UNIT-I

AXIOMATIC STUDY OF REAL NUMBERS

We could use the axioms of set theory as the foundation for real analysis. To carry this out, we would start by defining the set of natural numbers

$$\mathrm{N} = \{1, 2, 3, \ldots\} \text{ and } \omega = \{0\} \cup \mathrm{N} = \{0, 1, 2, 3, \ldots\},$$

together with the usual operations of addition and multiplication. We could then define an equivalence relation $\sim$ on the Cartesian product $\omega \times \omega$ by

$$(m, n) \sim (q, r) \quad \Leftrightarrow \quad m + r = q + n.$$

The set of integers Z is then the set of equivalence classes. If $[m, n]$ is the equivalence class of the pair $(m, n) \in \mathrm{N} \times \mathrm{N}$, then

$$\mathrm{Z} = \{\ldots, -2 = [0, 2], -1 = [0, 1], 0 = [0, 0], 1 = [1, 0], 2 = [2, 0], \ldots\}.$$

We then define the usual addition and multiplication on Z and show that it satisfies the familiar properties. Finally, we define an equivalence relation $\sim$ on the Cartesian product $\mathrm{Z} \times (\mathrm{Z} - \{0\})$ by

$$(m, n) \sim (q, r) \quad \Leftrightarrow \quad mr = qn.$$

We would think of the equivalence class $[m, n]$ are representing the fraction m/n as a rational number. We would then establish as theorems all the rules you learned for arithmetic with rational numbers in grade school. The process would be long and time-consuming, and you might wonder whether it isn't a bit pedantic to carry this out with so much rigor. But the point to understand is that all of the familiar rules of arithmetic for rational numbers can in fact be established in this way.

The last stage is developing the real numbers R, which can be thought of as limits of sequences of rational numbers. For example π is the limit of the sequence

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots, 3.14159265358979, \ldots).$$

It is precisely the notion of defining the limit of such a sequence which is the major difficulty in developing real analysis. It would take a long time just to define the real numbers in this manner. So for a first treatment of real analysis, most authors take a shortcut, and formulate a collection of axioms which characterize the real numbers. One assumes these axioms as the starting point of real analysis, rather than just the axioms of set theory. (Since one does want to use the properties of sets in discussing real numbers, a full formal development of analysis in this shortened form would require both the axioms of set theory and the axioms of real numbers. On the other hand, many authors, such as [1] just use set theory as a basic language whose basic properties are intuitively clear; this is more or less the way mathematicians thought about set theory prior to its axiomatization.)

The axioms for real numbers fall into three groups, the axioms for fields, the order axioms and the completeness axiom.

1 Field axioms

Definition. A *field* is a set F together with two operations (functions)

$$f : F \times F \to F, \qquad f(x,y) = x + y$$

and

$$g : F \times F \to F, \qquad g(x, y) = xy,$$

called addition and multiplication, respectively, which satisfy the following axioms:

- F1. addition is commutative: $x + y = y + x$, for all $x, y \in F$.
- F2. addition is associative: $(x + y) + z = x + (y + z)$, for all $x, y, z \in F$.
- F3. existence of additive identity: there is a unique element $0 \in F$ such that $x + 0 = x$, for all $x \in F$.
- F4. existence of additive inverses: if $x \in F$, there is a unique element $-x \in F$ such that $x + (-x) = 0$.
- F5. multiplication is commutative: $xy = yx$, for all $x, y \in F$.
- F6. multiplication is associative: $(xy)z = x(yz)$, for all $x, y, z \in F$.

- F7. existence of multiplicative identity: there is a unique element $1 \in F$

  such that $1 \; f = 0$ and $x1 = x$, for all $x \in F$.

- F8. existence of multiplicative inverses: if $x \in F$ and $x \; f = 0$, there is aunique element $(1/x) \in F$

  such that $x \cdot (1/x) = 1$.

- F9. distributivity: $x(y + z) = xy + xz$, for all $x, y, z \in F$.

Note the similarity between axioms F1-F4 and axioms F5-F8. In the language of algebra, axioms F1-F4 state that F with the addition operation f is an *abelian group*. (The group axioms are studied further in the first part of abstract algebra, which is devoted to group theory.) Axioms F5-F8 state that F $\{0\}$ with the multiplication operation g is also an abelian group. Axiom F9 ties the two field operations together.

Most important examples. The key examples of fields are the set of rational numbers Q, the set of real numbers R and the set of complex numbers C, in all cases taking f and g to be the usual addition and multiplication operations. On the other hand, the set of integers Z is NOT a field, because integers do not always have multiplicative inverses.

Other useful examples. Another example is the field Z/pZ, where p is a prime ≥ 2, which consists of the elements $\{0,1,2,.....,p-1\}$. In this case, we define addition or multiplication by first forming the sum or product in the usual sense and then taking the remainder after division by p, so as to arrive back in the set $\{0,1,2,.....,p-1\}$. This is often referred to as mod p addition and multiplication. Thus for example,

$$Z/5Z = \{0, 1, 2, 3, 4\}$$

and within Z/5Z,

$$3 + 4 = 7 \bmod 5 = 2, \quad 3 \cdot 4 = 12 \bmod 5 = 2.$$

On the other hand, if n is not a prime, then Z/nZ with mod n addition and multiplication is NOT a field. Indeed, in Z/4Z,

$2 + 2 = 4 \bmod 4 = 0$,

so 2 does not have a multiplicative inverse in Z/4Z, contradicting Axiom F8.

Starting with the field axioms, one can prove that the usual rules for addition and multiplication hold. As long as we are within ANY field, we can then use those rules with perfect confidence. For example, we could begin by giving a complete proof of the cancellation law:

Proposition. If F is a field and $x, y, z \in F$, then

$$x + z = y + z \quad \Rightarrow \quad x = y.$$

Proof: Suppose that $x + z = y + z$. Let $(-z)$ be an additive inverse to z, which exists by Axiom F4. Then

$$(x + z) + (-z) = (y + z) + (-z).$$

By associativity of addition (Axiom F2),

$$x + (z + (-z)) = y + (z + (-z)).$$

Then by Axiom F4, $x + 0 = y + 0$ and by Axiom F3, $x = y$.

Proposition. If F is a field and $x \in F$, then $x \cdot 0 = 0$.

Proof: By Axiom F3, $x \cdot 0 = x \cdot (0+0)$. By distributivity (Axiom F9), $x \cdot (0+0) = x \cdot 0 + x \cdot 0$. By Axiom F3 again,

$0 + x \cdot 0 = x \cdot 0 + x \cdot 0$,

and by Axiom F1,

$x \cdot 0 + 0 = x \cdot 0 + x \cdot 0$.

Hence $0 = x \cdot 0$ by the preceding proposition.

References:

1. Math 117: Axioms for the Real Numbers, John Douglas Moore, October 11, 2010.