A

Construction of the Real Numbers

In Chapter 1 we have described the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ of natural numbers, integers, and rational numbers, respectively, as follows:

$$\mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$

and

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0 \right\}.$$

The set $\mathbb{R}$ of "real numbers" was then introduced as a set containing $\mathbb{Q}$ and the "irrational numbers" in such a way that the elements of $\mathbb{R}$ are in one-to-one correspondence with the points on the "number line". But $\mathbb{R}$ defies a simplistic description such as that given above for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. Thus, while we can conceive easily what a rational number is, it is a little harder to say precisely what a real number is. For this reason, we made an assumption that there exists a set $\mathbb{R}$ containing $\mathbb{Q}$ that satisfies the three sets of properties given in Section 1.1, namely the Algebraic Properties A1-A5, the Order Properties O1-O2, and the Completeness Property. The main aim of this appendix is to show that such a set $\mathbb{R}$ does indeed exist and is essentially unique. The approach that we shall take is due to Cantor, and uses Cauchy sequences of rational numbers. In what follows, we shall assume familiarity with the set $\mathbb{Q}$ and the usual algebraic operations on $\mathbb{Q}$ as well as the usual order relation that permits us to talk of the subset $\mathbb{Q}^+$ of positive rational numbers in such a way that the properties A1-A5 and O1-O2 in Section 1.1 are satisfied if we replace $\mathbb{R}$ by $\mathbb{Q}$ throughout. This appendix is divided into three sections, which are organized as follows. In the first section below, we discuss some preliminaries about equivalence relations and equivalence classes. Then in the next section, we outline a construction of $\mathbb{R}$ using Cauchy sequences of rational numbers. The "uniqueness" of $\mathbb{R}$ is formally established in the last section.

A.1 Equivalence Relations

The notion of an equivalence relation is basic to much of mathematics, and it will be useful in our formal construction of $\mathbb{R}$ from $\mathbb{Q}$. The most basic equivalence relation on any set is that of equality denoted by =. Fundamental properties of this relation motivate the following definition.

S. R. Ghorpade, B. V. Limaye, *A Course in Calculus and Real Analysis*,
Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1

Let S be a set. A **relation** on S is a subset of $S \times S$. If $\sim$ is a relation on S and $a, b \in S$, then we usually write $a \sim b$ to indicate that the ordered pair (a, b) is an element of the subset $\sim$ of $S \times S$. A relation $\sim$ on S is called an **equivalence relation** if (i) $\sim$ is **reflexive**, that is, $a \sim a$ for all $a \in S$, (ii) $\sim$ is **symmetric**, that is, $b \sim a$ whenever $a, b \in S$ satisfy $a \sim b$, and (iii) $\sim$ is **transitive**, that is, $a \sim c$ whenever $a, b, c \in S$ satisfy $a \sim b$ and $b \sim c$.

If $\sim$ is an equivalence relation on a set S and if $a \in S$, then the set $\{x \in S : x \sim a\}$ is called the **equivalence class** of a and is denoted[1] by $[a]$; in general, a subset E of S is called an **equivalence class** (with respect to $\sim$) if $E = [a]$ for some $a \in S$. A key fact about equivalence relations is the following result, which basically says that an equivalence relation on a set partitions the set into disjoint equivalence classes.

Proposition A.1. *Let S be a set and let $\sim$ be an equivalence relation on S. Then any two equivalence classes (with respect to $\sim$) are either disjoint or identical. Consequently, if $\mathcal{E}$ denotes the collection of distinct equivalence classes with respect to $\sim$, then*

$$S = \bigcup_{E \in \mathcal{E}} E,$$

where the union is disjoint.

Proof. Let $a, b \in S$ and suppose the equivalence classes $[a]$ and $[b]$ are not disjoint, that is, there exists $c \in [a] \cap [b]$. Then $c \sim a$ and $c \sim b$. Hence using the fact that $\sim$ is an equivalence relation, we see that for every $x \in S$,

$$x \in [a] \iff x \sim a \iff x \sim c \iff x \sim b \iff x \in [b].$$

This shows that $[a] = [b]$. Thus any two equivalence classes are either disjoint or identical. Finally, since $a \in [a]$ for each $a \in S$, we obtain $S = \bigcup_{a \in S} [a]$. $\square$

We give several examples of equivalence relations and corresponding equivalence classes below. The detailed verification of the assertions made in these examples is left to the reader.

Examples A.2. (i) On the set $\mathbb{N}$, define a relation $\sim$ by

$$m \sim n \iff m \text{ and } n \text{ have the same } \textbf{parity}, \text{ that is, } (-1)^m = (-1)^n.$$

Then $\sim$ is an equivalence relation. There are exactly two equivalence classes with respect to $\sim$, namely the set of odd positive integers and the set of even positive integers.

[1] When $S \subseteq \mathbb{Q}$, the notation $[a]$ for the equivalence class of an element a of S conflicts with the notation used in the text for the integer part of a. To avoid any possible confusion, we shall always use in this appendix the notation $\lfloor a \rfloor$ for the integer part of a.

(ii) On the set $\mathbb{N} \times \mathbb{N}$, define a relation $\sim$ by

$$(a,b) \sim (c,d) \iff a+d=b+c \quad \text{for } (a,b),(c,d) \in \mathbb{N} \times \mathbb{N}.$$

Then $\sim$ is an equivalence relation, and the equivalence classes with respect to $\sim$ are in one-to-one correspondence with the set $\mathbb{Z}$ of all integers.

(iii) Let $S = \{(m,n) : m,n \in \mathbb{Z} \text{ and } n \neq 0\}$. The relation $\sim$ on S defined by

$$(a,b) \sim (c,d) \iff ad = bc \quad \text{for } (a,b),(c,d) \in S$$

is an equivalence relation on S, and the equivalence classes with respect to $\sim$ are in one-to-one correspondence with the set $\mathbb{Q}$ of all rational numbers.

(iv) Let $n \in \mathbb{N}$. Consider the relation $\sim$ on $\mathbb{Z}$ defined by

$$a \sim b \iff a-b \text{ is divisible by } n \qquad \text{for } a,b \in \mathbb{Z}.$$

Then $\sim$ is an equivalence relation, called **congruence modulo** n. There are exactly n distinct equivalence classes with respect to $\sim$ given by $C_0, C_1, \ldots, C_{n-1}$, where for $0 \leq i < n$, the set C_i consists of integers that leave remainder i when divided by n. These equivalence classes are known as **residue classes modulo** n, and the set $\{C_0, C_1, \ldots, C_{n-1}\}$ of all residue classes modulo n is sometimes denoted by $\mathbb{Z}/n\mathbb{Z}$.

We remark that examples (ii) and (iii) above can be used to formally construct $\mathbb{Z}$ from $\mathbb{N}$, and to construct $\mathbb{Q}$ from $\mathbb{Z}$. For an axiomatic treatment of $\mathbb{N}$, we refer to the book of Landau [54].

A.2 Cauchy Sequences of Rational Numbers

We shall now define the notion of a Cauchy sequence in $\mathbb{Q}$. This is completely analogous to the notion discussed in Chapter 2, except that we will refrain from using real numbers anywhere. In particular, ϵ will denote a positive rational number, that is, $\epsilon \in \mathbb{Q}^+$. Note that since it is well understood what positive rational numbers are, the notion of the absolute value of a rational number is well-defined and satisfies the basic properties given in Proposition 1.8.

A **sequence** in $\mathbb{Q}$ is a function from $\mathbb{N}$ to $\mathbb{Q}$. We usually write (a_n) to denote the sequence $\mathbf{a} : \mathbb{N} \to \mathbb{Q}$ defined by $\mathbf{a}(n) := a_n$ for $n \in \mathbb{N}$. The rational number a_n is called the n**th term** of the sequence (a_n). A sequence (a_n) of rational numbers is said to be

1. **bounded above** if there exists $\alpha \in \mathbb{Q}$ such that $a_n \leq \alpha$ for all $n \in \mathbb{N}$,
2. **bounded below** if there exists $\beta \in \mathbb{Q}$ such that $a_n \geq \beta$ for all $n \in \mathbb{N}$,
3. **bounded** if it is bounded above as well as bounded below,
4. **Cauchy** if for every $\epsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m,n \in \mathbb{N}$ with $m,n \geq n_0$.

We shall also say that a sequence (c_n) of rational numbers is **null**, and write $c_n \to 0$, if for every $\epsilon \in \mathbb{Q}^+$, there is $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon$ for all $n \geq n_0$.

Examples A.3. (i) Let (a_n) be the sequence in $\mathbb{Q}$ defined by $a_n := 1/n$ for $n \in \mathbb{N}$. Then (a_n) is a null sequence. Indeed, given any $\epsilon \in \mathbb{Q}^+$, say $\epsilon = p/q$, where $p, q \in \mathbb{N}$, the positive integer $n_0 := q+1$ satisfies $n_0 > 1/\epsilon$. Hence $|a_n| < \epsilon$ for all $n \geq n_0$.

(ii) Let (a_n) be the sequence in $\mathbb{Q}$ defined by $a_n := (n-1)/n$ for $n \in \mathbb{N}$. Then (a_n) is a Cauchy sequence. To see this, let $\epsilon = p/q \in \mathbb{Q}^+$ be given, where $p, q \in \mathbb{N}$. Now the positive integer $n_0 := 2(q+1)$ satisfies $n_0 > 2/\epsilon$. Hence

$$|a_n - a_m| = \left|\frac{n-1}{n} - \frac{m-1}{m}\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{2}{n_0} < \epsilon \quad \text{for all } m, n \geq n_0.$$

Note that although (a_n) is a Cauchy sequence, it is not a null sequence. In fact, $|a_n| \geq 1/2$ for all $n \geq 2$.

Proposition A.4. (i) *Every Cauchy sequence of rational numbers is bounded.*
(ii) *Every null sequence of rational numbers is Cauchy.*
(iii) *Let (a_n) be a Cauchy sequence of rational numbers that is not a null sequence. Then there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n - a_{n_0}| < \epsilon_0$ and $|a_n| \geq \epsilon_0$ for all $n \geq n_0$.*

Proof. (i) Let (a_n) be a Cauchy sequence. Then there exists $k \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $m, n \geq k$. Consequently,

$$|a_n| \leq \alpha \text{ for all } n \in \mathbb{N}, \quad \text{where} \quad \alpha := \max\{|a_1|, \ldots, |a_{k-1}|, |a_k| + 1\}.$$

Hence (a_n) is bounded.

(ii) Let (c_n) be a null sequence. Given any $\epsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon/2$ for all $n \geq n_0$. Then

$$|c_n - c_m| \leq |c_n| + |c_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq n_0.$$

Hence (c_n) is Cauchy.

