$\lim_{k\to\infty} a_k = 0\,.$$

Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ for which there exists M such that

$$\left|\sum_{k=0}^{n} b_k\right| \le M \quad \textit{for every } n \in \mathbb{N}\,. \tag{3.7}$$

Then

$$\sum_{k=0}^{\infty} a_k b_k \quad \textit{converges}\,.$$

Remark. The Dirichlet Test implies the convergence conclusion of the Alternating Series Test. Indeed, if we set $b_k = (-1)^k$ then

$$\sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (-1)^k = \begin{cases} 1 & \text{for } n \text{ even}\,, \\ 0 & \text{for } n \text{ odd}\,. \end{cases}$$

Hence, bound (3.7) holds with $M = 1$. The Dirichlet Test then tells us that for every positive, nonincreasing real sequence $\{a_k\}_{k\in\mathbb{N}}$ such that $a_k \to 0$ as $k \to \infty$ the series

$$\sum_{k=0}^{\infty} (-1)^k a_k \quad \text{converges}\,.$$

As the next example illustrates, the Dirichlet Test is far more powerful than the Alternating Series Test.

Example. Consider the problem of determining all $x, p \in \mathbb{R}$ for which the Fourier p-series

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^p} \quad \text{converges}\,.$$

Because $|\cos(kx)| \le 1$ for every $k \in \mathbb{Z}_+$, direct comparison with the regular p-series shows that this series converges absolutely for $p > 1$. However this argument says nothing about what happens when $p \le 1$.

First observe that when $x \in \{2m\pi : m \in \mathbb{Z}\}$ one has $\cos(kx) = 1$ for every $k \in \mathbb{Z}_+$. In this case the Fourier p-series reduces to a regular p-series, which diverges for every $p \le 1$.

Next, observe that when $x \in \{(2m+1)\pi : m \in \mathbb{Z}\}$ one has $\cos(kx) = (-1)^k$ for every $k \in \mathbb{Z}_+$. In this case the Fourier p-series reduces to an alternating p-series, which (by the Alternating Series Test) converges for every $p > 0$.

We now use the Dirichlet Test to analyze the more general case when $x \notin \{2m\pi : m \in \mathbb{Z}\}$. Let $a_k = 1/k^p$ and $b_k = \cos(kx)$. Clearly the sequence $\{a_k\}$ is positive, decreasing, and vanishes

as $k \to \infty$. The hard step is to show that the partial sums associated with the sequence $\{b_k\}$ satisfy (3.7). To do this we use the trigonometric identity

$$2\sin(\tfrac{1}{2}x)\cos(kx) = \sin((k+\tfrac{1}{2})x) - \sin((k-\tfrac{1}{2})x)\,,$$

and the fact $\sin(\frac{1}{2}x) \neq 0$ when $x \notin \{2m\pi : m \in \mathbb{Z}\}$ to obtain (by a telescoping sum) the formula

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \cos(kx) = \sum_{k=1}^{n} \frac{\sin((k+\frac{1}{2})x) - \sin((k-\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} = \frac{\sin((n+\frac{1}{2})x) - \sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}\,.$$

It is clear from this formula that

$$\left|\sum_{k=1}^{n} b_k\right| = \left|\frac{\sin((n+\frac{1}{2})x) - \sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}\right| \leq \frac{|\sin((n+\frac{1}{2})x)| + |\sin(\frac{1}{2}x)|}{2\,|\sin(\frac{1}{2}x)|} \leq \frac{1}{|\sin(\frac{1}{2}x)|}\,.$$

Hence, bound (3.7) holds with $M = 1/|\sin(\frac{1}{2}x)|$. The Dirichlet Test then implies that when $x \notin \{2m\pi : m \in \mathbb{Z}\}$ the Fourier p-series converges for every $p > 0$.

Finally, you can use the Divergence Test to show that the Fourier p-series diverges for every $p \leq 0$. This follows easily once you know that

$$\limsup_{k\to\infty} \cos(kx) > 0 \quad \text{for every } x \in \mathbb{R}\,.$$

We leave the details as an exercise. □

Remark. When applying the Dirichlet test to a given series, one must identify the sequences $\{a_k\}$ and $\{b_k\}$, and check that all the hypotheses on them are satisfied. The hypotheses on $\{a_k\}$ are easy to check, so do that first: the sequence $\{a_k\}$ must be positive, nonincreasing, and vanish as $k \to \infty$. The hypothesis on $\{b_k\}$ is typically much harder to check: the associated partial sums must satisfy (3.7). The key to checking this in the above example was to write $b_k = c_{k+1} - c_k$ (by using a trigonometric identity) for some bounded sequence $\{c_k\}$, whereby the partial sums telescoped as

$$\sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (c_{k+1} - c_k) = c_{n+1} - c_0\,.$$

This telescoping approach can be taken for a variety of other $\{b_k\}$ too.

