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Use of analogies in teaching the concept of function: Relation between knowledge of topics and knowledge of mathematics teaching

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This paper addresses the knowledge about the function concept and the knowledge on how to teach this concept using an analogy. The analysis of an episode of one lesson in which the analogy between a washing machine and the concept of a function is shown allows identifying specialised knowledge about the function concept and teaching strategies. The study findings reveal links between knowledge of the topic and knowledge of mathematics teaching, permitting identification of potentialities and limitations of the analogy used.

Keywords: Function concept, mathematics teachers' specialised knowledge, analogies.

Introduction

Function is one of the most important concepts in mathematics, fundamental for development of mathematical analysis and mathematics in general (Ponte, 1992). This concept is not only present in many areas of mathematics, but is prevalent in the school curriculum and is, of course, studied as a part of the mathematics teacher training programs. Given its importance and great complexity (Dubinsky & Harel, 1992), it is essential that the mathematics teacher's knowledge considers both the discipline-specific knowledge of the function concept, as well as knowledge about how it is taught and learned.

Teachers' knowledge has been widely studied from different perspectives, using a wide range of theoretical models (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008; Carrillo et al., 2014). The Mathematics Teacher's Specialised Knowledge (MTSK) model is presented both as a conceptualisation for mathematics teachers' knowledge and as an analytical tool for acquiring this knowledge (Flores, Escudero, & Aguilar, 2013). Studying the subdomains and categories proposed by the MTSK model, and their relationships, allows us to advance the understanding and analysis of teachers' knowledge (Sosa, Flores-Medrano, & Carrillo, 2015). Part of this knowledge is related to the depth of the teacher's understanding of a concept (in this case the function) as a mathematical concept and as an object of teaching. Several studies conducted to date focused on the understanding of the function concept on students, pre-service teachers, and practicing teachers (e.g., Even, 1990; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Other studies highlight the importance of fully understanding the concept (e.g., Sierpinska, 1992), identification of its representations (e.g., Even, 1990), as well as difficulties in learning (e.g., Dubinsky & Harel, 1992) and relating their different representations (e.g. Ponte, 1992; Figueiredo, Contreras, & Blanco, 2015). The focus of the present investigation is on the relationship between teachers' knowledge about the function concept and the knowledge about its teaching from the perspective of the MTSK, particularly when a teacher uses an analogy to make the function concept understandable. In this study we ask what knowledge about

functions and their teaching can be inferred from the use of an analogy? How are these knowledge types related?

Analogies

According to Treagust, Duit, Joslin, and Lindauer (1992), analogy is achieved through a comparison of structures in two distinct domains, one of which is familiar (source or analogue), while the other is unfamiliar (target).

> An analogy refers to comparisons of structures between domains. An analogy is a relation between parts of the structures of two conceptual domains and may be viewed as a comparison statement on the grounds that these structures bear some resemblance to one another. (Treagust et al., 1992, p. 413)

The use of analogies in teaching, particularly as a didactic strategy in the teaching-learning process, has been extensively studied (e.g., Duit, 1991; Treagust et al., 1992). For example, Curtis and Reigeluth (1984) analysed instructional text and provided a classification of the ways in which the relationship between the source and the target domain is established. According to the authors, the relationship can be (1) *structural*, referring to physical similarity or similar construction; (2) *functional*, referring to the way of functioning of both structures; and (3) *structural-functional*, formed by combining the previous two. They add that analogies occur in two forms: verbal and pictorial-verbal, whereby the former is achieved solely via the use of words, while in the latter words are complemented by an image.

Teachers tend to produce analogies automatically when answering questions or explaining the concepts they are teaching (Ünver, 2009). According to Figueiredo et al. (2015), function as a machine is an example of such analogies and this comparison will show only some aspect of the concept. Function can be descripting operationally as a computational process or structurally as a set of ordered pairs (Sfard, 1991). To present the function trough this analogy conducts to understand the concept as an input-output process in an operational way.

Mathematics Teacher's Specialised Knowledge

The MTSK model (Figure 1), in the spirit of contributions by Shulman (1986) and Ball et al. (2008), proposes, within the teacher's knowledge, a discipline-specific component (MK, mathematical knowledge) and a didactic component (PCK, pedagogical content knowledge). A further component related to beliefs about mathematics and about teaching and learning it is introduced in the middle of the model.