(iii) Since (a_n) is not a null sequence, there exists $\epsilon \in \mathbb{Q}^+$ such that for every $k \in \mathbb{N}$, there exists $n_1 \geq k$ satisfying $|a_{n_1}| \geq \epsilon$. Also, since (a_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon/2$ for all $m, n \geq n_0$. Let $k := n_0$, and find $n_1 \geq n_0$ such that $|a_{n_1}| \geq \epsilon$. Then

$$\epsilon \leq |a_{n_1}| \leq |a_{n_1} - a_n| + |a_n| \leq \frac{\epsilon}{2} + |a_n|, \text{ and hence } |a_n| \geq \frac{\epsilon}{2} \text{ for all } n \geq n_0.$$

Thus $\epsilon_0 := \epsilon/2 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ have the desired property. □

Now let us define

$$\mathscr{C} := \text{the set of all Cauchy sequences of rational numbers.}$$

Further, consider the relation $\sim$ on $\mathscr{C}$ defined by

$$(a_n) \sim (b_n) \iff (a_n - b_n) \text{ is a null sequence}, \quad \text{where } (a_n), (b_n) \in \mathscr{C}.$$

Proposition A.5. *The relation $\sim$ is an equivalence relation on $\mathscr{C}$.*

Proof. Clearly, $\sim$ is reflexive and symmetric. Suppose $(a_n), (b_n), (c_n) \in \mathscr{C}$ are such that $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Given any $\epsilon \in \mathbb{Q}^+$, the number $\epsilon/2$ is also in $\mathbb{Q}^+$. Hence there exist $n_1, n_2 \in \mathbb{N}$ such that

$$|a_n - b_n| < \frac{\epsilon}{2} \text{ for all } n \geq n_1 \quad \text{and} \quad |b_n - c_n| < \frac{\epsilon}{2} \text{ for all } n \geq n_2.$$

Now if $n_0 = \max\{n_1, n_2\}$, then for each $n \geq n_0$,

$$|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $(a_n) \sim (c_n)$. Thus $\sim$ is transitive as well. $\square$

We are now ready to define a model for $\mathbb{R}$ that we seek to construct. Let

$$\mathcal{R} := \text{the set of all equivalence classes of elements of } \mathscr{C} \text{ with respect to } \sim.$$

As in the previous section, the equivalence class of a Cauchy sequence (a_n) in $\mathscr{C}$ with respect to $\sim$ will be denoted by $[(a_n)]$. Given any $r \in \mathbb{Q}$, the constant sequence (r), that is, the sequence (r_n) with $r_n = r$ for all $n \in \mathbb{N}$, is clearly Cauchy. We will denote by $\mathcal{Q}$ the subset of $\mathcal{R}$ consisting of the equivalence classes of constant sequences of rational numbers. It is clear that the map from $\mathbb{Q}$ to $\mathcal{Q}$ given by $r \longmapsto [(r)]$ is one-one and onto. Thus we can, and will, identify $\mathcal{Q}$ with $\mathbb{Q}$. In particular, the equivalence classes of the constant sequences (0) and (1) will be denoted simply by 0 and 1, respectively.

We now define addition and multiplication on the set $\mathcal{R}$ as follows.

$$[(a_n)] + [(b_n)] = [(a_n + b_n)] \text{ and } [(a_n)] \cdot [(b_n)] = [(a_n b_n)] \quad \text{for } (a_n), (b_n) \in \mathscr{C}.$$

Proposition A.6. *The operations of addition and multiplication on $\mathcal{R}$ are well-defined and satisfy the following algebraic properties:*

A1. $a + (b + c) = (a + b) + c$ *and* $a(bc) = (ab)c$ *for all* $a, b, c \in \mathcal{R}$.
A2. $a + b = b + a$ *and* $ab = ba$ *for all* $a, b \in \mathcal{R}$.
A3. $a + 0 = a$ *and* $a \cdot 1 = a$ *for all* $a \in \mathcal{R}$.
A4. *Let* $a \in \mathcal{R}$. *Then there exists* $a' \in \mathcal{R}$ *such that* $a + a' = 0$. *Further, if* $a \neq 0$, *then there exists* $a^* \in \mathcal{R}$ *such that* $aa^* = 1$.
A5. $a(b + c) = ab + ac$ *for all* $a, b, c \in \mathcal{R}$.

Proof. To show that the operations of addition and multiplication on $\mathcal{R}$ are well-defined, it suffices to show that for all (a_n), (a'_n), (b_n), $(b'_n) \in \mathscr{C}$,

$$(a_n) \sim (a'_n) \text{ and } (b_n) \sim (b'_n) \Longrightarrow (a_n + b_n) \sim (a'_n + b'_n) \text{ and } (a_n b_n) \sim (a'_n b'_n).$$

The assertion $(a_n + b_n) \sim (a'_n + b'_n)$ follows from the definition, since $|(a_n + b_n) - (a'_n + b'_n)| \leq |a_n - a'_n| + |b_n - b'_n|$. To see that $(a_n b_n) \sim (a'_n b'_n)$, we use part (i) of Proposition A.4 and obtain $\alpha', \beta \in \mathbb{Q}^+$ such that $|a'_n| \le \alpha'$ and $|b_n| \le \beta$ for all $n \in \mathbb{N}$, so that

$$|a_n b_n - a'_n b'_n| = |(a_n - a'_n)b_n + a'_n(b_n - b'_n)| \le \beta|a_n - a'_n| + \alpha'|b_n - b'_n|.$$

Now since $a_n - a'_n \to 0$ and $b_n - b'_n \to 0$, it is readily seen that $a_n b_n - a'_n b'_n \to 0$.

Having established that addition and multiplication on $\mathcal{R}$ are well-defined, we see that properties A1, A2, A3, and A5 are immediate consequences of the definition and the corresponding properties of rational numbers.

Moreover, for every $a = [(a_n)] \in \mathcal{R}$, the element $a' := [(-a_n)]$ is clearly in $\mathcal{R}$ and it satisfies $a + a' = 0$. Finally, suppose $a = [(a_n)] \in \mathcal{R}$ is such that $a \neq 0$. Then by part (iii) of Proposition A.4, there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n| \ge \epsilon_0$ for all $n \ge n_0$. In particular, $a_n \neq 0$ for all $n \ge n_0$. Define the sequence (a_n^*) in $\mathbb{Q}$ by $a_n^* := 1$ for $1 \le n < n_0$ and $a_n^* = 1/a_n$ for $n \ge n_0$. Then $|a_n^* - a_m^*| \le (1/\epsilon_0^2)|a_n - a_m|$ for all $n, m \ge n_0$. Since (a_n) is a Cauchy sequence, this implies that (a_n^*) is also a Cauchy sequence. Moreover, $a_n a_n^* - 1 = 0$ for all $n \ge n_0$, and so $(a_n a_n^*) \sim (1)$. This proves A4. □

As noted in Section 1.1, several simple properties (such as, $a \cdot 0 = 0$ for all $a \in \mathcal{R}$) are formal consequences of properties A1–A5 proved in Proposition A.6, and these will now be tacitly assumed; also uniqueness of the additive inverse $a' \in \mathcal{R}$ for $a \in \mathcal{R}$, and of the multiplicative inverse $a^* \in \mathcal{R}$ for $a \in \mathcal{R}$ with $a \neq 0$ as in A4, is a formal consequence of Proposition A.6, and we will adopt the usual notation $-a$ for a', and $1/a$ or a^{-1} for a^*. We remark also that as a consequence of Proposition A.6, the addition and multiplication on $\mathcal{R}$ are compatible with the usual addition and multiplication on $\mathbb{Q}$ when $\mathbb{Q}$ is identified with the subset $\mathcal{Q}$ of $\mathcal{R}$ as before.

Now let us turn to order properties. We shall say that a sequence $(a_n) \in \mathscr{C}$ is **positive** if it satisfies the following property:

$$\text{There exist } r \in \mathbb{Q}^+ \text{ and } n_0 \in \mathbb{N} \text{ such that } a_n \ge r \text{ for all } n \ge n_0.$$

Note that if $(a'_n) \in \mathscr{C}$ is such that $(a'_n) \sim (a_n)$ and (a_n) satisfies the above property, then so does (a'_n). Indeed, since $(a'_n) \sim (a_n)$, there exists $n_1 \in \mathbb{N}$ such that $|a'_n - a_n| < r/2$ for all $n \ge n_1$. Now if we let $r' := r/2$ and $n_2 := \max\{n_0, n_1\}$, then we obtain

$$a'_n > a_n - \frac{r}{2} \ge r - \frac{r}{2} = r' \text{ for all } n \ge n_2.$$

With this in view, we define $\mathcal{R}^+$ to be the set of all equivalence classes of positive sequences in $\mathscr{C}$. It is clear that $\mathcal{R}^+$ is a well-defined subset of $\mathcal{R}$.

Proposition A.7. *The set $\mathcal{R}^+$ satisfies the following order properties:*

O1. *Given any $a \in \mathcal{R}$, exactly one of the following statements is true:*

$$a \in \mathcal{R}^+; \qquad a = 0; \qquad -a \in \mathcal{R}^+.$$

O2. *If $a, b \in \mathcal{R}^+$, then $a + b \in \mathcal{R}^+$ and $ab \in \mathcal{R}^+$.*

Proof. Let $(a_n) \in \mathscr{C}$ be such that $[(a_n)] \neq 0$. By part (iii) of Proposition A.4, there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n - a_{n_0}| < \epsilon_0$ and $|a_n| \geq \epsilon_0$ for all $n \geq n_0$. In particular, $a_{n_0} \neq 0$. Now if $a_{n_0} > 0$, then the above inequalities imply $a_n = a_{n_0} + (a_n - a_{n_0}) > \epsilon_0 - \epsilon_0 = 0$ for all $n \geq n_0$, and consequently, $a_n \geq \epsilon_0$ for all $n \geq n_0$. Likewise, if $a_{n_0} < 0$, then $-a_n \geq \epsilon_0$ for all $n \geq n_0$. Thus $[(a_n)] \in \mathcal{R}^+$ or $-[(a_n)] = [(-a_n)] \in \mathcal{R}^+$. This proves O1.

Next, if $(a_n), (b_n) \in \mathscr{C}$ are such that $[(a_n)], [(b_n)] \in \mathcal{R}^+$, then there exist $r_1, r_2 \in \mathbb{Q}^+$ and $n_1, n_2 \in \mathbb{N}$ such that

$$a_n > r_1 \text{ for all } n \geq n_1 \quad \text{and} \quad b_n > r_2 \text{ for all } n \geq n_2.$$

Now if we let $n_0 = \max\{n_1, n_2\}$, then we clearly have

$$a_n + b_n > r_1 + r_2 \text{ for all } n \geq n_0 \quad \text{and} \quad a_n b_n > r_1 r_2 \text{ for all } n \geq n_0.$$

Since $r_1 + r_2,\ r_1 r_2 \in \mathbb{Q}^+$, we obtain $[(a_n)] + [(b_n)] \in \mathcal{R}^+$ and $[(a_n)][(b_n)] \in \mathcal{R}^+$. This proves O2. $\square$

Using the set $\mathcal{R}^+$, we can define an order relation on $\mathcal{R}$ exactly as in Section 1.1, namely, for all $a, b \in \mathcal{R}$, we write $a < b$ or $b > a$ if $b - a \in \mathcal{R}^+$. Moreover, we shall write $a \leq b$ or $b \geq a$ to mean that either $a < b$ or $a = b$. The usual properties of this order relation, as listed in (i), (ii), and (iii) on page 3 (with $\mathbb{R}$ replaced by $\mathcal{R}$), and also the fact that $1 > 0$ are formal consequences of O1 and O2, and will thus be tacitly assumed. Moreover, the notions of a subset of $\mathcal{R}$ being bounded above, bounded below, or bounded as well as the notions of upper bound, lower bound, supremum, and infimum for subsets of $\mathcal{R}$ can now be defined exactly as they were defined for subsets of $\mathbb{R}$ in Chapter 1. Note also that the order relation on $\mathcal{R}$ that we have just defined is compatible with the known order relation on the set $\mathbb{Q}$, that is, if $r, s \in \mathbb{Q}$ and if $[(r)], [(s)]$ are the corresponding elements of $\mathcal{Q}$, then $r < s$ if and only if $[(r)] < [(s)]$.