Our proof of the Dirichlet Test uses an identity that is a discrete analog of the integration-by-parts formula from calculus. Because it has many other applications, this identity gets its own proposition.

Proposition 3.15. (Summation-by-Parts Identity) *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}$. Let $B_{-1} = 0$ and*

$$B_n = \sum_{k=0}^{n} b_k \quad \textit{for every } n \in \mathbb{N}\,.$$

Then for every $m, n \in \mathbb{N}$ with $m \leq n$ one has identity

$$\sum_{k=m}^{n} a_k b_k = a_n B_n - a_m B_{m-1} + \sum_{k=m}^{n-1} (a_k - a_{k+1}) B_k\,,$$

with the understanding that the last sum is zero when $m = n$.

Remark. This is called the summation-by-parts identity because it is a discrete analog of the integration-by-parts formula

$$\int_m^n a(x)b(x)\,\mathrm{d}x = a(x)B(x)\Big|_m^n - \int_m^n a'(x)B(x)\,\mathrm{d}x\,,$$

where $B'(x) = b(x)$.

Proof. Because $b_k = B_k - B_{k-1}$ we have

$$\begin{aligned}\sum_{k=m}^n a_k b_k &= \sum_{k=m}^n a_k(B_k - B_{k-1})\\ &= \sum_{k=m}^n a_k B_k - \sum_{k=m}^n a_k B_{k-1}\\ &= \sum_{k=m}^n a_k B_k - \sum_{k=m-1}^{n-1} a_{k+1} B_k\\ &= a_n B_n - a_m B_{m-1} + \sum_{k=m}^{n-1}(a_k - a_{k+1})B_k\,.\end{aligned}$$

To get from the second to the third line above we re-indexed the last sum. All the other steps are straightforward algebra. □

We now turn to the proof of the Dirichlet Test.

Proof of Dirichlet Test. Let

$$s_n = \sum_{k=0}^n a_k b_k\,.$$

We will show the sequence $\{s_k\}_{k\in\mathbb{N}}$ is Cauchy, and therefore convergent.

Let $\epsilon > 0$. We seek $N_\epsilon \in \mathbb{N}$ such that

$$m, n \geq N_\epsilon \implies |s_n - s_m| < \epsilon\,.$$

For $m = n$ this is always true. Suppose $m < n$. (For the case $n < m$ simply reverse the roles of m and n.) Then

$$\begin{aligned}|s_n - s_m| = \Big|\sum_{k=m+1}^n a_k b_k\Big| &= \Big|a_n B_n - a_{m+1}B_m + \sum_{k=m+1}^{n-1}(a_k - a_{k+1})B_k\Big|\\ &\leq a_n|B_n| + a_{m+1}|B_m| + \sum_{k=m+1}^{n-1}(a_k - a_{k+1})|B_k|\\ &\leq a_n M + a_{m+1}M + \sum_{k=m+1}^{n-1}(a_k - a_{k+1})M = 2a_{m+1}M\,.\end{aligned}$$

Here we have used the summation-by-parts identity in the second step, the triangle inequality and the fact that $\{a_k\}_{k\in\mathbb{N}}$ is positive and nonincreasing in the third step, the bound $|B_k| \leq M$ in the fourth step, and evaluated the telescoping sum in the last step. Because $a_k \to 0$ as $k \to \infty$, we can choose N_ϵ so that $m \geq N_\epsilon$ implies $2a_{m+1}M < \epsilon$. Hence, for every $n > m \geq N_\epsilon$ the above inequalities imply $|s_n - s_m| < \epsilon$. □

4. Sets of Real Numbers

4.1. **Closure, Closed, and Dense.** The notions of *closure, closed,* and *dense* pertain to sets as they relate to the limit process. The closure of a set is all points that are the limit of some convergent sequence that lies within the set. If the closure of a set is the set itself then the set is said to be closed. Simply put, limits do not get out of closed sets. If the closure of a set is everything then the set is said to be dense. Simply put, limits can go anywhere from dense sets. Here we make these notions precise for subsets of $\mathbb{R}$.

4.1.1. *Closure.* We begin with the definition.

