Limits of Real Numbers

Drawn from *Real analysis with real applications*
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1. Limits

The notion of a limit is *the* basic notion of analysis. Limits are the culmination of an infinite process. It is the concern with limits in particular that separates analysis from algebra. In this section, we will deal with limits of a sequence of real numbers. Later we will need other kinds of limits, such as limits of functions possibly with vectors or more general values.

In the 1680s, Newton and Leibniz independently developed calculus. But it is not calculus as we know it today. Their writings about limits were vague and depended on physical reasoning that was somewhat circular and certainly was imprecise. In the late eighteenth century, some mathematicians, such as d'Alembert, saw the need to develop a precise notion of limit, while other great mathematicians, such as Lagrange, tried to develop calculus without dependence on this notion. Gauss in 1812 was the first mathematician to concern himself with tests for convergence of infinite series as necessary before attempting to evaluate the limit. It was not until 1829 that Cauchy gave a definition of limit that is close to the modern one we use today.

Intuitively, to say that a sequence a_n converges to a limit L means that eventually *all* the terms of the (tail of the) sequence approximate the limit value L to *any* desired accuracy. To make this precise, we introduce a subtle definition.

1.1. Definition of the Limit of a Sequence. A real number L is the **limit** of a sequence of real numbers $(a_n)_{n=1}^{\infty}$ if for *every* $\varepsilon > 0$, there is an integer $N = N(\varepsilon) > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq N.$$

We say that the sequence $(a_n)_{n=1}^{\infty}$ **converges** to L, and we write $\lim_{n\to\infty} a_n = L$. A sequence which does not converge is said to **diverge**.

The important issue in this definition is that for any desired accuracy, there is a point in the sequence such that *every* element after that point approximates the limit L to the desired accuracy. Clearly, the N that works (meaning it satisfies the condition in definition) for, say, $\varepsilon = 1/10$ will work for any larger value of ε. It suffices to consider only values for ε of the form $\frac{1}{2}10^{-k}$. The statement $|a_n - L| < \frac{1}{2}10^{-k}$ means that a_n and L agree to at least k decimal places. Thus a sequence converges to L precisely when for every k, no matter how large, eventually all the terms of the sequence agree with L to at least k decimals of accuracy.

1.2. Example. Consider the sequence $(a_n) = (n/(n+1))_{n=1}^{\infty}$, which we claim converges to 1. If the definition agrees with our intuitive idea of convergence, we should be able to pick N for any ε. Suppose $\varepsilon = .05$. We need to find some N so that

$$\left| \frac{n}{n+1} - 1 \right| < .05 \quad \text{for all} \quad n \geq N.$$

First we simplify the left-hand side of this equation: $\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1}$. If $n \geq 20$, then

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} \leq \frac{1}{21} < .05.$$

So it is enough to choose $N = 20$.

We could also choose $N = 73$. It is not necessary to find the best choice for N. However, as we shall see in connection with the analysis of numerical methods, better estimates can lead to better algorithms for computation.

Observe that $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1}$. So if $\varepsilon = \frac{1}{2}10^{-k}$, we can choose $N = 2 \cdot 10^k$. Then for all $n \geq N$,

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} \leq \frac{1}{2 \cdot 10^k + 1} < \tfrac{1}{2}10^{-k} = \varepsilon.$$

We could also choose $N = 73 \cdot 10^k$. It is not necessary to find the best choice for N. But in practice, better estimates can lead to better algorithms for computation.

1.3. Example. Consider the sequence (a_n) with $a_{2n-1} = \pi + \frac{1}{n}$ and $a_{2n} = \pi$ for $n \geq 1$. This sequence converges to π. Indeed, given $\varepsilon > 0$, choose a large positive integer N so that $\frac{1}{N} < \varepsilon$. Then if $n > 2N$, we may write $n = 2k - 1$ or $n = 2k$ for some $k > N$. In the first case,

$$|a_n - \pi| = |a_{2k-1} - \pi| = \frac{1}{k} < \frac{1}{N} < \varepsilon,$$

while in the second case,

$$|a_n - \pi| = |a_{2k} - \pi| = 0 < \varepsilon.$$

Note that some terms of a convergent sequence may actually equal the limit exactly.