We shall now proceed to prove that the set $\mathcal{R}$, which we have constructed from $\mathbb{Q}$, has the completeness property. As a preliminary step, we will first show that $\mathcal{R}$ has the archimedean property. It may be recalled that in Proposition 1.3, the archimedean property of $\mathbb{R}$ was deduced from the assumption that $\mathbb{R}$ has the completeness property. Here we will give a direct proof to show that $\mathcal{R}$ has the archimedean property, and later, use it to derive the completeness property of $\mathcal{R}$.

Proposition A.8. *Given any $a \in \mathcal{R}$, there is some $k \in \mathbb{N}$ such that $k > a$.*

Proof. First note that if $a \in \mathcal{Q}$, then a corresponds to a unique rational number p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Hence $k := |p| + 1$ clearly satisfies $k > a$. Now suppose $a = [(a_n)]$ is an arbitrary element of $\mathcal{R}$, where (a_n) is a Cauchy sequence of rational numbers. Then for $\epsilon = 1/2$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < 1/2$ for all $n, m \geq n_0$. Since the archimedean property holds for rational numbers, there exists $\ell \in \mathbb{N}$ such that $|a_{n_0}| < \ell$. Hence for each $n \geq n_0$, we obtain

$$|a_n| \leq |a_n - a_{n_0}| + |a_{n_0}| < \frac{1}{2} + \ell \quad \text{and hence} \quad |(1+\ell) - a_n| \geq (1+\ell) - |a_n| > \frac{1}{2}.$$

It follows that $[(a_n)] < [(1+\ell)]$, that is, $k := 1 + \ell \in \mathbb{N}$ satisfies $k > a$. □

The archimedean property proved in Proposition A.8 enables us to define the integer part of every $x \in \mathcal{R}$ exactly as in the paragraph following Proposition 1.3 of Chapter 1, and this, in turn, permits us to deduce that between any two elements of $\mathcal{R}$, there is a rational number. It is important to note that this result uses only the algebraic and order properties together with the archimedean property (Propositions A.6, A.7, and A.8).

Proposition A.9. *Given any $a, b \in \mathcal{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. The proof is identical to that of Proposition 1.6 and hence omitted. A more general result (Proposition A.12) is proved in the next section. □

Corollary A.10. *Let $(r_n) \in \mathscr{C}$ and let $a = [(r_n)]$ be the corresponding element of $\mathcal{R}$. If there exists $n_0 \in \mathbb{N}$ such that $r_n \geq 0$ for all $n \geq n_0$, then $a \geq 0$. More generally, if there exist $\alpha, \beta \in \mathcal{R}$ and $n_0 \in \mathbb{N}$ such that $\beta \leq r_n \leq \alpha$ for all $n \geq n_0$, then $\beta \leq a \leq \alpha$.*

Proof. Suppose there exists $n_0 \in \mathbb{N}$ such that $r_n \geq 0$ for all $n \geq n_0$. Let, if possible, $a < 0$. Then by Proposition A.9, there exists $s \in \mathbb{Q}$ such that $a < s < 0$. Now $-s > 0$ and $r_n - s \geq -s$ for all $n \geq n_0$. So it follows from the definition of $\mathcal{R}^+$ that $a - s > 0$, which is a contradiction. This proves that $a \geq 0$.

Next suppose $\alpha, \beta \in \mathcal{R}$ and $n_0 \in \mathbb{N}$ are such that $\beta \leq r_n \leq \alpha$ for all $n \geq n_0$. By Proposition A.9, there exist $\alpha_n, \beta_n \in \mathbb{Q}$ such that $\beta - \frac{1}{n} < \beta_n < \beta$ and $\alpha < \alpha_n < \alpha + \frac{1}{n}$ for each $n \in \mathbb{N}$. This implies that $(\alpha_n), (\beta_n) \in \mathscr{C}$. Moreover, $\alpha = [(\alpha_n)]$ and $\beta = [(\beta_n)]$. (Verify!) Now applying the first assertion in the corollary to $(r_n - \beta_n)$ and $(\alpha_n - r_n)$, we obtain $\beta \leq a \leq \alpha$. □

We are now ready to prove that the set $\mathcal{R}$ has the completeness property.

Proposition A.11. *Every nonempty subset of $\mathcal{R}$ that is bounded above has a supremum.*

Proof. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$ that is bounded above. Since $\mathcal{S}$ is nonempty, there is some $a_0 \in \mathcal{S}$, and since $\mathcal{S}$ is bounded above, there is some $\alpha_0 \in \mathcal{R}$ such that α_0 is an upper bound of $\mathcal{S}$, that is, $a \leq \alpha_0$ for all $a \in \mathcal{S}$. Now let $\beta_1 := (a_0 + \alpha_0)/2$. If β_1 is an upper bound of $\mathcal{S}$, we let $a_1 := a_0$ and $\alpha_1 := \beta_1$, whereas if β_1 is not an upper bound of $\mathcal{S}$, then there exists $b \in \mathcal{S}$

such that $\beta_1 < b$, and in this case, we let $a_1 := b$ and $\alpha_1 := \alpha_0$. In any case, $a_0 \le a_1$ and $\alpha_0 \ge \alpha_1$, and moreover,

$$a_1 \in \mathcal{S}, \quad \alpha_1 \text{ is an upper bound of } \mathcal{S}, \quad \text{and} \quad 0 \le \alpha_1 - a_1 \le \frac{\alpha_0 - a_0}{2}.$$

Next, we replace (a_0, α_0) by (a_1, α_1) and proceed as before. In general, given $n \in \mathbb{N}$ and $a_i \in \mathcal{S}$ and upper bounds α_i of $\mathcal{S}$ with $0 \le (\alpha_i - a_i) \le (\alpha_0 - a_0)/2^i$ for $0 \le i \le n-1$ and with $a_0 \le a_1 \le \cdots \le a_{n-1}$ and $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_{n-1}$, we choose $a_n \in \mathcal{S}$ and an upper bound α_n of $\mathcal{S}$ as follows. Let $\beta_n := (a_{n-1} + \alpha_{n-1})/2$. If β_n is an upper bound of $\mathcal{S}$, we let $a_n := a_{n-1}$ and $\alpha_n := \beta_n$, whereas if β_n is not an upper bound of $\mathcal{S}$, then there exists $b \in \mathcal{S}$ such that $\beta_n < b$, and in this case, we let $a_n := b$ and $\alpha_n := \alpha_{n-1}$. In any case, $a_{n-1} \le a_n$ and $\alpha_{n-1} \ge \alpha_n$, and moreover,

$$a_n \in \mathcal{S}, \quad \alpha_n \text{ is an upper bound of } \mathcal{S}, \quad \text{and} \quad 0 \le \alpha_n - a_n \le \frac{\alpha_0 - a_0}{2^n}.$$

Note that if $a_n = \alpha_n$ for some $n \ge 0$, then clearly α_n is the supremum of $\mathcal{S}$.

Now suppose $a_n < \alpha_n$ for all $n \ge 0$. By Proposition A.9, for each $n \in \mathbb{N}$, there exists $r_n \in \mathbb{Q}$ such that $a_n < r_n < \alpha_n$. We claim that (r_n) is a Cauchy sequence. To see this, let $\epsilon \in \mathbb{Q}^+$ be given. Applying Proposition A.8 to $a = (\alpha_0 - a_0)/\epsilon$, we see that there exists $k \in \mathbb{N}$ such that

$$\frac{\alpha_0 - a_0}{k} < \epsilon \quad \text{and hence} \quad \frac{\alpha_0 - a_0}{2^k} < \epsilon,$$

where the last inequality follows by noting that $2^j \ge j$ for all $j \in \mathbb{N}$, as can be seen easily by induction on j. Now given any $m, n \in \mathbb{N}$ with $m \ge n \ge k$, since $a_m < r_m < \alpha_m$, $a_n < r_n < \alpha_n$, and $a_m \ge a_n$, we obtain

$$r_n - r_m < \alpha_n - r_m < \alpha_n - a_m \le \alpha_n - a_n.$$

In a similar way, since $\alpha_m \le \alpha_n$, we obtain

$$r_n - r_m > a_n - r_m > a_n - \alpha_m \ge a_n - \alpha_n.$$

It follows that

$$|r_n - r_m| < \alpha_n - a_n \le \frac{\alpha_0 - a_0}{2^n} \le \frac{\alpha_0 - a_0}{2^k} < \epsilon \quad \text{for all } m, n \ge k.$$

Thus $(r_n) \in \mathscr{C}$, and so $\alpha := [(r_n)] \in \mathcal{R}$. We shall now show that α is the supremum of $\mathcal{S}$. To this end, let us first observe that for every fixed $m \in \mathbb{N}$, the inequalities $a_m \le a_n < r_n < \alpha_n \le \alpha_m$ hold for all $n \ge m$, and so by Corollary A.10, we see that $a_m \le \alpha \le \alpha_m$.

Now suppose, if possible, α is not an upper bound of $\mathcal{S}$. Then there is $a \in \mathcal{S}$ such that $a > \alpha$. By Proposition A.9, there exists $\delta \in \mathbb{Q}^+$ such that $\delta < a - \alpha$. Further, by Proposition A.8, there is $m \in \mathbb{N}$ such that $m > (\alpha_0 - a_0)/\delta$, and so

$$0 \le \alpha_m - a_m \le \frac{\alpha_0 - a_0}{2^m} \le \frac{\alpha_0 - a_0}{m} < \delta.$$

Hence $\alpha_m < a_m + \delta \le \alpha + \delta < a$. But this contradicts the fact that α_m is an upper bound of $\mathcal{S}$. Hence α is an upper bound of $\mathcal{S}$.

Next, suppose α is not the least upper bound of $\mathcal{S}$. Then there exists $\beta \in \mathcal{R}$ with $\beta < \alpha$ such that β is an upper bound of $\mathcal{S}$. Again, choose $\delta \in \mathbb{Q}^+$ such that $0 < \delta < \alpha - \beta$ and $m \in \mathbb{N}$ such that $0 \le \alpha_m - a_m < \delta$. Then $a_m > \alpha_m - \delta \ge \alpha - \delta > \beta$, which is a contradiction, since $a_m \in \mathcal{S}$ and β is an upper bound of $\mathcal{S}$. It follows that α is the supremum of $\mathcal{S}$. □

A.3 Uniqueness of a Complete Ordered Field

In the previous section, we have shown that the set $\mathcal{R}$ possesses all the properties that were postulated for $\mathbb{R}$ in Chapter 1. In other words, we have established the existence of the set of all real numbers. We will now prove its "uniqueness". First, we introduce some useful terminology.

By a **field** we shall mean a set F that has operations of addition and multiplication (that is, maps from $F \times F$ to F that associate elements $a + b$ and ab of F to $(a, b) \in F \times F$) and has distinct elements 0_F and 1_F in it such that the five algebraic properties A1–A5 in Proposition A.6 are satisfied with $\mathcal{R}$ replaced throughout by F, and with 0 and 1 replaced by 0_F and 1_F. It is easy to see that in a field F, elements 0_F and 1_F satisfying A3 are unique, and these are sometimes called the additive identity and the multiplicative identity of F, respectively. Note that $\mathbb{Q}$ and $\mathcal{R}$ are examples of fields. A special case of Example A.2 (iv), namely the set $\mathbb{Z}/p\mathbb{Z}$ of residue classes modulo a prime number p, is a field having only finitely many elements.