According to Carrillo et al. (2014), MK corresponds to knowledge specific to the discipline being taught, and comprises of three subdomains: *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KSM), and *Knowledge of the Practice of Mathematics* (KPM). KoT considers the phenomenology, definitions, properties, procedures, and foundations of the topic, as well as the ways of recording and representing it. On the other hand, KSM pertains to the conceptual connections among mathematical concepts, relating a concept to prior contents (simplification), later contents (adding complexity), or to contents with a common property (transverse connections), and the auxiliary connections among objects. Finally, KPM is related to knowledge about the characteristics

of mathematical work, namely how to proceed and create knowledge in mathematics, practices linked to mathematics in general, and practices linked to a specific topic.

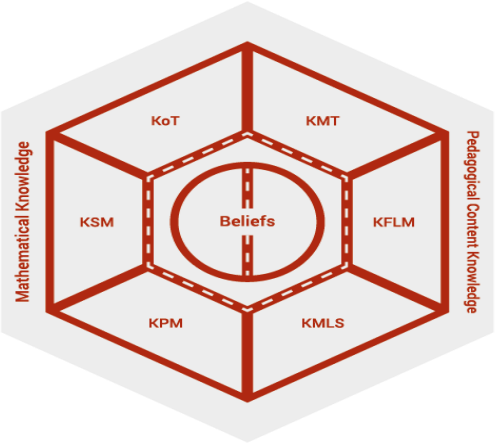


Figure 1: Sub-domains of the MTSK model (Carrillo et al., 2014)

PCK corresponds to didactic knowledge specific to teaching work in the process of teaching and learning mathematics. Once again, it comprises of three subdomains, namely *Knowledge of Features of Learning Mathematics* (KFLM), *Knowledge of Mathematics Teaching* (KMT) and *Knowledge of Mathematics Learning Standards* (KMLS). KFLM considers teachers' knowledge of their students' learning styles, strengths and difficulties associated with learning, way of interacting with mathematical content, students' conceptions of mathematics, and personal or institutional theories of mathematics learning. KMT is knowledge about mathematical content conditioned by its teaching, including knowledge about personal or institutional teaching theories, physical and virtual resources, and strategies, activities, examples, and help. Finally, KMLS pertains to knowledge about required mathematical concepts to be taught, knowledge about the level of conceptual and procedural development expected, and the sequencing of the various topics.

The aforementioned division into subdomains allows us to deepen our understanding of the elements of knowledge that are utilised in an integrated and interconnected manner. The MTSK model is a suitable analytical tool for meeting the objective of the present study because, in addition to highlighting mathematics, its categories, and subdomains, it allows focusing on teachers' knowledge about the function concept and its teaching (for example, the definition, its properties, representation, and the strategies used when teaching).

Methods

This research is grounded in an interpretative paradigm and is based on the instrumental case study design (Stake, 2007). The aim is to investigate from the perspective of MTSK the knowledge manifested by a high-school teacher when teaching the concept of function. The teacher that is in the focus of the study, henceforth referred to as Arturo, has ten years of teaching experience, teaching classes from fifth to twelfth grade. He is also a university teacher in first-year classes for engineering students and for pre-service teachers. He has also worked as a teacher in continuing education courses for primary-school teachers and has taken courses connected to university teaching, curricular updates in geometry, and curricular reform. At the time of this study, Arturo was teaching ninth-grade classes,

where he had planned to introduce the function concept. According to the information provided by Arturo, his group of students is familiar with the use of algebraic language, equation solving techniques, and Cartesian plane, among others.

To collect the data, classes in which Arturo planned to introduce the function concept were observed and video-recorded. The videos were transcribed and the transcripts served as the principal source of information. The resulting data was subjected to content analysis (Bardin, 1996), whereby class episodes were determined according to the tacit or explicit goals of the teacher. The units of analysis correspond to the teacher's interventions and responses provided by his students. In addition, only those that present evidence of teacher's specialised knowledge have been considered (Flores et al., 2013). In the first class, Arturo introduces the concept of function and its definition, and provides some examples of functions. An episode was selected from this class, in which the teacher uses an analogy to promote students' understanding of the definition of function. The episode was analysed in relation to the KoT and KMT sub-categories of the MTSK.