What does it mean for a sequence to diverge? To contradict the definition of limit, one must show that for every L, the definition of limit fails. This means that we need to find one value of $\varepsilon > 0$ so that we cannot satisfy the definition; which means that for every N, there is an $n \geq N$ so that $|a_n - L| \geq \varepsilon$.

We can understand this symbolically. Notice that $(a_n)_{n=1}^{\infty}$ converges means

$$\exists_L \ \forall_{\varepsilon>0} \ \exists_N \ \forall_{n\geq N} \ |a_n - L| < \varepsilon.$$

The negation flips the for all and exists quantifiers:

$$\forall_L \ \exists_{\varepsilon>0} \ \forall_N \ \exists_{n\geq N} \ |a_n - L| \geq \varepsilon.$$

1.4. Example. Consider the sequence (a_n) with $a_n = (-1)^n$. Since this flips back and forth between two values that are always distance 2 apart, intuition says that it does not converge. To show this using our definition, we need to show that the definition of limit fails for *every* choice of L. However, for each choice of L, we need find *only one* value of ε that violates the definition.

Split the argument into two cases:

Case 1, $L \geq 0$. Take $\varepsilon = 1$. Then for any (large) positive integer N, pick $n = 2N + 1 \geq N$. Then

$$|a_n - L| = |-1 - L| = L + 1 \geq 1.$$

Case2: $L < 0$. Take $\varepsilon = 1$. Then for any (large) positive integer N, pick $n = 2N \geq N$. Then

$$|a_n - L| = |1 - L| = |L| + 1 \geq 1.$$

Consequently, this sequence does not converge.

1.5. The Squeeze Theorem.
Suppose that three sequences (a_n), (b_n), and (c_n) satisfy

$$a_n \le b_n \le c_n \quad \textit{for all} \quad n \ge 1 \qquad \textit{and} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim_{n\to\infty} b_n = L$.

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there is some N_1 such that

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \ge N_1,$$

or equivalently, $L - \varepsilon < a_n < L + \varepsilon$ for all $n \ge N_1$. There is also some N_2 such that

$$|c_n - L| < \varepsilon \quad \text{for all} \quad n \ge N_2$$

or $L - \varepsilon < c_n < L + \varepsilon$ for all $n \ge N_2$. Then, if $n \ge \max\{N_1, N_2\}$, we have

$$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon.$$

Thus $|b_n - L| < \varepsilon$ for $n \ge \max\{N_1, N_2\}$, as required. ■

In this proof, as in the examples, to show $\lim_{n\to\infty} b_n = L$, we must show that there is an 'N' that works for every possible value of ε. On the other hand, we know $\lim_{n\to\infty} a_n$ exists so we get to pick any value of ε we like and the existance of the limit is the guarantee that there is an 'N' that works for that value. When we know a limit exists, we will take full advantage of this freedom to pick any value we like for ε.

1.6. Example. Consider the sequence $((\sin n)/n)_{n=1}^{\infty}$. The numerator oscillates, but it remains bounded between ± 1 while the denominator goes off to infinity. We obtain the estimates

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

We know that $\lim_{n\to\infty} 1/n = 0 = \lim_{n\to\infty} -1/n$. By the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Exercises for Section 1

A. In each of the following, compute the limit. Then, using $\varepsilon = 10^{-6}$, find an integer N that satisfies the limit definition.

(a) $\lim_{n\to\infty} \dfrac{\sin n^2}{\sqrt{n}}$ (b) $\lim_{n\to\infty} \dfrac{1}{\log\log n}$ (c) $\lim_{n\to\infty} \dfrac{3^n}{n!}$

(d) $\lim_{n\to\infty} \dfrac{n^2+2n+1}{2n^2-n+2}$ (e) $\lim_{n\to\infty} \sqrt{n^2+n} - n$ f) $\lim_{n\to\infty} \cos \frac{1}{n}$

B. Prove from the definition that the sequence $a_n = L$ for $n \ge 1$ has a limit.

C. Show that $\lim_{n\to\infty} \sin \frac{n\pi}{2}$ does not exist using the definition of limit.

D. Prove that if $a_n \le b_n$ for $n \ge 1$, $L = \lim_{n\to\infty} a_n$, and $M = \lim_{n\to\infty} b_n$, then $L \le M$.