If a field F contains a subset F^+ satisfying the two order properties O1–O2 with $\mathcal{R}^+$ replaced throughout by F^+, then F is called an **ordered field**; in this case, for every $a, b \in F$, we write $a < b$ or $b > a$ if $b - a \in F^+$; also, we write $a \le b$ or $b \ge a$ if either $a < b$ or $a = b$. The notions of boundedness, supremum, etc. are defined for subsets of an ordered field in exactly the same way as in the case of $\mathbb{R}$. Note that $\mathbb{Q}$ and $\mathcal{R}$ are ordered fields, but $\mathbb{Z}/p\mathbb{Z}$ is not. In fact, an ordered field F cannot be finite. Indeed, $1_F > 0_F$ (because otherwise $-1_F > 0_F$, and so $1_F = (-1_F)(-1_F) > 0_F$, which would be a contradiction). Hence for every $n \in \mathbb{N}$, if we let $n_F := 1_F + \cdots + 1_F$ (n times), then $n_F > 0_F$; moreover, $0_F < 1_F < 2_F < \cdots$, and so F contains infinitely many elements. Furthermore, in an ordered field F, for every $m \in \mathbb{Z}$ with $m < 0$, we let m_F denote the additive inverse of $(-m)_F$, that is, the unique element of F satisfying $(-m)_F + m_F = 0_F$. For every $r = m/n$ in $\mathbb{Q}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we let $r_F := (m_F)(n_F)^{-1}$. It is easily seen that $r \longmapsto r_F$ gives a well-defined, one-one map of $\mathbb{Q} \to F$, which preserves algebraic and order operations, that is, for all $r, s \in \mathbb{Q}$,

$$(r + s)_F = r_F + s_F, \quad (rs)_F = r_F s_F, \quad \text{and} \quad r < s \Longrightarrow r_F < s_F.$$

Thus F contains a copy of $\mathbb{Q}$, namely $\mathbb{Q}_F := \{r_F : r \in \mathbb{Q}\}$. In an ordered field F, the **absolute value** of an element can be defined as in the case of $\mathbb{Q}$ or $\mathbb{R}$. Thus for every $a \in F$, we let $|a| := a$ if $a \geq 0_F$ and $|a| := -a$ if $a < 0_F$. It is easily seen that $|ab| = |a||b|$ and $|a+b| \leq |a| + |b|$ for all $a, b \in F$.

Let F be an ordered field. We say that F is **archimedean** if for every $a \in F$, there exists $n \in \mathbb{N}$ such that $n_F > a$. and we say that F is **complete** if every nonempty subset of F that is bounded above has a supremum in F. For example, both $\mathbb{Q}$ and $\mathcal{R}$ are archimedean ordered fields, and $\mathcal{R}$ is complete (thanks to Proposition A.11), but $\mathbb{Q}$ is not. In general, by arguing as in the proof of Proposition 1.3, we readily see that a complete ordered field is archimedean.

Let F be an archimedean ordered field and let $a \in F$. Then there exist $m, n \in \mathbb{N}$ such that $m_F > -a$ and $n_F > a$, that is, $-m_F < a < n_F$. Thus if k is the largest among the finitely many integers satisfying $-m \leq k \leq n$ and $k_F \leq a$, then k_F is called the **integer part** of a in F, and is denoted by $\lfloor a \rfloor$. Note that $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1_F$. The following result is similar to Proposition A.9. The proof is similar to that of Proposition 1.6, but this time we include it.

Proposition A.12. *Let F be an archimedean ordered field and let $a_1, a_2 \in F$ satisfy $a_1 < a_2$. Then there is $r \in \mathbb{Q}$ such that $a_1 < r_F < a_2$.*

Proof. Since F is archimedean, there exists $n \in \mathbb{N}$ such that $n_F > (a_2 - a_1)^{-1}$, that is, $(n_F)^{-1} < (a_2 - a_1)$. Let $m \in \mathbb{N}$ be such that $m_F = \lfloor n_F a_1 \rfloor + 1_F$. Then $m_F - 1_F \leq n_F a_1 < m_F$. Hence

$$a_1 < m_F (n_F)^{-1} \leq (n_F a_1 + 1_F)(n_F)^{-1} = a_1 + (n_F)^{-1} < a_1 + (a_2 - a_1) = a_2.$$

Thus $r = m/n \in \mathbb{Q}$ satisfies $a_1 < r_F < a_2$. □

Corollary A.13. *Let F be an archimedean ordered field. Suppose $a \in F$ satisfies $|a| < \epsilon_F$, that is, $-\epsilon_F < a < \epsilon_F$, for all $\epsilon \in \mathbb{Q}^+$. Then $a = 0_F$.*

Proof. In case $a > 0_F$, by Proposition A.12, there exists $r \in \mathbb{Q}$ such that $0_F < r_F < a$. Thus the hypothesis is contradicted if we take $\epsilon = r$. Likewise, we arrive at a contradiction if $a < 0_F$. Hence $a = 0_F$. □

Let K and F be ordered fields. A map $f : K \to F$ is called an **order isomorphism** if f is both one-one and onto, and f preserves algebraic and order operations, that is, for all $x, y \in K$,

$$f(x+y) = f(x) + f(y), \quad f(xy) = f(x)f(y), \quad \text{and} \quad x < y \Longrightarrow f(x) < f(y).$$

If such a map exists, then we say that F is **order isomorphic** to K.

Proposition A.14. *Every complete ordered field is order isomorphic to $\mathcal{R}$.*

Proof. Let F be a complete ordered field. Define $f : \mathcal{R} \to F$ by

$$f(x) := \sup F_x, \quad \text{where} \quad F_x := \{r_F : r \in \mathbb{Q} \text{ and } r \le x\} \quad \text{for } x \in \mathcal{R}.$$

Note that f is well-defined. Indeed, given any $x \in \mathcal{R}$, by Proposition A.9, there exist $s, t \in \mathbb{Q}$ such that $x - 1 < s < x < t < x + 1$. It follows that $s_F \in F_x$ and t_F is an upper bound of F_x. Thus $\sup F_x$ exists, since F is complete. Note also that $f(r) = r_F$ for all $r \in \mathbb{Q}$. Indeed, $f(r) < r_F$ as well as $f(r) > r_F$ will both lead to a contradiction using Proposition A.12.

Let $x, y \in \mathcal{R}$ be such that $x < y$. Then $x < (x + y)/2 < y$, and using Corollary A.9, we can find $u, v \in \mathbb{Q}$ such that $x < u < (x + y)/2 < v < y$. Now u_F is an upper bound of F_x and v_F is an element of F_y. Hence we obtain $f(x) \le u_F < v_F \le f(y)$. Thus f is order-preserving, and therefore one-one.

To show that f is onto, suppose $a \in F$. In case $a = r_F \in \mathbb{Q}_F$ for some $r \in \mathbb{Q}$, then $a = f(r)$. Now suppose $a \notin \mathbb{Q}_F$. Let $\mathbb{Q}_a := \{r \in \mathbb{Q} : r_F \le a\}$. Since F is complete, it is archimedean, and therefore by Proposition A.12, there exist $r, s \in \mathbb{Q}$ such that $a - 1 < r_F < a$ and $a < s_F < a + 1$. This implies that the set $\mathbb{Q}_a$ is nonempty and bounded above. Hence $x := \sup \mathbb{Q}_a$ is a well-defined element of $\mathcal{R}$. We shall now show that $f(x) = a$.

First, suppose $x \in \mathbb{Q}$. Then $f(x) = x_F$. Now if $x_F < a$, then by Proposition A.12, there exists $r \in \mathbb{Q}$ such that $x_F < r_F < a$, and this leads to a contradiction, because on the one hand $x < r$, since $x, r \in \mathbb{Q}$ and $x_F < r_F$, but on the other hand, $r \le x$, since $r_F < a$ implies $r \in \mathbb{Q}_a$ and $x = \sup \mathbb{Q}_a$. Likewise, if $x_F > a$, then by Proposition A.12, there exists $s \in \mathbb{Q}$ such that $a < s_F < x_F$, but then s is an upper bound of $\mathbb{Q}_a$ (because $r \in \mathbb{Q}$ and $r_F \le a$ implies $r_F < s_F$ and hence $r < s$), and therefore $x = \sup \mathbb{Q}_a \le s$, which implies $x_F \le s_F$, and this contradicts $s_F < x_F$. Thus $f(x) = a$ when $x \in \mathbb{Q}$.

Next, suppose $x \notin \mathbb{Q}$. Let $r \in \mathbb{Q}$ with $r \le x$. Since $x \notin \mathbb{Q}$, we obtain $r < x$, and since $x = \sup \mathbb{Q}_a$, there exists $s \in \mathbb{Q}_a$ such that $r < s \le x$. Consequently, $r_F < s_F \le a$. It follows that a is an upper bound of F_x. Hence $f(x) \le a$. Furthermore, if $f(x) < a$, then by Proposition A.12, there exists $t \in \mathbb{Q}$ such that $f(x) < t_F < a$. But then $t \in \mathbb{Q}_a$, and so $t \le x$, which implies $t_F \le f(x)$, and so we obtain a contradiction. It follows that $f(x) = a$. Thus f is onto.

It remains to show that f preserves the algebraic operations. Let $x, y \in \mathcal{R}$ and let $\epsilon \in \mathbb{Q}^+$ be given. By Proposition A.9, there exist $r, s, u, v \in \mathbb{Q}$ such that

$$x - \frac{\epsilon}{4} < r < x < s < x + \frac{\epsilon}{4} \quad \text{and} \quad y - \frac{\epsilon}{4} < u < y < v < y + \frac{\epsilon}{4}.$$

Then $0 < s - r < \epsilon/2$ and $0 < v - u < \epsilon/2$. Since f is order-preserving, $r_F < f(x) < s_F$ and $u_F < f(y) < v_F$. Hence $r_F + u_F < f(x) + f(y) < s_F + v_F$. Moreover, $r + u < x + y < s + v$, and again since f is order-preserving, $r_F + u_F < f(x + y) < s_F + v_F$. Consequently,

$$f(x + y) - f(x) - f(y) < (s_F - r_F) + (v_F - u_F) < \frac{\epsilon_F}{2_F} + \frac{\epsilon_F}{2_F} = \epsilon_F.$$

By a similar argument, $f(x + y) - f(x) - f(y) > -\epsilon_F$. Now by Corollary A.13, we obtain $f(x + y) = f(x) + f(y)$. Thus, f preserves addition.