Definition 4.1. *Given any $A \subset \mathbb{R}$ its* closure *is given by*

$$A^c = \Big\{ a \in \mathbb{R} \,:\, a \text{ is the limit of a sequence in } A \Big\} .$$

It is clear that $A \subset A^c$ for every $A \subset \mathbb{R}$. Indeed, every $a \in A$ is the limit of the constant sequence $\{a_k\}$ with $a_k = a$ for every $k \in \mathbb{N}$. As we will now see, sometimes $A^c = A$, but in general A^c will be larger than A.

Examples. It is easy to show that $\emptyset^c = \emptyset$ and $\mathbb{R}^c = \mathbb{R}$.

Examples. If $a < b$ then the closures of the intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, and $(-\infty, b]$ are given by

$$(a, b)^c = (a, b]^c = [a, b)^c = [a, b]^c = [a, b] ,$$
$$(a, \infty)^c = [a, \infty)^c = [a, \infty) , \qquad (-\infty, b)^c = (-\infty, b]^c = (-\infty, b] .$$

You should be able to prove these facts.

We have the following propositions.

Proposition 4.1. *If $A \subset \mathbb{R}$ is nonempty and bounded above (below) then* $\sup\{A\} \in A^c$ *(*$\inf\{A\} \in A^c$*).*

Proof. You have to show that if $\sup\{A\} \notin A$ then there exists a sequence $\{a_k\} \subset A$ such that $a_k \to \sup\{A\}$ as $k \to \infty$. The details are left as an exercise. □

Proposition 4.2. *For every A, $B \subset \mathbb{R}$ one has that*

$$\begin{aligned} &(i) \quad A \subset B \quad \Longrightarrow \quad A^c \subset B^c , \\ &(ii) \quad (A \cup B)^c = A^c \cup B^c , \\ &(iii) \quad (A \cap B)^c \subset A^c \cap B^c . \end{aligned}$$

Proof. Exercise.

An important fact is that the closure of $\mathbb{Q}$ is $\mathbb{R}$. In other words, every real number is the limit of a sequence of rational numbers.

Proposition 4.3. $\mathbb{Q}^c = \mathbb{R}$.

Proof. Let $a \in \mathbb{R}$. Consider the sequence of intervals $\{I_k\}_{k \in \mathbb{N}}$ where each I_k is given by

$$I_k = \big(a - \tfrac{1}{2^k} \,,\, a + \tfrac{1}{2^k} \big) .$$

For each $k \in \mathbb{N}$ the third assertion of Proposition 1.15 implies there exists a $a_k \in I_k \cap \mathbb{Q}$. The step of showing that $a_k \to a$ as $k \to \infty$ is left as an exercise. It then follows that $a \in \mathbb{Q}^c$, whereby the assertion follows. □

4.1.2. *Closed.* We are ready for the next definition.

Definition 4.2. *A subset A of $\mathbb{R}$ is said to be* closed *when $A = A^c$.*

Examples. The empty set $\emptyset$ is closed.

Examples. If $a < b$ then intervals of the form $[a, a]$, $[a, b]$, $[a, \infty)$, $(-\infty, b]$, and $\mathbb{R} = (-\infty, \infty)$ are closed, while intervals of the form (a, b), $(a, b]$, $[a, b)$, (a, ∞), and $(-\infty, b)$ are not.

Our terminology seems to demand that closures should be closed. This is indeed the case.

Proposition 4.4. *Let $A \subset \mathbb{R}$. Then A^c is closed (i.e. $(A^c)^c = A^c$).*

Proof. Let $a \in (A^c)^c$. We must show that $a \in A^c$. Because $a \in (A^c)^c$ there exists a sequence $\{b_i\}_{i\in\mathbb{N}}$ in A^c such that $b_i \to a$ as $i \to \infty$. If $b_i = a$ for some $i \in \mathbb{N}$ then $a = b_i \in A^c$.