E. Prove that if $L = \lim_{n\to\infty} a_n$, then $L = \lim_{n\to\infty} a_{2n}$ and $L = \lim_{n\to\infty} a_{n^2}$.

F. Define a sequence $(a_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_{n^2}$ exists but $\lim_{n\to\infty} a_n$ does not exist.

G. Sometimes, a limit is defined informally as follows: "As n goes to infinity, a_n gets closer and closer to L." Find as many faults with this definition as you can.
(a) Can a sequence satisfy this definition and still fail to converge?
(b) Can a sequence converge yet fail to satisfy this definition?

H. Suppose that $\lim\limits_{n\to\infty} a_n = L$ and $L \neq 0$. Prove there is some N such that $a_n \neq 0$ for all $n \geq N$.

I. Give a careful proof, using the definition of limit, that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ imply that $\lim\limits_{n\to\infty} 2a_n + 3b_n = 2L + 3M$.

J. For each $x \in \mathbb{R}$, determine whether $\left(\dfrac{1}{1+x^n}\right)_{n=1}^{\infty}$ has a limit, and compute it when it exists.

K. Let a_0 and a_1 be positive real numbers, and set $a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n}$ for $n \geq 0$.
(a) Show that there is N such that for all $n \geq N$, $a_n \geq 1$.
(b) Let $\varepsilon_n = |a_n - 4|$. Show that $\varepsilon_{n+2} \leq (\varepsilon_{n+1} + \varepsilon_n)/3$ for $n \geq N$.
(c) Prove that this sequence converges.

L. Show that the sequence $(\log n)_{n=1}^{\infty}$ does not converge.

M. Provide an example of three sequences, (a_n), (b_n), and (c_n), with $a_n \leq b_n \leq c_n$ such that both $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} c_n$ exist, but $\lim\limits_{n\to\infty} b_n$ does not exist.

2. Basic Properties of Limits

We have already developed a number of basic properties of limits in the examples and exercises of the previous section. For example, the Squeeze Theorem and Exercise 1.D show that limits respect order. It is also crucial that limits respect the arithmetic operations. Proving this is straightforward. We will prove parts (1) and (4). The other parts are done in a similar manner.

2.1. Theorem. *If* $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = M$, *and* $\alpha \in \mathbb{R}$, *then*

(1) $\lim\limits_{n\to\infty} a_n + b_n = L + M$,

(2) $\lim\limits_{n\to\infty} \alpha a_n = \alpha L$,

(3) $\lim\limits_{n\to\infty} a_n b_n = LM$, *and*

(4) *if* $M \neq 0$, *then there is an integer* N_0 *so that* $b_n \neq 0$ *for* $n \geq N_0$, *and* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{M}$.

(We only consider terms with $n \geq N_0$ *so that* $\dfrac{a_n}{b_n}$ *is defined.)*

Proof. (1) Notice that

$$|(a_n + b_n) - (L + M)| = |a_n - L + b_n - M| \leq |a_n - L| + |b_n - M|.$$

Since $\lim\limits_{n\to\infty} a_n = L$, we can find $N_1 > 0$ so that

$$|a_n - L| < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_1.$$

Similarly, we can find $N_2 > 0$ so that

$$|b_n - M| < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_2.$$

Define $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(4) Since $M \neq 0$, $|M|/2 > 0$. Using $\varepsilon = |M|/2$, we can find some N_0 so that

$$|b_n - M| < \frac{|M|}{2} \quad \text{for all} \quad n \geq N_0.$$

Therefore

$$|b_n| \geq |M| - |b_n - M| > |M|/2;$$

and hence $b_n \neq 0$ when $n \geq N_0$. Taking the reciprocals of both sides, we get

$$\frac{1}{|b_n|} \leq \frac{2}{|M|} \quad \text{for all} \quad n \geq N_0.$$

For $n \geq N_0$, we can write

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{L}{M}\right| &= \left|\frac{a_n M - L b_n}{b_n M}\right| \\
&= \left|\frac{a_n M - LM + LM - L b_n}{b_n M}\right| \\
&\leq \left|\frac{a_n M - LM}{b_n M}\right| + \left|\frac{LM - L b_n}{b_n M}\right| \\
&= |a_n - L| \left|\frac{1}{b_n}\right| + |M - b_n| \left|\frac{L}{b_n M}\right| \\
&\leq |a_n - L| \frac{2}{|M|} + |M - b_n| \frac{2|L|}{M^2}.
\end{aligned}$$