To show that f preserves multiplication, first consider $x, y \in \mathcal{R}$ with $x > 0$ and $y > 0$. Let $\epsilon \in \mathbb{Q}^+$ be given. By Proposition A.8, there exists $n \in \mathbb{N}$ such that $n > x$, $n > y$, and $n > \epsilon/6$. Now using Proposition A.9 and the assumption that $x > 0$ and $y > 0$, we obtain $r, s, u, v \in \mathbb{Q}^+$ such that

$$x - \frac{\epsilon}{6n} < r < x < s < x + \frac{\epsilon}{6n} \quad \text{and} \quad y - \frac{\epsilon}{6n} < u < y < v < y + \frac{\epsilon}{6n}.$$

Next, we use the usual trick of adding and subtracting suitable terms to write

$$xy - ru = (x-r)y + r(y-u) \quad \text{and} \quad sv - xy = (s-x)(v-y) + x(v-y) + y(s-x).$$

Consequently, $xy - ru < (\epsilon/6n)y + r(\epsilon/6n) < (\epsilon/6) + (\epsilon/6) < \epsilon/2$ and

$$sv - xy < \left(\frac{\epsilon}{6n}\right)^2 + x\frac{\epsilon}{6n} + y\frac{\epsilon}{6n} < \frac{\epsilon}{6n} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \le \frac{\epsilon}{6} + \frac{\epsilon}{3} = \frac{\epsilon}{2}.$$

Thus $sv - (\epsilon/2) < xy < ru + (\epsilon/2)$, and since f is order-preserving, we obtain

$$s_F v_F - \frac{\epsilon_F}{2_F} < f(xy) < r_F u_F + \frac{\epsilon_F}{2_F}.$$

Again, since f is order-preserving, by arguing as before, we obtain

$$s_F v_F - \frac{\epsilon_F}{2_F} < f(x)f(y) < r_F u_F + \frac{\epsilon_F}{2_F}.$$

It follows that $-\epsilon_F < f(xy) - f(x)f(y) < \epsilon_F$. By Corollary A.13, we obtain $f(xy) = f(x)f(y)$. Finally, since f preserves addition, it is easily seen that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathcal{R}$. Hence the result just proved, namely $f(xy) = f(x)f(y)$ for all positive $x, y \in \mathcal{R}$, implies that $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{R}$. So f preserves multiplication as well. □

Remark A.15. In view of the results of the previous section and the uniqueness result in Proposition A.14, it makes sense to refer to any set satisfying the algebraic, order, and completeness properties as **the** set of all real numbers. The construction of $\mathcal{R}$ given in the previous section is one of the ways of constructing $\mathbb{R}$. Several other constructions are possible. The most prominent among these is a construction due to Dedekind, where the basic idea is to determine a real number x by means of the pair (L_x, R_x) of subsets of $\mathbb{Q}$, where $L_x := \{r \in \mathbb{Q} : r < x\}$ and $R_x := \{r \in \mathbb{Q} : r \ge x\}$. Such a pair is called a **Dedekind cut**, or simply a **cut**. More formally, a **cut** is a pair (L, R) of nonempty disjoint subsets of $\mathbb{Q}$ such that (i) $L \cup R = \mathbb{Q}$, (ii) L is downwards closed, that is, $s \in L$ whenever $s < t$ for some $t \in L$, (iii) R is upwards closed, that is, $s \in R$ whenever $s > t$ for some $t \in R$, and (iv) L has no maximum element. One defines addition, multiplication, and an order on the set of all cuts, and shows that this set is a complete ordered field. For more details about this approach, one can refer to the essays of Dedekind [24] or the appendix to Chapter 1 of Rudin [71]. For a host of other constructions for $\mathbb{R}$, we refer to the article of Weiss [87]. At any rate, by Proposition A.14, all these constructions yield essentially the same ordered field. ◇

Remark A.16. In this book, we have used the word completeness (of $\mathbb{R}$, or more generally, of any ordered field F) to mean that every nonempty subset that is bounded above has a supremum. This is sometimes referred to as **order completeness**, especially when contrasted with other notions such as **monotone completeness** and **Cauchy completeness** that are defined as follows. An ordered field F is said to be **monotone complete** (resp. **Cauchy complete**) if every monotonic bounded sequence (resp. Cauchy sequence) in F is convergent. Note that in an ordered field, the notions of absolute value and of a sequence being monotonic, bounded, convergent, or Cauchy are defined in the same way as in the case of $\mathbb{R}$. Arguing as in the proofs of Propositions 2.8 and 2.22, we readily see that for every ordered field F,

$$F \text{ is complete} \Longrightarrow F \text{ is monotone complete} \Longrightarrow F \text{ is Cauchy complete.}$$

It can be shown that Cauchy completeness implies (order) completeness, provided that the ordered field is archimedean. (Compare Exercise 2.42 of Chapter 2.) Thus for an archimedean ordered field, the three notions of completeness are equivalent. Moreover, an ordered field that is monotone complete is necessarily archimedean (because otherwise, the sequence (n_F) would be convergent, and if n_F converges to a, then there exists $n_0 \in \mathbb{N}$ such that $a - 1_F < n_F < a + 1_F$ for $n \geq n_0$, but then $(n+2)_F > a + 1_F$, which would be a contradiction!) and consequently, order complete. However, there do exist ordered fields that are Cauchy complete, but not archimedean and therefore neither order complete nor monotone complete; see, for instance, Ex. 4 and 7 in Chapter 1 of Gelbaum and Olmsted [32]. For more on various notions of completeness and related matters, see the article of Hall and Todorov [36]. Finally, we remark that the notion of Cauchy completeness can be readily defined for any field F that (is not necessarily ordered, but) has an "absolute value function", that is, a map from F to $\mathbb{R}^+$ given by $a \longmapsto |a|$ satisfying for all $a, b \in F$, the following: (i) $|a| = 0 \iff a = 0_F$, (ii) $|ab| = |a||b|$, and (iii) $|a + b| \leq |a| + |b|$. In the next section, we will formally introduce the field $\mathbb{C}$ of complex numbers and show that $\mathbb{C}$ is not an ordered field, but $\mathbb{C}$ has an absolute value function. Moreover, it is easy to show (using Proposition 2.22) that $\mathbb{C}$ is Cauchy complete. $\diamond$

B

Fundamental Theorem of Algebra

Although the set $\mathbb{R}$ of all real numbers is "complete", it has a lacuna from an algebraic point of view. Namely, there are polynomials with real coefficients that have no root in $\mathbb{R}$. The simplest among these is the polynomial x^2+1. By adjoining to $\mathbb{R}$ an "imaginary" root i of x^2+1, we obtain the complex numbers, which are sometimes "defined" as the numbers of the form $x + iy$, where $x, y \in \mathbb{R}$. It is not difficult to give a formal and precise definition of complex numbers and in particular, of the number i. We do this in Section B.1, and then outline how several of the notions about real-valued functions can be extended to complex-valued functions. Next, in Section B.2, we prove a remarkable result known as the Fundamental Theorem of Algebra, which basically says that if one can solve $x^2 + 1 = 0$, then one can solve every polynomial equation in one variable with real or complex coefficients!

B.1 Complex Numbers and Complex Functions

A **complex number** is defined as an ordered pair of real numbers. The set of all complex numbers is denoted by $\mathbb{C}$. Addition and multiplication in $\mathbb{C}$ are defined as follows. For all $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, let

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2,\ y_1 + y_2), \text{ and}\\ (x_1, y_1)(x_2, y_2) &:= (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).\end{aligned}$$

It is easily seen that with respect to these algebraic operations, $\mathbb{C}$ is a field. Moreover, the map given by $x \longmapsto (x, 0)$ gives a one-one map from $\mathbb{R}$ to $\mathbb{C}$, which preserves the algebraic operations. With this in view, we regard $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifiyng a real number x with the ordered pair $(x, 0)$ in $\mathbb{C}$. We define $i := (0, 1)$. With the identification of $\mathbb{R}$ with a subset of $\mathbb{C}$ as above, we can write any $(x, y) \in \mathbb{C}$ as $x + iy$. Note that $i^2 = -1$, where we have again identified -1 with $(-1, 0)$. Let $z \in \mathbb{C}$. As noted above, $z = x + iy$ for unique $x, y \in \mathbb{R}$. We call x the **real part** of z and denote it by $\Re(z)$, and we call y the **imaginary part** of z and denote it by $\Im(z)$. The complex number $x - iy$ is called the **conjugate** of $z = x + iy$ and is denoted by $\overline{z}$. We also define the **absolute value** or the **modulus** of z to be the nonnegative real number $|z| := \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$. Note that this definition is consistent with

S. R. Ghorpade, B. V. Limaye, *A Course in Calculus and Real Analysis*,
Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1

that of the absolute value of a real number, and that the **Triangle Inequality** $|z_1 + z_2| \leq |z_1| + |z_2|$ holds for all $z_1, z_2 \in \mathbb{C}$. Also, note that

$$\max\{|\Re(z)|, |\Im(z)|\} \leq |z| \leq |\Re(z)| + |\Im(z)| \quad \text{for all } z \in \mathbb{C}.$$

Observe that $\mathbb{C}$ is not an ordered field. Indeed, if $\mathbb{C}$ had a subset $\mathbb{C}^+$ satisfying the two order properties O1 and O2 (as in Proposition A.7 with $\mathcal{R}^+$ replaced by $\mathbb{C}^+$ and $\mathcal{R}$ replaced by $\mathbb{C}$), then by O1, either $i \in \mathbb{C}^+$ or $-i \in \mathbb{C}^+$. Now O2 implies that $-1 = (\pm i)^2 \in \mathbb{C}^+$ and $1 = (-1)^2 \in \mathbb{C}^+$. This contradicts O1.

The **complex exponential** of a complex number z is defined by

$$e^z := e^x(\cos y + i \sin y), \quad \text{where } x = \Re(z) \text{ and } y = \Im(z).$$

Note that $|e^z| = e^{\Re(z)}$ and therefore $e^z \neq 0$ for all $z \in \mathbb{C}$. Note also that $e^{i\theta} = \cos\theta + i\sin\theta$ for all $\theta \in \mathbb{R}$. In particular, $e^{i\pi} + 1 = 0$.

We now consider **complex-valued** functions of a real variable, that is, functions whose codomain is $\mathbb{C}$ and whose domain is a subset of $\mathbb{R}$. For example, $h : \mathbb{R} \to \mathbb{C}$ defined by $h(t) := e^{it}$ is a complex-valued function of a real variable. In general, if $D \subseteq \mathbb{R}$ and if $h : D \to \mathbb{C}$ is a complex-valued function of a real variable, then there are unique functions $f, g : D \to \mathbb{R}$ such that $h(t) = f(t) + ig(t)$ for $t \in D$. We call f the **real part** and g the **imaginary part** of h, and write $h = f + ig$. The notions of boundedness, continuity, differentiability and Riemann integration extend to complex-valued functions of a real variable in a straightforward manner. In fact, we define $h : D \to \mathbb{C}$ to be

- **bounded** on D if both f and g are bounded on D,
- **continuous** at a point c of D if both f and g are continuous at c,
- **differentiable** at an interior point c of D if both f and g are differentiable at c; in this case, the complex number $h'(c) := f'(c) + ig'(c)$ is called the **derivative** of h at c,

In case $D := [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$, and $h : D \to \mathbb{C}$ is bounded, then we define h to be **integrable** on $[a, b]$ if both f and g are integrable $[a, b]$, and in this case, the complex number $\int_a^b f(t)dt + i\int_a^b g(t)dt$ is called the **integral** of h over $[a, b]$ and denoted by $\int_a^b h(t)dt$.