On the other hand, if $b_i \neq a$ for every $i \in \mathbb{N}$ then because $b_i \in A^c$ for each $i \in \mathbb{N}$ there exists a sequence $\{b_{(i,j)}\}_{j\in\mathbb{N}}$ in A such that $b_{(i,j)} \to b_i$ as $j \to \infty$. The picture is

$$\begin{array}{ccccccccc}
b_{(0,0)}\,, & b_{(0,1)}\,, & b_{(0,2)}\,, & \cdots & b_{(0,j)}\,, & \cdots & \to & b_0 \\
b_{(1,0)}\,, & b_{(1,1)}\,, & b_{(1,2)}\,, & \cdots & b_{(1,j)}\,, & \cdots & \to & b_1 \\
b_{(2,0)}\,, & b_{(2,1)}\,, & b_{(2,2)}\,, & \cdots & b_{(2,j)}\,, & \cdots & \to & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
b_{(i,0)}\,, & b_{(i,1)}\,, & b_{(i,2)}\,, & \cdots & b_{(i,j)}\,, & \cdots & \to & b_i \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 & & & & & & & \downarrow \\
 & & & & & & & a\,,
\end{array}$$

Because for each $i \in \mathbb{N}$ we have $b_{(i,j)} \to b_i$ as $j \to \infty$ and $|b_i - a| > 0$, there exists a $j_i \in \mathbb{N}$ such that

$$\left|b_{(i,j_i)} - b_i\right| < |b_i - a|\,.$$

Set $a_i = b_{(i,j_i)}$ for each $i \in \mathbb{N}$. It is clear that the sequence $\{a_i\}_{i\in\mathbb{N}}$ lies within A. The step of showing that $a_i \to a$ as $i \to \infty$ is left as an exercise. It then follows that $a \in A^c$, whereby the assertion follows. $\square$

Proposition 4.5. *Let $A \subset \mathbb{R}$. Then A^c is the smallest closed set that contains A.*

Proof. The previous proposition shows that A^c is closed. Earlier we showed that $A \subset A^c$. Now let B be any closed set that contains A. We see from (i) of Proposition 4.2 that $A \subset B$ implies $A^c \subset B^c$. Because B is closed we know that $B^c = B$. It follows that $A^c \subset B^c = B$. Therefore A^c is the smallest closed set that contains A. $\square$

The property of being closed is preserved by certain set operations.

Proposition 4.6. *If A and B are closed subsets of $\mathbb{R}$ then $A \cap B$ and $A \cup B$ are closed. If $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of closed subsets of $\mathbb{R}$ then*

$$\bigcap_{k\in\mathbb{N}} A_k \quad \textit{is closed}\,.$$

Proof. Exercise.

Remark. By repeated application of the first assertion above, we see that the union and intersection of any finite collection of closed sets is again closed. The second assertion states that

the intersection of any countable collection of closed sets is again closed. The analogous statment for unions is generally false. Indeed, consider the countable collection of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ where each I_k is given by

$$I_k = \left[-1 + \tfrac{1}{2^k} \, , \, 1 - \tfrac{1}{2^k} \right] .$$

You can easily show that

$$\bigcup_{k\in\mathbb{N}} I_k = (-1, 1) \, ,$$

which is not closed.

4.1.3. *Dense.* Finally, we have the concept of a set being dense in a larger one.

Definition 4.3. *Let $A \subset B \subset \mathbb{R}$. Then A is said to be* dense *in B if $B \subset A^c$.*

Examples. Proposition 4.3 states the $\mathbb{Q}$ is dense in $\mathbb{R}$. In a similar manner one can show that $(a, b) \cap \mathbb{Q}$ is dense in $[a, b]$, that $(a, \infty) \cap \mathbb{Q}$ is dense in $[a, \infty)$, and $(-\infty, b) \cap \mathbb{Q}$ is dense in $(-\infty, b]$.

Proposition 4.7. *If $A \subset B \subset C \subset D \subset \mathbb{R}$ and A is dense in D then B is dense in C.*

Proof. Exercise.

Proposition 4.8. *Let $A \subset \mathbb{R}$. Then A is dense in $\mathbb{R}$ if and only if for every interval (a, b) one has $A \cap (a, b) \neq \emptyset$.*

Proof. Exercise.

4.2. **Completeness.** Completeness is a central notion in analysis. As such, it arises in many settings. Here we introduce it in the setting of $\mathbb{R}$, where it is easily characterized. The basic notion of completeness is as follows.

Definition 4.4. *A set $S \subset \mathbb{R}$ is said to be* complete *if every Cauchy sequence contained in S has a limit that is in S.*

The Cauchy Criterion, Proposition 2.19, immediately implies that $\mathbb{R}$ is complete. Moreover, it easily yields the following characterization of all complete subsets of $\mathbb{R}$.

Proposition 4.9. *A subset of $\mathbb{R}$ is complete if and only if it is closed.*

Proof. Exercise.

