Derivation of the Kinetic Energy of the Rigid Body in Spatial Motion

Sandipan Bandyopadhyay

Department of Engineering Design

Indian Institute of Technology Madras, Chennai 600 036

Email: sandipan@iitm.ac.in

Web: `http://www.ed.iitm.ac.in/~sandipan`

The following derivation can be found in many textbooks, e.g., [1]. Consider a system of n particles, and let, $\boldsymbol{r}_i$ be the position vector of the ith particle relative to a point $\boldsymbol{o}$ fixed in an inertial frame, as shown in Fig. 1.

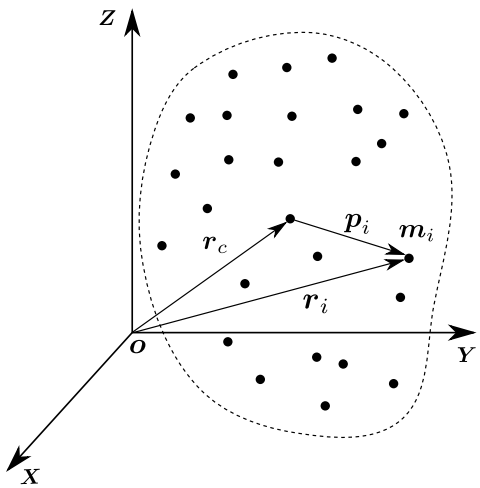

Figure 1: Position vectors for a system of particles

The total kinetic energy of the system, T, is the sum of individual kinetic energies of the particles:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \|\dot{\boldsymbol{r}}_i\|^2, \tag{1}$$

where m_i is the mass of the ith particle. Position of the ith particle can be expressed relative to the position of the *centre of mass* of the system, $\boldsymbol{r}_c$:

$$\boldsymbol{r}_i = \boldsymbol{r}_c + \boldsymbol{p}_i. \tag{2}$$

Using Eq. (2) in Eq. (1), one obtains:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i(\|\dot{\boldsymbol{r}}_c\|^2 + 2\dot{\boldsymbol{r}}_c \cdot \dot{\boldsymbol{p}}_i + \|\dot{\boldsymbol{p}}_i\|^2). \tag{3}$$

However,

$$\sum_{i=1}^{n} m_i \boldsymbol{p}_i = \boldsymbol{0}, \tag{4}$$

since $\boldsymbol{p}_i$ are measured from the centre of mass. In terms of the total mass of the system, $m = \sum_{i=1}^{n} m_i$, Eq. (3) reduces to:

$$T = \frac{1}{2}m\|\dot{\boldsymbol{r}}_c\|^2 + \frac{1}{2}\sum_{i=1}^{n} m_i\|\dot{\boldsymbol{p}}_i\|^2. \tag{5}$$

Equation (5) is the mathematical form of König's theorem (see, e.g., [2]), which states that the total kinetic energy of a system of particles is equal to the sum of:

1. The kinetic energy due to a particle having a mass equal to the total mass of the system and moving with the velocity of the centre of mass; and

2. The kinetic energy due to the motion of the system *relative* to its centre of mass.

Equation (5) was derived for a system of particles. However, it can be adapted to the case of a rigid body in general spatial motion, by considering a small volume element $\mathrm{d}V$ having a density ρ, as shown in Fig. 2.

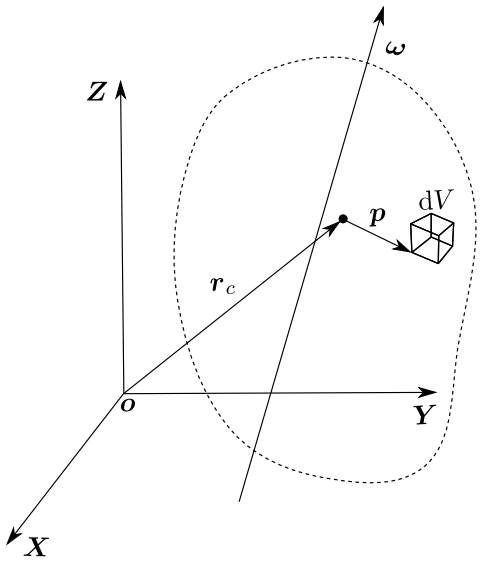

Figure 2: Volume element in a rigid body

The total kinetic energy is given by:

$$T = \frac{1}{2}m\|\boldsymbol{v}_c\|^2 + \frac{1}{2}\int_V \rho\|\dot{\boldsymbol{p}}\|^2 \mathrm{d}V, \text{ where } \boldsymbol{v}_c = \dot{\boldsymbol{r}}_c. \tag{6}$$