For example, if $D = [0, \pi]$ and $h : D \to \mathbb{R}$ is defined by $h(t) := e^{it}$, then h is differentiable at each $t \in D$ and $h'(t) = ie^{it}$. Also, h is integrable and

$$\int_a^b h(t)dt = \int_0^\pi \cos t dt + i\int_0^\pi \sin t dt = 0 - i((-1) - 1) = 2i = \frac{1}{i}\left(e^{i\pi} - e^0\right).$$

In a similar way, we can consider complex-valued functions of two or more real variables. Thus, for example, if $D \subseteq \mathbb{R}^2$ and $h : D \to \mathbb{C}$ is a complex-valued function of two real variables, then there are unique real-valued functions $f, g : D \to \mathbb{R}$ such that $h(t, u) = f(t, u) + ig(t, u)$ for $(t, u) \in D$. As in Section 10.6, we can fix one variable and differentiate or integrate with respect to another. Thus we let

$$D_1h(t,u) = D_1f(t,u) + iD_1g(t,u) \text{ and } D_2h(t,u) = D_2f(t,u) + iD_2g(t,u),$$

where D_1 indicates differentiation with respect to the first variable t (treating u as a constant) and D_2 indicates differentiation with respect to the second variable u (treating t as a constant). Likewise, if $D \subseteq \mathbb{R}^2$ is of the form $D = [a,b] \times E$ for some $a, b \in \mathbb{R}$ with $a \le b$ and $E \subseteq \mathbb{R}$, and if $h : D \to \mathbb{C}$ is continuous in t (which means both f and g are continuous in t), then we can consider the integral function $H : E \to \mathbb{C}$ given by

$$H(u) = \int_a^b h(t,u)dt := \int_a^b f(t,u)dt + i\int_a^b g(t,u)dt \quad \text{for } u \in E.$$

Moreover, the following analogue of Proposition 10.52 holds.

Proposition B.1. *Let E be an interval in $\mathbb{R}$, and let $a, b \in \mathbb{R}$ with $a \le b$. Suppose $h : [a,b] \times E \to \mathbb{C}$ is a complex-valued function of two real variables such that D_2h exists and is bounded on $[a,b] \times E$, and for each $u \in E$, the integral $\int_a^b D_2h(t,u)dt$ exists. Then the integral function $H : E \to \mathbb{C}$ given by $H(u) := \int_a^b h(t,u)dt$ is differentiable and $H'(u) = \int_a^b D_2h(t,u)dt$ for $u \in E$.*

Proof. Use Proposition 10.52 for the real and imaginary parts of h. □

B.2 Polynomials and Their Roots

In this section, we shall give a proof due to Paul Loya [60] of the Fundamental Theorem of Algebra. The statement involves polynomials (in one variable) with complex coefficients, which are defined in the same way as polynomials with real coefficients. (See Section 1.3.) Notions of the degree and of the leading coefficient of a nonzero polynomial are also defined in the same way. In particular, by a **nonconstant polynomial** we mean a polynomial of positive degree. Also, by a **monic polynomial** we mean a polynomial whose leading coefficient is 1. We will begin by proving an elementary property of polynomials that will be useful later.

Lemma B.2. *Let $p(z)$ be a polynomial with coefficients in $\mathbb{C}$. Then for every $z_0 \in \mathbb{C}$, there exists $\delta_0 > 0$ such that $|p(z)| \ge |p(z_0)|/2$ for all $z \in \mathbb{C}$ with $|z - z_0| < \delta_0$.*

Proof. Write $p(z) = c_dz^d + c_{d-1}z^{d-1} + \cdots + c_1z + c_0$, where $c_0, c_1, \ldots, c_d \in \mathbb{C}$. Let us first consider $z_0 := 0$, so that $p(z_0) = c_0$. Note that if $c_0 = 0$, then there is nothing to prove. Let us now assume that $c_0 \ne 0$. By writing $c_0 = p(z) - z\left(c_dz^{d-1} + c_{d-1}z^{d-2} + \cdots + c_1\right)$, we see that the triangle inequality gives $|p(z)| \ge |c_0| - |z|\left(|c_d||z|^{d-1} + \cdots + |c_1|\right)$. Consider $\delta_0 \in \mathbb{R}$ defined by $\delta_0 := |c_0|/2\left(|c_d| + \cdots + |c_1| + |c_0|\right)$. Then $0 < \delta_0 \le 1/2 < 1$ and

$$|z| < \delta_0 \Longrightarrow |z|\left(|c_d||z|^{d-1} + \cdots + |c_1|\right) \le \delta_0\left(|c_d| + \cdots + |c_1|\right) < \frac{|c_0|}{2}.$$

Consequently, $|p(z)| \ge |c_0|/2 = |p(0)|/2$ whenever $|z| < \delta_0$.

Next, consider any $z_0 \in \mathbb{C}$. Substituting $z = (z - z_0) + z_0$ in $p(z)$, we see that $p(z) = p^*(z - z_0)$, where $p^*(w) = c_d^* w^d + \cdots + c_1^* w + c_0^*$ is a polynomial in w whose coefficients $c_0^*, \ldots, c_d^* \in \mathbb{C}$ are determined by $c_0, \ldots, c_d$ and z_0, and moreover, $c_0^* = p(z_0)$. From the first part of the proof, we find $\delta_0 > 0$ such that $|p^*(w)| \ge |c_0^*|/2$ for all $w \in \mathbb{C}$ with $|w| < \delta_0$. Puting $w = z - z_0$, we obtain the desired result. □

We remark that the above lemma can also be deduced from the continuity of polynomial functions of a complex variable. However, since we have not discussed continuity of functions of a complex variable, we have chosen to give a direct proof.

Proposition B.3 (Fundamental Theorem of Algebra). *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Let $p(z)$ be a nonconstant polynomial of degree d with coefficients in $\mathbb{C}$. Dividing $p(z)$ by its leading coefficient, we may assume that $p(z)$ is monic. Thus we can write $p(z) = z^d + c_{d-1}z^{d-1} + \cdots + c_1 z + c_0$, where $c_0, \ldots, c_{d-1} \in \mathbb{C}$. Suppose $p(z)$ has no root in $\mathbb{C}$, that is, $p(z_0) \neq 0$ for all $z_0 \in \mathbb{C}$. In particular, $c_0 = p(0) \neq 0$. Consider the function $h : [-\pi, \pi] \times \mathbb{R} \to \mathbb{C}$ defined by

$$h(t, u) := \frac{1}{p\left(ue^{it}\right)} = \frac{1}{u^d e^{idt} + \cdots + c_1 u e^{it} + c_0} \quad \text{for } t \in [-\pi, \pi] \text{ and } u \in \mathbb{R}.$$

Clearly, h is differentiable in each of the two variables t and u. Moreover, a direct computation shows that

$$D_2 h(t, u) = \frac{-\left(du^{d-1}e^{idt} + \cdots + c_1 e^{it}\right)}{\left(u^d e^{idt} + \cdots + c_1 u e^{it} + c_0\right)^2} \quad \text{for } t \in [-\pi, \pi] \text{ and } u \in \mathbb{R}.$$

It is clear that for each fixed $u \in \mathbb{R}$, the function from $[-\pi, \pi]$ to $\mathbb{C}$ given by $t \longmapsto D_2 h(t, u)$ is continuous. Now let $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then

$$\left|du^{d-1}e^{idt} + \cdots + c_1 e^{it}\right| \le d\alpha^{d-1} + \cdots + |c_1| \quad \text{for } t \in [-\pi, \pi] \text{ and } u \in [-\alpha, \alpha].$$

Next, we show that there exists $\delta > 0$ such that $\left|p\left(ue^{it}\right)\right| \ge \delta$ for all $t \in [-\pi, \pi]$ and $u \in [-\alpha, \alpha]$. Suppose this is not the case. Then there are sequences (t_n) in $[-\pi, \pi]$ and (u_n) in $[-\alpha, \alpha]$ such that $\left|p\left(u_n e^{it_n}\right)\right| < 1/n$ for all $n \in \mathbb{N}$. By the Bolzano–Weierstrass Theorem (Proposition 2.17), there exist a subsequence (t_{n_k}) of (t_n) and $t_0 \in [-\pi, \pi]$ such that $t_{n_k} \to t_0$. In turn, there exist a subsequence $(u_{n_{k_j}})$ of (u_{n_k}) and $u_0 \in [-\alpha, \alpha]$ such that $u_{n_{k_j}} \to u_0$. Let $z_j := u_{n_{k_j}} e^{it_{n_{k_j}}}$ for $j \in \mathbb{N}$ and let $z_0 := u_0 e^{it_0}$. Then $p(z_0) \neq 0$, and by Lemma B.2, there exists $\delta_0 > 0$ such that $|p(z)| \ge |p(z_0)|/2$ for all $z \in \mathbb{C}$ with

$|z - z_0| < \delta_0$. Since $z_j \to z_0$, there exists $j_0 \in \mathbb{N}$ such that $|z_j - z_0| < \delta_0$ for $j \geq j_0$, and hence

$$0 < \frac{|p(z_0)|}{2} \leq |p(z_j)| < \frac{1}{n_{k_j}} \quad \text{for all } j \geq j_0.$$

But this is impossible, since $n_{k_j} \to \infty$ as $j \to \infty$. Hence there exists $\delta > 0$ such that $\left|p\left(ue^{it}\right)\right| \geq \delta$ for all $t \in [-\pi, \pi]$ and $u \in [-\alpha, \alpha]$. It follows that

$$|D_2 h(t, u)| \leq \frac{d\alpha^{d-1} + \cdots + |c_1|}{\delta^2} \quad \text{for all } t \in [-\pi, \pi] \text{ and } u \in [-\alpha, \alpha].$$

Thus the hypothesis of Proposition B.1 is satisfied with $E = [-\alpha, \alpha]$. Since $\alpha > 0$ is arbitrary, we see that the integral function $H : \mathbb{R} \to \mathbb{C}$ given by $H(u) = \int_{-\pi}^{\pi} h(t, u)dt$ is differentiable and

$$H'(u) = \int_{-\pi}^{\pi} D_2 h(t, u)dt \quad \text{ for all } u \in \mathbb{R}.$$

If $u = 0$, then $D_2 h(t, u) = D_2 h(t, 0) = -c_1 e^{it}/c_0^2$ for $t \in [-\pi, \pi]$, and so $H'(0) = -\left(c_1/c_0^2\right) \int_{-\pi}^{\pi} e^{it}dt = -\left(c_1/ic_0^2\right)\left(e^{i\pi} - e^{-i\pi}\right) = 0$, whereas

$$D_2 h(t, u) = \frac{-\left(u^d ide^{idt} + \cdots + c_1 uie^{it}\right)}{iu\left(u^d e^{idt} + \cdots + c_1 ue^{it} + c_0\right)^2} = \frac{1}{iu} D_1 h(t, u), \quad \text{if } u \neq 0.$$

Consequently, for all $u \in \mathbb{R}$ with $u \neq 0$,

$$H'(u) = \frac{1}{iu} \int_{-\pi}^{\pi} D_1 h(t, u)dt = \frac{1}{iu}\left[h(\pi, u) - h(-\pi, u)\right] = 0,$$

where the second equality follows from the Fundamental Theorem of Calculus (part (ii) of Proposition 6.24) and the last equality follows since $e^{\pi i} = e^{-\pi i}$. Thus $H'(u) = 0$ for all $u \in \mathbb{R}$. Hence by Corollary 4.23, H is a constant function, and so

$$H(u) = H(0) = \int_{-\pi}^{\pi} h(t, 0)dt = \frac{2\pi}{c_0} \neq 0 \quad \text{ for all } u \in \mathbb{R}.$$

On the other hand, we show that $H(n) \to 0$ as $n \to \infty$. To this end, let

$$g_n(t) := h(t, n) = \frac{1}{p(ne^{it})} \quad \text{ for } n \in \mathbb{N} \text{ and } t \in [-\pi, \pi].$$