Remark. In more general settings the notions of complete and closed do not coincide. For example, consider the set $\mathbb{Q}$ equipped with the usual notion of distance. A sequence in $\mathbb{Q}$ is said to be convergent in $\mathbb{Q}$ if it is convergent as a sequence in $\mathbb{R}$ and its limit is in $\mathbb{Q}$. A sequence in $\mathbb{Q}$ is said to be Cauchy in $\mathbb{Q}$ if it is Cauchy as a sequence in $\mathbb{R}$. Because there are sequences in $\mathbb{Q}$ that are Cauchy in $\mathbb{Q}$ but not convergent in $\mathbb{Q}$ the set $\mathbb{Q}$ is not complete. On the other hand, the set $\mathbb{Q}$ is closed because it contains all possible limit points of sequences that are convergent in $\mathbb{Q}$.

4.3. **Sequential Compactness.** Compactness is another central notion in analysis where it plays the principle role in many existence proofs. As such, it comes in many varieties. Fortunately, these varieties coincide in the setting of $\mathbb{R}$. We will take advantage of this coincidence by presenting only the concept of sequential compactness, for which we have all the tools at hand.

Definition 4.5. *A set $A \subset \mathbb{R}$ is said to be* sequentially compact *if every sequence in A has a subsequence that converges to a limit in A.*

Example. The interval $[0, \infty)$ is not sequentially compact because the incresing sequence $\{k\}_{k\in\mathbb{N}}$ diverges, and therefore has no convergent subseqence.

Example. The interval $(0, 1)$ is not sequentially compact because the limit of the convergent sequence $\{2^{-k}\}_{k\in\mathbb{N}}$ is 0, which is not in $(0, 1)$.

It is clear that every sequentially compact set must be closed, for otherwise there would be a convergent sequence within it whose limit lies outside it. Intuitively, a sequentially compact set must also be "small enough" that every sequence within it has a convergent subsequence. The following characterization of sequentially compact subsets of $\mathbb{R}$ uses the Bolzano-Weierstrass Theorem to show that "small enough" is simply that the set is bounded.

Proposition 4.10. *A set $A \subset \mathbb{R}$ is sequentially compact if and only if A is closed and bounded.*

Proof. ($\Rightarrow$) Suppose that A is either not bounded or not closed. Here we give the proof for the case when A is not closed. The proof for the case when A is not bounded is left as an exercise.

Because A is not closed there exists a sequence $\{a_k\}$ in A and a point $a \notin A$ such that $a_k \to a$ as $k \to \infty$. By Proposition 2.6 every subsequence of $\{a_k\}$ also converges to the point a, which is not in A. Therefore A is not sequentially compact.

($\Leftarrow$) Let $A \subset \mathbb{R}$ be closed and bounded. Let $\{a_k\}$ be an arbitrary sequence in A. Because A is bounded, the sequence $\{a_k\}$ is bounded. By the Bolzano-Weierstrass Theorem, $\{a_k\}$ has a converging subsequence $\{a_{n_k}\}$. Let a be the limit of this subsequence. Because A is closed and $\{a_{n_k}\}$ is in A, the limit a must also be in A. By the arbitrariness of $\{a_k\}$, we conclude that every sequence in A has a subsequence that converges to a limit in A. Therefore A is sequentially compact. $\square$

An immediate consequence of this characterizarion is the following.

Proposition 4.11. *Let $A \subset \mathbb{R}$ be sequentially compact and $B \subset \mathbb{R}$ be closed. Then $A \cap B$ is sequentially compact. In particular, every closed subset of A is sequentially compact.*

Proof. Exercise.

Sequentially compact subsets of $\mathbb{R}$ have the following important covering property.

Proposition 4.12. *Let $A \subset \mathbb{R}$ be sequentially compact. Let $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ be a countable collection of open intervals that covers A — i.e. such that*

$$A \subset \bigcup_{k\in\mathbb{N}} (a_k, b_k)\,.$$

Then there exist $n \in \mathbb{N}$ such that

$$A \subset \bigcup_{k=0}^{n} (a_k, b_k)\,.$$

Proof. Suppose not. Then for every $n \in \mathbb{N}$ there exists an $x_n \in A$ such that

$$x_n \notin \bigcup_{k=0}^{n} (a_k, b_k)\,. \tag{4.1}$$

Because $\{x_n\}_{n\in\mathbb{N}} \subset A$ while A is sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to a limit $x \in A$. Because $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ covers A, there exists $m \in \mathbb{N}$ such that $x \in (a_m, b_m)$. Because $x_{n_k} \to x$ as $k \to \infty$, this implies that $x_{n_k} \in (a_m, b_m)$ eventually as $k \to \infty$. But this contradicts the fact seen from (4.1) that $x_{n_k} \notin (a_m, b_m)$ for every $n_k \geq m$. $\square$

There is a converse of the previous proposition that we state without proof.