Results and discussion

Arturo defined function as *a correspondence between elements of two sets in which each element of the input set corresponds to a single element in the output set*. Knowledge of this definition is part of his KoT. To make this definition understandable, Arturo introduces a washing machine as an analogy for a function, alluding to a family context for his students. To know this context is not part of the MTSK, but it allows us to reflect on the scope and applicability of the analogy according to the type of students to whom they are presented. In the following extract, Arturo presents the analogy between function and washing machine:

Arturo: Before giving you more names, the function works like a kind of machine. An example could be a washing machine. A washing machine carries out a function. What is its function?

Student: Washing!

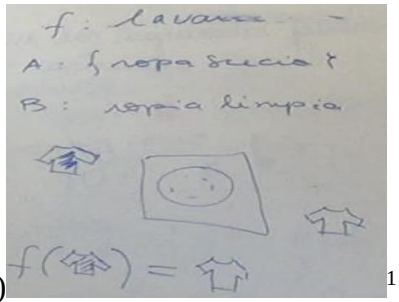
Arturo: What do you do? You take an article of clothing. It's dirty. You put it in the washing machine. How does it come out?

Student: Clean.

Arturo: Did the washing machine fulfil its function? Yes. The dirty article of clothing would be a member of the input set, and the clean article of clothing would be a member of the output set. This is what the function does. Here [he points to a diagram] we would have the dirty article of clothing. The function does what it does, depending on the machine, and arrives at the other side. In the case of a washing machine, it arrives clean.

Knowledge of analogies, as elements that enhance the teaching of a concept, is part of the teacher's specialised knowledge (Carrillo et al., 2014). The use of analogy shows teacher's knowledge about when to give any specific help to his students (KMT). The use of analogy favours the understanding and visualisation of abstract objects in students, besides being a motivation for a new theme. This analogy presents the function as a *process* and allows the students to better understand this concept (Sfard, 1991; Figueiredo et al., 2015). Moreover, different components can be identified, namely

domain, co-domain, pre-image, and image explaining the connection between the source and the target domain.

a) 

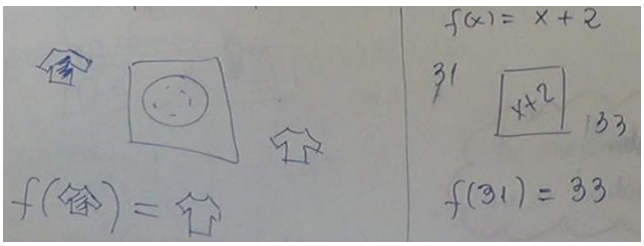
[1] b) 

Figure 2a: Presentation of the analogy. 2b: Relationship between the source and the target domain.

The analogy is presented in two formats: in the intervention described (verbal) and when the teacher draws a washing machine on the whiteboard (pictorial). This illustration (Figure 2a) shows the function as an input-output process, in which the object that enters is modified (in this case, a dirty article of clothing comes out clean). In this example, the objects are the same at entry and exit (T-shirts in both cases). Other analogies for this concept can relate objects of different nature. For example, function can be represented as a coffee dispensing machine into which money is entered in order to obtain a cup of coffee, highlighting the arbitrariness of related sets (Even, 1990). In this sense, the analogy of the washing machine impedes the association of arbitrary sets, which may result in students gaining a partial understanding of the concept. Likewise, the washing machine does not show other conceptions for the function as, for example, the co-variation of magnitudes. Similarly, it does not facilitate representation of more complex functions or the complexity of the concept itself (function algebra, composition).

Arturo takes advantage of this relation between input and output to clarify the definition of the concepts of image and pre-image, which are parts of his KoT. Similarly, evidence of his knowledge about the domain and co-domain of the function appears in the analogy as "dirty clothes" and "clean clothes." This knowledge and the exposed relations between the source and target domains account for the use of analogy as a strategy for teaching the function, evidencing a relationship between its KoT and its KMT. Figure 2a shows Arturo's knowledge of the notation $f(x) = y$ (knowledge about representations as a part of his KoT) that allows him to show the relationship between two domains of the analogy (as knowledge of strategies - KMT) and to introduce new ways to represent the function.

In the following excerpt, Arturo explains the relationship between the linked domains, source and target, using the analogy, in which we interpret the *structural* and *functional* character of the analogy presented (Curtis & Reigeluth, 1984).