Now for all $n \in \mathbb{N}$ and $t \in [-\pi, \pi]$,

$$\begin{aligned}
|p(ne^{it})| &= |n^d e^{idt} + \cdots + c_1 ne^{it} + c_0| \\
&\geq \left|n^d e^{idt}\right| - \left|c_{d-1} n^{d-1} e^{i(d-1)t} + \cdots + c_1 ne^{it} + c_0\right| \\
&= n^d \left(1 - \left|\frac{c_{d-1}}{n} e^{i(d-1)t} + \cdots + \frac{c_1}{n^{d-1}} e^{it} + \frac{c_0}{n^d}\right|\right) \\
&\geq n^d \left(1 - \frac{|c_{d-1}|}{n} - \cdots - \frac{|c_1|}{n^{d-1}} - \frac{|c_0|}{n^d}\right) \\
&\geq \frac{n^d}{2}, \quad \text{provided } n \geq 2\left(|c_0| + |c_1| + \cdots + |c_{d-1}|\right).
\end{aligned}$$

This implies that $g_n(t) \to 0$ as $n \to \infty$ for each $t \in [-\pi, \pi]$, and also that the sequence $(|g_n|)$ is uniformly bounded on $[-\pi, \pi]$. So by applying the Arzelà Bounded Convergence Theorem (Proposition 10.40) to the real and imaginary parts of g_n, we obtain

$$H(n) = \int_{-\pi}^{\pi} h(t, n)dt = \int_{-\pi}^{\pi} g_n(t)dt \longrightarrow \int_{-\pi}^{\pi} 0\, dt = 0.$$

But since $H(u) = 2\pi/c_0 \neq 0$, this is a contradiction. It follows that $p(z)$ must have a root in $\mathbb{C}$. □

Corollary B.4. (i) *Every nonzero polynomial $p(z)$ of degree d with coefficients in $\mathbb{C}$ can be factored into d linear factors as*

$$p(z) = c(z - \alpha_1) \cdots (z - \alpha_d),$$

where $c \in \mathbb{C}$ with $c \neq 0$ and $\alpha_1, \ldots, \alpha_d$ are complex numbers.

(ii) *Every nonzero polynomial $p(z)$ of degree d with coefficients in $\mathbb{C}$ is a product of powers of distinct linear factors as*

$$p(z) = c(z - \lambda_1)^{m_1} \cdots (z - \lambda_k)^{m_k},$$

where $c \in \mathbb{C}$ with $c \neq 0$ and $\lambda_1, \ldots, \lambda_k$ are distinct complex numbers and $m_1, \ldots, m_k$ are positive integers satisfying $m_1 + \cdots + m_k = d$.

(iii) **(Real Fundamental Theorem of Algebra)** *Every nonzero polynomial $p(x)$ with coefficients in $\mathbb{R}$ can be factored as a finite product of linear polynomials and quadratic polynomials with negative discriminants.*

Proof. (i) The result is obvious when $d = 0$, since an empty product equals 1. For $d \geq 1$, it follows from Proposition B.3 by induction on d.

(ii) This follows from (i) by collating equal linear factors.

(iii) Note that if $\alpha \in \mathbb{C}$ is a root of a polynomial $p(x)$ with real coefficients, that is, if $p(\alpha) = 0$, then $p(\overline{\alpha}) = \overline{p(\alpha)} = 0$, and hence the conjugate $\overline{\alpha}$ of α is also a root of $p(x)$. Thus linear factors of $p(x)$ of the form $(x - \alpha)$, where α is a nonreal complex number, occur in conjugate pairs. Since $(x - \alpha)(x - \overline{\alpha})$ is the polynomial $x^2 - 2\Re(\alpha)x + |\alpha|^2$ with real coefficients whose discriminant is negative, we see that (i) implies (iii). □

Recall that part (iii) of Corollary B.4 was stated earlier in Chapter 1.

Remark B.5. Numerous proofs of the Fundamental of Algebra are known, and in fact, the theorem can be proved using techniques developed in many of the advanced courses in mathematics such as complex analysis, topology, and Galois theory. The proof that we have chosen is closer to the spirit of this book and uses some of the ideas developed in Chapter 10. For a variety of other proofs, see the nice book of Fine and Rosenberger [30]. ◇

References

1. S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, American Mathematical Society, Providence, RI, 1990.
2. T. Apostol, *Mathematical Analysis*, first ed. and second ed., Addison-Wesley, Reading, MA, 1957 and 1974.
3. T. Apostol, *Calculus*, second ed., vols. 1 and 2, John Wiley, New York, 1967.
4. T. Apostol et al. (eds.), *A Century of Calculus*, vols. I and II, Mathematical Association of America, Washington, D.C., 1992.
5. J. Arndt and C. Haenel, π *Unleashed*, Springer-Verlag, New York, 2001.
6. E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, New York, 1964.
7. C. Arzelà, Sulla integrazione per serie, *Atti Acc. Lincei Rend.* **1** (1885), Part 1: pp. 532–537 and Part 2: 596–599.
8. A. Baker, *Transcendental Number Theory*, Cambridge University Press, Cambridge, 1975.
9. R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis*, third ed., John Wiley, New York, 2000.
10. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, New York, 1965.
11. S. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, *Comm. Soc. Math. Kharkow* **13** (1912), pp. 1–2.
12. L. Bers, On avoiding the mean value theorem, *Amer. Math. Monthly* **74** (1967), p. 583. [Reprinted in [4]: Vol. I, p. 224.]
13. G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra*, third ed., Macmillan, New York, 1965.
14. R. P. Boas, Who needs those mean-value theorems, anyway? *College Math. J.* **12** (1981), pp. 178–181. [Reprinted in [4]: Vol. II, pp. 182–186.]
15. C. B. Boyer, *The History of Calculus and Its Conceptual Development*, Dover, New York, 1959.
16. T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, third ed., Chelsea, New York, 1991.
17. P. S. Bullen, D. S. Mitrinovic, and P. M. Vasić, *Means and Their Inequalities*, D. Reidel, Dordrecht, 1988.

S. R. Ghorpade, B. V. Limaye, *A Course in Calculus and Real Analysis*,
Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1

18. G. Chrystal, *Algebra: An Elementary Text-Book for the Higher Classes of Secondary Schools and for Colleges*, sixth ed., parts I and II, Chelsea, New York, 1959.
19. L. Cohen, On being mean to the mean value theorem, *Amer. Math. Monthly* **74** (1967), pp. 581–582.
20. T. Cohen and W. J. Knight, Convergence and divergence of $\sum_{n=1}^{\infty} 1/n^p$, *Math. Mag.* **52** (1979), p. 178. [Reprinted in [4]: Vol. II, p. 400.]
21. R. Courant and F. John, *Introduction to Calculus and Analysis*, vols. I and II, Springer-Verlag, New York, 1989.
22. D. A. Cox, The arithmetic-geometric mean of Gauss, *Enseign. Math. (2)* **30** (1984), pp. 275–330.
23. P. J. Davis, Leonhard Euler's integral: A historical profile of the gamma function, *Amer. Math. Monthly* **66** (1959), pp. 849–869.
24. R. Dedekind, *Essays on the Theory of Numbers*, Dover Publications, New York, 1963.
25. P. Dienes, *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable*, Dover, New York, 1957.
26. C. H. Edwards Jr., *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.
27. H. B. Enderton, *Elements of Set Theory*, Academic Press, New York, 1977.
28. L. Féjer, Sur les functions bornées et intégrables, *C. R. Acad. Sci. Paris*, **131** (1900), pp. 984–987.
29. G. Fichtenholz, Un théorème sur l'intégration sous le signe intégrale, *Rend. Circ. Mat. Palermo* **36** (1913), pp. 111–114.
30. B. Fine and G. Rosenberger, *The Fundamental Theorem of Algebra*, Springer-Verlag, New York, 1997.
31. A. R. Forsyth, *A Treatise on Differential Equations*, Reprint of the sixth (1929) ed., Dover, New York, 1996.
32. B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, Corrected reprint of the second ed., Dover Pubications, Mineola, New York, 2003.
33. S. R. Ghorpade and B. V. Limaye, *A Course in Multivariable Calculus and Analysis*, Springer, New York, 2010.
34. C. Goffman, *Introduction to Real Analysis*, Harper & Row, New York-London, 1966.
35. R. Goldberg, *Methods of Real Analysis*, second ed., John Wiley, New York, 1976.
36. J. F. Hall and T. D. Todorov, Ordered fields, the purge of infinitesimals from mathematics and the rigorousness of infinitesimal calculus, *Bulg. J. Phys.* **42** (2015), pp. 99–127.
37. P. R. Halmos, *Naive Set Theory*, Springer-Verlag, New York, 1974.
38. R. W. Hamming, An elementary discussion of the transcendental nature of the elementary transcendental functions, *Amer. Math. Monthly* **77** (1972), pp. 294–297. [Reprinted in [4]: Vol. II, pp. 80–83.]
39. G. H. Hardy, *A Course of Pure Mathematics*, Reprint of the (1952) tenth ed., Cambridge University Press, Cambridge, 1992.
40. G. H. Hardy, *Divergent Series*, second ed., Chelsea, New York, 1992.
41. G. H. Hardy, D. E. Littlewood and G. Pólya, *Inequalities*, second ed., Cambridge University Press, Cambridge, 1952.

42. J. Havil, *Gamma: Exploring Euler's Constant*, Princeton University Press, Princeton, NJ, 2003.
43. T. Hawkins, *Lebesgue's Theory of Integration: Its Origins and Development*, Reprint of the (1979) corrected second ed., AMS Chelsea Publishing, Providence, RI, 2001.
44. K. D. Joshi, *Introduction to General Topology*, John Wiley, New York, 1983.
45. R. V. Kadison and Z. Liu, Bernstein polynomials and approximation, *Online notes* (accessed: March 31, 2018) available at: http://www.math.upenn.edu/∼kadison/bernstein.pdf.
46. R. Kaplan, *The Nothing That Is: A Natural History of Zero*, Oxford University Press, New York, 2000.
47. G. Klambauer, *Aspects of Calculus*, Springer-Verlag, New York, 1986.
48. M. Kline, *Mathematical Thought from Ancient to Modern Times*, vols. 1, 2, and 3, second ed., Oxford University Press, New York, 1990.
49. K. Knopp, *Theory and Application of Infinite Series*, second ed., Dover, New York, 1990.
50. D. E. Knuth, Two notes on notation, *Amer. Math. Monthly* **99** (1992), pp. 403–422.
51. J. Korevaar, *Tauberian Theory: A Century of Developments*, Springer-Verlag, Berlin, 2004.
52. P. P. Korovkin, *Linear Operators and Approximation Theory*, first revised ed., Hindustan Publishing Corporation, New Delhi, 2017 (Translated from the Russian edition of 1959).
53. P. P. Korovkin, *Inequalities*, Little Mathematics Library No. 5, Mir Publishers, Moscow, 1975. [A reprint is distributed by Imported Publications, Chicago, 1986.]
54. E. Landau, *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers* (translated by F. Steinhardt), Chelsea, New York, 1951.
55. J. W. Lewin, A truly elementary approach to the bounded convergence theorem, *Amer. Math. Monthly* **93** (1986), pp. 395–397.
56. J. W. Lewin, Some applications of the bounded convergence theorem for an introductory course in analysis, *Amer. Math. Monthly* **94** (1987), pp. 988–993.
57. J. Lewin, *An Interactive Introduction to Mathematical Analysis*, third ed., Cambridge University Press, Cambridge, 2014.
58. L. Lichtenstein, Über die Integration eines bestimmten Integrals in Bezug auf einen Parameter, *Nach. Akad. Wiss. Göttingen, Math.-Phys. Kl.* (1910), pp. 468–475.
59. B. V. Limaye, *Functional Analysis*, revised third ed., New Age International, New Delhi, 2017.
60. P. Loya, An(other) elementary proof of the fundamental theorem of algebra, *Online Notes* (accessed: March 31, 2018) available at: http://people.math.binghamton.edu/loya/papers/LoyaFTA.pdf.
61. J. Lu, Is the composite function integrable? *Amer. Math. Monthly* **106** (1999), pp. 763–766.
62. W. A. J. Luxemburg, Arzelà's dominated convergence theorem for the Riemann integral, *Amer. Math. Monthly* **78** (1971), pp. 970–979.

