

B

Kinetic Energy in Generalized Coordinates

For a system of N point particles in three spatial dimensions the kinetic energy expressed in Cartesian coordinates is always given by

$$T = T(\dot{\mathbf{r}}) = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{x}_i^2 \quad (\text{B.1})$$

with x_i ($i = 1, \dots, 3N$) being the i th Cartesian component. For the masses m_i an obvious notation has been employed. If there are R constraints on the system, the Cartesian coordinates may be replaced by $f = 3N - R$ suitably chosen generalized coordinates q_k according to Eq. (2.44), or in a more compact notation

$$x_i = x_i(q, t), \quad \forall i = 1, 2, \dots, 3N \quad (\text{B.2})$$

This coordinate transformation does *not* in general depend on the velocities \dot{q} . Accordingly the total time derivatives of the Cartesian coordinates read

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{k=1}^f \frac{\partial x_i}{\partial q_k} \dot{q}_k + \frac{\partial x_i}{\partial t} = \dot{x}_i(q, \dot{q}, t), \quad \forall i = 1, 2, \dots, 3N \quad (\text{B.3})$$

and depend on both the generalized coordinates and velocities and the time. Inserting this expression into (B.1) one obtains

$$\begin{aligned} T = & \frac{1}{2} \sum_{k,l=1}^f \sum_{i=1}^{3N} m_i \underbrace{\left(\frac{\partial x_i}{\partial q_k} \right) \left(\frac{\partial x_i}{\partial q_l} \right)}_{m_{kl}(q, t)} \dot{q}_k \dot{q}_l + \sum_{k=1}^f \sum_{i=1}^{3N} m_i \underbrace{\left(\frac{\partial x_i}{\partial q_k} \right) \left(\frac{\partial x_i}{\partial t} \right)}_{a_k(q, t)} \dot{q}_k \\ & + \frac{1}{2} \sum_{i=1}^{3N} m_i \underbrace{\left(\frac{\partial x_i}{\partial t} \right)^2}_{b(q, t)} = T(q, \dot{q}, t) \end{aligned} \quad (\text{B.4})$$

This is the most general expression for the kinetic energy T in terms of the generalized coordinates. As soon as other than Cartesian coordinates are em-

ployed, T may no longer depend only on the velocities but also on the coordinates itself. Because of its definition by Eq. (B.4) the *mass matrix* is symmetric, $m_{kl} = m_{lk}$.

Equation (B.4) is simplified to a large extent if the constraints do not explicitly depend on time, i.e., the coordinate transformation given by Eq. (B.2) itself does not explicitly depend on time,

$$x_i = x_i(q(t)), \quad \forall i = 1, 2, \dots, 3N \quad (\text{B.5})$$

Then the partial derivatives $\partial x_i / \partial t$ vanish and the kinetic energy is given by

$$T(q, \dot{q}) = \frac{1}{2} \sum_{k,l=1}^f m_{kl}(q) \dot{q}_k \dot{q}_l \quad (\text{B.6})$$

If furthermore the potential U does *not* depend on the velocities, $U = U(q, t)$, one finds

$$\sum_{k=1}^f \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = \sum_{k=1}^f \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k \stackrel{(\text{B.6})}{=} \sum_{k,l=1}^f m_{kl} \dot{q}_k \dot{q}_l = 2T(q, \dot{q}) \quad (\text{B.7})$$

Under those circumstances the energy of the system is thus given by

$$\sum_{k=1}^f \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \stackrel{(\text{B.7})}{=} 2T - T + U = T + U = E \quad (\text{B.8})$$