Now we are set to use the convergence of a_n and b_n. Choose N_1 so that

$$|a_n - L| < \frac{\varepsilon |M|}{4} \quad \text{for all} \quad n \geq N_1;$$

and choose N_2 so that

$$|b_n - M| < \frac{\varepsilon |M|^2}{4|L| + 1} \quad \text{for all} \quad n \geq N_2.$$

Let $N = \max\{N_0, N_1, N_2\}$. If $n \geq N$, then since $b_n \geq |M|/2 > 0$, we know b_n is not zero and we can use our estimate to conclude

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{L}{M}\right| &\leq \left|a_n - L\right| \frac{2}{|M|} + \left|M - b_n\right| \frac{2|L|}{M^2} \\
&< \frac{\varepsilon |M|}{4} \frac{2}{|M|} + \frac{\varepsilon M^2}{4|L| + 1} \frac{2|L|}{M^2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

■

Another important observation is that convergent sequences are bounded.

2.2. Proposition. *If $(a_n)_{n=1}^{\infty}$ is a convergent sequence of real numbers, then it is bounded.*

Proof. Let $L = \lim_{n\to\infty} a_n$. If we set $\varepsilon = 1$, then by the definition of limit, there is some $N > 0$ such that $|a_n - L| < 1$ for all $n \geq N$. In other words,

$$L - 1 < a_n < L + 1 \quad \text{for all} \quad n \geq N.$$

Let $M = \max\{a_1, a_2, \dots, a_{N-1}, L+1\}$ and $m = \min\{a_1, a_2, \dots, a_{N-1}, L-1\}$. Clearly, for all n, we have $m \leq a_n \leq M$. ■

There is no special reason to use 1 in this proof except convenience. We could have picked $\varepsilon = 1/2$ or $\varepsilon = 42$ and the argument would still work.

Exercises for Section 2

A. Prove parts (2) and (3) of Theorem 2.1.

B. Compute the following limits.

(a) $\displaystyle\lim_{n\to\infty} \frac{3n^5 - n^4 + 175}{7n^5 + 100N^3 - 32n}$ (b) $\displaystyle\lim_{n\to\infty} \frac{2^{100+5n}}{e^{4n-10}}$ (c) $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} + \frac{2\arctan n}{\log n}$

C. If $\lim_{n\to\infty} a_n = L > 0$, prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. Be sure to discuss the issue of when $\sqrt{a_n}$ makes sense. Hint: Express $|\sqrt{a_n} - \sqrt{L}|$ in terms of $|a_n - L|$.

D. Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be two sequences of real numbers such that $|a_n - b_n| < \frac{1}{n}$. Suppose that $L = \lim_{n\to\infty} a_n$ exists. Show that $(b_n)_{n=1}^\infty$ converges to L also.

E. Find $\displaystyle\lim_{n\to\infty} \frac{\log(2+3^n)}{2n}$. Hint: $\log(2+3^n) = \log 3^n + \log \frac{2+3^n}{3^n}$

F. (a) Let $x_n = \sqrt[n]{n} - 1$. Use the fact that $(1+x_n)^n = n$ to show that $x_n^2 \leq 2/n$.
Hint: Use the Binomial Theorem and throw away most terms.
(b) Hence compute $\lim_{n\to\infty} n^{1/n}$.

G. Show that the set of rational numbers is **dense** in $\mathbb{R}$, meaning that every real number is a limit of rational numbers.

H. (a) Show that $\frac{b-1}{b} \leq \log b \leq b - 1$. Hint: Integrate $1/x$ from 1 to b.
(b) Apply this to $b = \sqrt[n]{a}$ to show that $\log a \leq n(\sqrt[n]{a} - 1) \leq \sqrt[n]{a} \log a$.
(c) Hence evaluate $\lim_{n\to\infty} n\big(\sqrt[n]{a} - 1\big)$.