63. R. Lyon and M. Ward, The limit for e, *Amer. Math. Monthly* **59** (1952), pp. 102–103 [Reprinted in [4]: Vol. I, pp. 432–433.]
64. E. Maor, *e: The Story of a Number*, Princeton University Press, Princeton, NJ, 1994.
65. J. E. Marsden, A. Tromba, and A. Weinstein, *Basic Multivariable Calculus*, Springer-Verlag, Heidelberg, 1993.
66. M. D. Meyerson, Every power series is a Taylor series, *Amer. Math. Monthly* **88** (1981), pp. 51–52.
67. D. J. Newman, A simplified version of the fast algorithms of Brent and Salamin, *Math. Comp.* **44** (1985), pp. 207–210.
68. A. Pinkus, Weierstrass and approximation theory, *J. Approx. Theory* **107** (2000), pp. 1–66.
69. K. A. Ross, *Elementary Analysis: The Theory of Calculus*, second ed., Springer-Verlag, New York, 2013.
70. H. L. Royden, *Real Analysis*, third ed., Prentice Hall, Englewood Cliffs, NJ, 1988.
71. W. Rudin, *Principles of Mathematical Analysis*, third ed., McGraw-Hill, New York–Auckland–Düsseldorf, 1976.
72. W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill, New York, 1986.
73. J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, second ed., Springer, Cham, 2015.
74. G. F. Simmons, *Differential Equations, with Applications and Historical Notes*, McGraw-Hill, New York–Düsseldorf–Johannesburg, 1972.
75. R. S. Smith, Rolle over Lagrange – another shot at the mean value theorems, *College Math. J.* **17** (1986), pp. 403–406.
76. M. Spivak, *Calculus*, third ed., Publish or Perish Inc., Houston, 1994.
77. R. Sridharan, From lintearia to lemniscate, Parts I and II, *Resonance* **9** (2004), no. 4 and 6, pp. 21–29 and 11–20.
78. O. E. Stanaitis, *An Introduction to Sequences, Series and Improper Integrals*, Holden-Day, San Francisco, 1967.
79. E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, Princeton, NJ, 2003.
80. J. Stillwell, *Mathematics and Its History*, Springer, New York, 1989.
81. A. E. Taylor, L'Hospital's rule, *Amer. Math. Monthly* **59** (1952), pp. 20–24.
82. G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, ninth ed., Addison-Wesley, Reading, MA, 1996.
83. J. van Tiel, *Convex Analysis: An Introductory Text*, John Wiley, New York, 1984.
84. J.-P. Tignol, *Galois' Theory of Algebraic Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
85. N. Y. Vilenkin, *Method of Successive Approximations*, Little Mathematics Library, Mir Publishers, Moscow, 1979.
86. K. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen, *Sitzungsber. Akad. Berlin* **2** (1885), Part 1: pp. 633–639, Part 2: pp. 789–805.
87. I. Weiss, The real numbers – a survey of constructions, *Rocky Mountain J. Math.*, **45** (2015), pp. 737–762.

List of Symbols and Abbreviations

	Definition/Description	Page
$\mathbb{N}$	set of all positive integers	1
$\mathbb{Z}$	set of all integers	1
$\mathbb{Q}$	set of all rational numbers	1
$\mathbb{R}$	set of all real numbers	2
$\sum$	sum	3
$\prod$	product	3
$A := B$	A is defined to be equal to B	3
$\mathbb{R}^+$	set of all positive real numbers	3
$\sup S$	supremum of a subset S of $\mathbb{R}$	4
$\inf S$	infimum of a subset S of $\mathbb{R}$	4
$\max S$	maximum of a subset S of $\mathbb{R}$	5
$\min S$	minimum of a subset S of $\mathbb{R}$	5
$[x]$	integer part of a real number x	6
$\lfloor x \rfloor$	integer part or the floor of a real number x	6
$\lceil x \rceil$	ceiling of a real number x	6
$\sqrt[n]{a}$	nth root of a nonnegative real number a	7
$\sqrt{a}$	square root of a nonnegative real number a	7
$m \mid n$	m divides n	8
$m \nmid n$	m does not divide n	8
(a, b)	open interval $\{x \in \mathbb{R} : a < x < b\}$	9
$[a, b]$	closed interval $\{x \in \mathbb{R} : a \le x \le b\}$	9
$[a, b)$	semiopen interval $\{x \in \mathbb{R} : a \le x < b\}$	9
$(a, b]$	semiopen interval $\{x \in \mathbb{R} : a < x \le b\}$	9
(a, ∞)	semi-infinite interval $\{x \in \mathbb{R} : x > a\}$	9
$[a, \infty)$	semi-infinite interval $\{x \in \mathbb{R} : x \ge a\}$	9
$(-\infty, a)$	semi-infinite interval $\{x \in \mathbb{R} : x < a\}$	9
$(-\infty, a]$	semi-infinite interval $\{x \in \mathbb{R} : x \le a\}$	9
$\lvert a \rvert$	absolute value of a real number a	10

S. R. Ghorpade, B. V. Limaye, *A Course in Calculus and Real Analysis*,
Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1

	Definition/Description	**Page**
A.M.	arithmetic mean	12
G.M.	geometric mean	12
$D \setminus C$	$\{x \in D : x \notin C\}$	13
$D \times E$	$\{(x, y) : x \in D$ and $y \in E\}$	14
$f_{\vert C}$	restriction of $f : D \to E$ to a subset C of D	16
$g \circ f$	composite of g with f	16
f^{-1}	inverse of an injective function f	16
$f = 0$	f is the zero function (on its domain)	17
$f \leq g$	$f(x) \leq g(x)$ for all x	17
$f \geq 0$	$f(x) \geq 0$ for all x	17
$\mathbb{R}[x]$	set of all polynomials in x with coefficients in $\mathbb{R}$	18
$\deg p(x)$	degree of a nonzero polynomial $p(x)$	18
IVP	intermediate value property	29
H.M.	harmonic mean	34
GCD	greatest common divisor	37
LCM	least common multiple	37
(a_n)	sequence whose nth term is the real number a_n	41
$a_n \to a$	sequence (a_n) tends to a real number a	42
$\lim\limits_{n\to\infty} a_n$	limit of the sequence (a_n)	43
$a_n = O(b_n)$	(a_n) is big-oh of (b_n)	51
$a_n = o(b_n)$	(a_n) is little-oh of (b_n)	52
$a_n \sim b_n$	(a_n) is asymptotically equivalent to (b_n)	52
$a_n \to \infty$	sequence (a_n) tends to ∞	52
$a_n \to -\infty$	sequence (a_n) tends to $-\infty$	52
$\not\to$	does not tend to	53
$\limsup\limits_{n\to\infty} a_n$	limit superior of (a_n)	60
$\liminf\limits_{n\to\infty} a_n$	limit inferior of (a_n)	60
$\lim\limits_{x\to c} f(x)$	limit of $f(x)$ as x tends to c	84
$\lim\limits_{x\to c^-} f(x)$	left limit of $f(x)$ as x tends to c	89
$\lim\limits_{x\to c^+} f(x)$	right limit of $f(x)$ as x tends to c	90
$f(x) = O(g(x))$	$f(x)$ is big-oh of $g(x)$ as $x \to \infty$	92
$f(x) = o(g(x))$	$f(x)$ is little-oh of $g(x)$ as $x \to \infty$	92
$f(x) \sim g(x)$	$f(x)$ is asymptotically equivalent to $g(x)$	93
$f'(c),\ \left.\dfrac{df}{dx}\right\vert_{x=c}$	derivative of f at c	106
$f''(c),\ \left.\dfrac{d^2f}{dx^2}\right\vert_{x=c}$	second derivative of f at c	114
$f^{(n)}(c),\ \left.\dfrac{d^nf}{dx^n}\right\vert_{x=c}$	nth derivative of f at c	114

	Definition/Description	**Page**
$f'_-(c)$	left derivative of f at c	115
$f'_+(c)$	right derivative of f at c	115
MVT	Mean Value Theorem	122
$\approx$	approximately equal	126
L'HR	L'Hôpital's Rule	135, 136
P_n	partition of $[a, b]$ into n equal parts	182
$m(f)$	infimum of $\{f(x) : x \in [a, b]\}$	182
$M(f)$	supremum of $\{f(x) : x \in [a, b]\}$	182
$m_i(f)$	infimum of $\{f(x) : x \in [x_{i-1}, x_i]\}$	182
$M_i(f)$	supremum of $\{f(x) : x \in [x_{i-1}, x_i]\}$	182
$L(P, f)$	lower sum for f with respect to P	183
$U(P, f)$	upper sum for f with respect to P	183
$L(f)$	lower Riemann integral of f	183
$U(f)$	upper Riemann integral of f	183
$\int_a^b f(x)dx$	Riemann integral of f on $[a, b]$	185
f^+	positive part of f	202
f^-	negative part of f	202
FTC	Fundamental Theorem of Calculus	204
$\int f(x)dx$	an indefinite integral of f	206
$\big[F(x)\big]_a^b$, $F(x)\big\vert_a^b$	$F(b) - F(a)$	206
$S(P, \mathcal{T}, f)$	Riemann sum for f corresponding to a partition P and a tag set $\mathcal{T}$	211
$\mu(P)$	mesh of a partition P	211
$\ell(D)$	length of a bounded subset D of $\mathbb{R}$	221
ln	logarithmic function	234
e	unique real number such that $\ln e = 1$	234
exp	exponential function	236
arctan	arctangent function	246
π	$2 \sup\{\arctan x : x \in (0, \infty)\}$	247
$\angle(OP_1, OP_2)$	angle between OP_1 and OP_2	269
$L_1 \parallel L_2$	lines L_1 and L_2 are parallel	271
$L_1 \nparallel L_2$	lines L_1 and L_2 are not parallel	271
$\measuredangle(L_1, L_2)$	(acute) angle between L_1 and L_2	271
$L_1 \perp L_2$	lines L_1 and L_2 are perpendicular	272
$L_1 \not\perp L_2$	lines L_1 and L_2 are not perpendicular	272
$\measuredangle(C_1, C_2; P)$	angle at P between C_1 and C_2	273
Area (R)	area of a region R	296
Vol (D)	volume of a solid body D	303, 307
$\ell(C)$	length of a curve C	316
Area (S)	area of a surface S	325
Av(f)	average of f	329
Av$(f; w)$	weighted average of f with respect to w	329

Index

S. R. Ghorpade, B. V. Limaye, *A Course in Calculus and Real Analysis*,
Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1