Proposition 4.13. *Let $A \subset \mathbb{R}$ such that for every countable collection of open intervals $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ that covers A there exist $n \in \mathbb{N}$ such that*

$$A \subset \bigcup_{k=0}^{n} (a_k, b_k)\,.$$

Then A is sequentially compact.

Remark. By combining Propositions 4.12 and 4.13, we see that the covering property stated in the hypothesis of Proposition 4.13 characterizes sequential compactness for subsets of $\mathbb{R}$. This property is closely related to the properties of *countable compactness* and *compactness*, which are also covering properties that characterize sequential compactness for subsets of $\mathbb{R}$. In more general settings these notions of campactness can differ from each other.

4.4. **Connectedness.** Connectedness is another central notion in analysis. As such, it also comes in many varieties. Fortunately, these varieties coincide in the setting of $\mathbb{R}$. The basic notion of connectedness is as follows.

Definition 4.6. *A set $S \subset \mathbb{R}$ is said to be* disconnected *there exists nonempty A, $B \subset S$ such that*

$$A \cup B = S\,, \qquad A^c \cap B = A \cap B^c = \emptyset\,. \tag{4.2}$$

Otherwise S is said to be connected.

If a set S is disconnected then the nonempty sets A and B that arise in Definition 4.6 have the property that any convergent sequence that lies within one them will have a limit that is not in the other. In other words, if $\{x_n\} \subset A$ is convergent and $x_n \to x$ then $x \notin B$.

Example. The set $(-\infty, 0) \cup (0, \infty)$ is disconnected because (4.2) is satisfied by the sets $A = (-\infty, 0)$ and $B = (0, \infty)$.

It is clear from Definition 4.6 that every disconnected set has at least two points in it. In particular, the empty set or any singleton set (a set containing only a single point) is connected. The following proposition shows that the connected subsets of $\mathbb{R}$ are precisely the intervals. We will use the Interval Characterization Theorem, Proposition 1.18.

Proposition 4.14. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Proof. ($\Longrightarrow$) Let $S \subset \mathbb{R}$ be connected. We will show that S is then an interval by using the Interval Characterization Theorem, Proposition 1.18.

Let x, $y \in S$ such that $x < y$. We must show that $(x, y) \subset S$. Let $z \in (x, y)$. We must show that $z \in S$. Suppose not. Let $A = (-\infty, z] \cap S$ and $B = [z, \infty) \cap S$. Because $x \in A$ and

$y \in B$, these sets are nonempty. It is easy to check that A and B satisfy (4.2), whereby S is disconnected. But this contradicts the fact S is connected. Hence, $z \in S$. Therefore $(x, y) \subset S$. Because this is true for every x, $y \in S$ such that $x < y$, the Interval Characterization Theorem, Proposition 1.18, implies that S is an interval.

($\Longleftarrow$) Let $S \subset \mathbb{R}$ be an interval. Suppose that S is disconnected. Then there exists nonempty sets A, $B \subset S$ that satisfy (4.2). Let $x \in A$ and $y \in B$. Because we can always relabel the sets A and B, we can assume without loss of generality that $x < y$. Because x, $y \in S$ while S is an interval, we know from the Interval Characterization Theorem, Proposition 1.18, that $[x, y] \subset S$. Because $A \cup B = S$, we have $(A \cap [x, y]) \cup (B \cap [x, y]) = [x, y]$.

Now consider the point $z = \sup\{A \cap [x, y]\}$. By Propositions 4.1 and 4.2 we have

$$z \in (A \cap [x, y])^c \subset A^c \cap [x, y]\,.$$

Because $A^c \cap B = \emptyset$ and $y \in B$, it follows that $z \neq y$, which implies that $z < y$. Because $z = \sup\{A \cap [x, y]\} < y$ while $(A \cap [x, y]) \cup (B \cap [x, y]) = [x, y]$, we see that $(z, y] \subset B$. But then $z \in [z, y] \subset B^c$ by Proposition 4.2, so that $z \notin A$ because $A \cap B^c = \emptyset$. On the other hand, because $A^c \cap B = \emptyset$ while $(A \cap [x, y]) \cup (B \cap [x, y]) = [x, y]$, it follows that $z \in A \cap [x, y] \subset A$. This contradicts the conclusion of the sentence before it. Therefore S must be connected. $\square$