$$\Rightarrow T = \frac{1}{2}\boldsymbol{v}_c^\top m\boldsymbol{v}_c + \frac{1}{2}\int_V \rho\|\dot{\boldsymbol{p}}\|^2 \mathrm{d}V, \tag{7}$$

where, $\boldsymbol{p}$ is the position vector of the *volume element*, $\mathrm{d}V$, relative to the mass center, $\boldsymbol{r}_c$. The first term is called the *translational kinetic energy*, of the rigid body (denoted by T_{trans}) and the second term is called the *rotational kinetic energy* (denoted by T_{rot}). If one considers the reference point $\boldsymbol{r}_c$ to coincide with the center of mass and assume that the body is rotating with an angular velocity $\boldsymbol{\omega}$, then one can write:

$$\dot{\boldsymbol{p}} = \boldsymbol{\omega} \times \boldsymbol{p}. \tag{8}$$

Therefore

$$\|\dot{\boldsymbol{p}}\|^2 = (\boldsymbol{\omega} \times \boldsymbol{p}) \cdot (\boldsymbol{\omega} \times \boldsymbol{p}) = \boldsymbol{\omega} \cdot (\boldsymbol{p} \times (\boldsymbol{\omega} \times \boldsymbol{p})) \,. \tag{9}$$

Hence, the rotational kinetic energy of the body can be written in the form:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\int_V \rho \, (\boldsymbol{p} \times (\boldsymbol{\omega} \times \boldsymbol{p})) \, \mathrm{d}V \right). \tag{10}$$

Expanding the vector triple product, one obtains:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\int_V \rho \left(\|\boldsymbol{p}\|^2 \boldsymbol{\omega} - (\boldsymbol{p} \cdot \boldsymbol{\omega}) \boldsymbol{p} \right) \mathrm{d}V \right). \tag{11}$$

$$\tag{12}$$

Considering a *body-fixed* reference system, with its origin at $\boldsymbol{r}_c$, in terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, one can write:

$$\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{13}$$

$$\boldsymbol{\omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}. \tag{14}$$

Using Eqs. (13, 14) into Eq. (11), one obtains:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\int_V \rho \left(\left((y^2 + z^2)\omega_x - (xy\omega_y + xz\omega_z) \right) \boldsymbol{i} + \dots \right) \mathrm{d}V \right). \tag{15}$$

The integration over the body leads to:

$$T_{\text{rot}} = \frac{1}{2} I_{xx}\omega_x^2 + \frac{1}{2} I_{yy}\omega_y^2 + \frac{1}{2} I_{zz}\omega_z^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z, \tag{16}$$

where:

$$I_{xx} = \int_V \rho(y^2 + z^2)\mathrm{d}V, \tag{17}$$

$$I_{yy} = \int_V \rho(x^2 + z^2)\mathrm{d}V, \tag{18}$$

$$I_{zz} = \int_V \rho(x^2 + y^2)\mathrm{d}V. \tag{19}$$

Also,

$$I_{xy} = I_{yx} = -\int_V \rho xy \mathrm{d}V, \tag{20}$$

$$I_{xz} = I_{zx} = -\int_V \rho xz \mathrm{d}V, \tag{21}$$

$$I_{yz} = I_{zy} = -\int_V \rho yz \mathrm{d}V. \tag{22}$$

Using the matrix notation, Eq. (16) can be written in the form:

$$T_{\mathrm{rot}} = \frac{1}{2}\boldsymbol{\omega}_b^\top \boldsymbol{I}_b \boldsymbol{\omega}_b, \tag{23}$$

where $\boldsymbol{\omega}_b = [\omega_x, \omega_y, \omega_z]^\top$ is the *body-fixed* angular velocity, and

$$I_b = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{24}$$

is the *inertia tensor* in the body-fixed frame of reference. Therefore, we have obtained the expression for the total kinetic energy of a rigid body in terms of the translational motion of its mass centre, and the rotational motion of the body about the mass centre:

$$T = T_{\mathrm{trans}} + T_{\mathrm{rot}} = \frac{1}{2}\boldsymbol{v}_c^\top m \boldsymbol{v}_c + \frac{1}{2}\boldsymbol{\omega}_b^\top \boldsymbol{I}_b \boldsymbol{\omega}_b. \tag{25}$$

References

[1] D. T. Greenwood, *Advanced Dynamics*. New York: Cambridge University Press, 2003.

[2] S. Banach, *Mechanics*, vol. 25 of *Monografie Matematyczne*. Warszawa-Wroclaw: Polish Mathematical Society, 1951. `http://kielich.amu.edu.pl/Stefan_Banach/e-mechanics.html`.