Arturo: In our context, our function was the washing machine, washing. Set A would be dirty clothes and Set B clean clothes. If this is our washing machine, and it carries out its function; dirty clothes go in, and how do they come out?

[1] f: washing machine A: dirty clothes B: clean clothes

Student: Clean!

Arturo: The same as what we did here. The function was applied to this kind of T-shirt that I drew that was dirty. What will it equal?

Student: Clean, clean clothes.

Arturo: The same T-shirt, but clean. These two elements also have names. This element here is called the "image" of what I sent in. [...] And these elements here are called "pre image." What is the clean T-shirt?

Student: Image.

Arturo: The image of what?

Student: Of the dirty T-shirt.

The *structural* characteristic is shown in establishing the correspondences of the Set A with the dirty clothes, the Set B with the clean clothes, and the washing machine with the arithmetic process carried out by the function. That is, the structure of the laundry process is analogous to the evaluation process in the function. When Arturo presents the analogy "function as a washing machine," he also refers to its functional character, as he establishes a comparison between the operation of the machine and the function. After this intervention, Arturo represented verbally and as an algebraic expression (representations in his KoT) an example of function (Figure 2b), thus deepening the analogy between machine and function.

Arturo: With numbers, the function isn't going to do the washing. It's going to add two to whatever comes in [he writes "f(x) = x+2"]. Whatever comes into the function, to the machine, I add two to it. If this is my machine that adds two to whatever comes in, if a one enters, how does it come out? [Student: Three.]

It should be noted that, when Arturo teaches "with numbers," he aims to work in the target domain of functions. Consequently, the situation created becomes an example (knowledge of examples - KMT) that will allow him to present the *functional* characteristic of the analogy and propose a two-way process for understanding the relationship between a function and a washing machine (Figure 2b).

The relationship between an algebraic expression (function) and a washing machine presents functions as an input-output process, meaning of the concept that we consider part of his KoT. In the same way, the articulation and selection of representations for the function (the analogy, f(x) = y, algebraic expression and natural language) accounts for Arturo's KoT, relating to his KMT.

In this last intervention, by associating a function with a machine, Arturo highlights the process role of a function (part of his KoT). We do not have evidence of Arturo highlighting the object role of a function supported by the analogy studied here, although the structural character of the analogy may be the first approach to this conception.

Conclusion

According to the results yielded by the analyses presented above, the relations between the teacher's KoT and KMT are demonstrated in the articulation between knowledge of representations of the function and the choice and use of these representations as examples and analogies for teaching the

concept. In addition, the choice of analogies and examples given by Arturo reflect his KMT, which is nurtured and influenced by his KoT. The analogy shows the function as a correspondence in coherence with the definition given, while also permitting articulation of different representations: sagittal and algebraic diagram (Figure 2b). They also allow the teacher to produce other representations as a Cartesian graph or a table of values, thereby expanding the concept image for the students. Likewise, the analogy allows the students to appreciate the univalence character of the function (Even, 1990) and to extend the range of situations in which the concept of function is present (phenomenology - KoT).

The analogy utilised in this case is suitable for teaching and learning given that it is connected to the students' prior everyday experience. Moreover, its functional and structural character allows them to understand the concept from different perspectives and using different representations. Establishing a bidirectional relationship between the function and the washing machine can allow students to utilise the concept of a function they have learned to identify the concept in other areas of mathematics and other areas of knowledge. It should be noted that, in spite of the benefits presented by this analogy, students may have a partial notion of function if the conception of function as a machine is maintained. Thus, it is essential that other aspects of the concept be highlighted, such as the arbitrariness of the sets involved (Even, 1990). It is also important to relate it to other representations (Figueiredo et al., 2015). Lack of articulation between representations can cause a limitation in the development of the conception of function from the *structural* perspective (Sfard, 1991; Breidenbach et al., 1992). On the other hand, this presentation of the concept may constitute an obstacle to learning, for example, the algebra of functions or its composition (How do I sum two machines?). Likewise, students' direct experience with the laundry process can be detrimental in understanding the function: "the function did not fulfil its function." However, it is not the purpose of the study to evaluate the methodological proposal or Arturo's knowledge, but rather to approach it with the intention of improving our understanding of such knowledge, that of subdomain relationships in particular. In that sense, the analysis using the MTSK model is useful for advancing this understanding.

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