I. Suppose that $\lim_{n\to\infty} a_n = L$. Show that $\displaystyle\lim_{n\to\infty} \frac{a_1 + a_2 + \dots + a_n}{n} = L$.

J. Show that the set $S = \{n + m\sqrt{2} : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Hint: Find infinitely many elements of S in $[0, 1]$. Use the Pigeonhole Principle to find two that are close within 10^{-k}.

3. The Least Upper Bound Principle

After defining the least upper bound of a set of real numbers, we prove the Least Upper Bound Principle (3.3). This result depends crucially on our construction of the real numbers. It will be the basis for the deeper properties of the real line.

3.1. Definition. A set $S \subset \mathbb{R}$ is **bounded above** if there is a real number M such that $s \leq M$ for all $s \in S$. We call M an **upper bound** for S. Similarly, S is **bounded below** if there is a real number m such that $s \geq m$ for all $s \in S$, and we call m a **lower bound** for S. A set that is bounded above and below is called **bounded**.

Suppose a nonempty subset S of $\mathbb{R}$ is bounded above. Then L is the **supremum** or **least upper bound** for S if L is an upper bound for S that is smaller than all other upper bounds, i.e., for all $s \in S$, $s \le L$, *and* if M is another upper bound for S, then $L \le M$. It is denoted by $\sup S$.

Similarly, if S is a nonempty subset of $\mathbb{R}$ which is bounded below, the **infimum** or **greatest lower bound**, denoted by $\inf S$, is the number L such that L is an lower bound *and* whenever M is another lower bound for S, then $L \ge M$.

The supremum of a set, if it exists, is unique. We have not defined suprema or infima for sets that are not bounded above or bounded below, respectively. For example, $\mathbb{R}$ itself has neither a supremum nor an infimum. For a nonempty set $S \subseteq \mathbb{R}$, sometimes we write $\sup S = +\infty$ if S is not bounded above and $\inf S = -\infty$ if S is not bounded below. Finally, by convention, $\sup \varnothing = -\infty$ and $\inf \varnothing = +\infty$.

Note that $\sup S = L \in \mathbb{R}$ if and only if L is a upper bound for S and for all $K < L$, there is $x \in S$ with $K < x < L$. There is an equivalent characterization for $\inf S$.

Recall that the **maximum** of a set $S \subset \mathbb{R}$, *if it exists*, is an element $m \in S$ such that $s \le m$ for all $s \in S$. Thus, when the maximum of a set exists, it is the least upper bound. The situation for the **minimum** of a set and its infimum is the same. We use $\max S$ and $\min S$ to denote the maximum and minimum of S.

3.2. Examples.

(1) If $A = \{4, -2, 5, 7\}$, then any $L \le -2$ is a lower bound for A and any $M \ge 7$ is an upper bound. So, $\inf A = \min A = -2$ and $\sup A = \max A = 7$.

(2) If $B = \{2, 4, 6, \ldots\}$, then $\inf B = \min B = 2$ and $\sup B = +\infty$.

(3) If $C = \{\pi/n : n \in \mathbb{N}\}$, then $\sup C = \max C = \pi$. However, for any element of C, say π/n, we have a smaller element of C, such as $\pi/(2n)$. So C does not have a minimum. Clearly, 0 is a lower bound and for all $x > 0$, there is some $\pi/n \in C$ with $\pi/n < x$, showing that 0 is the greatest lower bound.

(4) If $D = \{(-1)^n n/(n+1) : n \in \mathbb{N}\}$, then D has neither a maximum nor a minimum. However, D has upper and lower bounds, and $\inf D = -1$ and $\sup D = 1$. Neither 1 nor -1 belongs to D.

In proving the Least Upper Bound Principle, the definition of the real numbers as *all* infinite decimals is essential. The principle is not true for some subsets of the rational numbers. For example, $\{s \in \mathbb{Q} : s^2 < 2\}$ is bounded above but has no least upper bound in $\mathbb{Q}$.

3.3. Least Upper Bound Principle.

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum. Similarly, every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.

Proof. We prove the second statement first, since it is more convenient. Let M be some lower bound for S with decimal expansion $M = m_0.m_1m_2\ldots$. Let s be some element of S with decimal expansion $s = s_0.s_1s_2\ldots$. Notice that since $m_0 \le M$, we have that m_0 is a lower bound for S. On the other hand, $s < s_0 + 2$. So $s_0 + 2$ is not a lower bound. There are only finitely many integers between m_0 and $s_0 + 1$. Pick the largest of these that is still a lower bound for S, and call it a_0. Since $a_0 + 1$ is not a lower bound, we may also choose an element x_0 in S such that $x_0 < a_0 + 1$.

Next pick the greatest integer a_1 such that $y_1 = a_0 + 10^{-1}a_1$ is a lower bound for S. Since $a_1 = 0$ works and $a_1 = 10$ does not, a_1 belongs to $\{0, 1, \ldots, 9\}$. To verify our choice, pick an element x_1 in S such that $a_0.a_1 \le x_1 < a_0.a_1 + 0.1$. Continue in this way recursively. Figure 3.1 shows how a_2 and x_2 would be chosen.

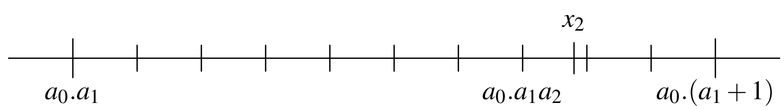

Figure 3.1. The second stage ($k = 2$) in the proof.

At the kth stage, we have a lower bound $y_{k-1} = a_0.a_1 \ldots a_{k-1}$ and an element $x_{k-1} \in S$ such that $y_{k-1} \le x_{k-1} < y_{k-1} + 10^{1-k}$. Select the largest integer a_k in $\{0, 1, \ldots, 9\}$ such that $y_k = a_0.a_1a_2 \ldots a_k$ is a lower bound for S. Since $y_k + 10^{-k}$ is not a lower bound, we also pick an element x_k in S such that $x_k < y_k + 10^{-k}$ to verify our choice.

We claim that $L = a_0.a_1a_2 \ldots$ is $\inf S$. If $L = y_k$ for some k, then L is a lower bound for S. Otherwise, $L > y_k$ for all k and, in particular, for each k there is $l > k$ with $y_l > y_k$. If $s = s_0.s_1s_2 \ldots$ is in S, then it follows that $s > y_k$ for each k. By the definition of the order, either $s_i = a_i$ for $1 \le i \le k$ or there is some j, $0 \le j \le k$, with $s_i = a_i$ for $1 \le i < j$ and $s_j > a_j$. If the latter occurs for some k, then $s > L$; if the former occurs for every k, then $s = L$. Either way, L is a lower bound for S.

To see that L is the greatest lower bound, suppose $M = b_0.b_1b_2 \ldots > L$. By the definition of the ordering, there is some first integer k such that $b_k > a_k$ and $b_i = a_i$ for all i with $0 \le i < k$. But then

$$M \ge a_0.a_1 \ldots a_{k-1}b_k \ge y_k + 10^{-k} > x_k.$$

So M is not a lower bound for S. Hence L is the greatest lower bound.

A simple trick handles upper bounds. Notice that $S \subset \mathbb{R}$ is bounded above if and only if $-S = \{-s : s \in S\}$ is bounded below and that L is an upper bound for S precisely when $-L$ is a lower bound for $-S$. Further, $M < L$ if and only if $-M > -L$, so M is an upper bound of S less than L exactly when $-M$ is a lower bound of $-S$ greater than $-L$. Thus $\sup S = -\inf(-S)$, so $\sup S$ exists. ■

Exercises for Section 3

A. Suppose that $S \subset \mathbb{R}$ is bounded above. When does S have a maximum? Your answer should be expressed in terms of $\sup S$.

B. For the following sets, find the supremum and infimum. Which have a max or min?
(a) $A = \{a + a^{-1} : a \in \mathbb{Q},\ a > 0\}$.
(b) $B = \{a + (2a)^{-1} : a \in \mathbb{Q},\ 0.1 \le a \le 5\}$.
(c) $C = \{\sin n : n \in \mathbb{Z}\}$.
(d) $D = \{xe^{-x} : x > 0\}$.

C. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of positive real numbers and $S = \left\{ \dfrac{a_n}{b_n} : n \ge 1 \right\}$ is bounded above. Prove that there is a constant M such that $a_n \le M b_n$ for all $n \ge 1$.

D. Let $\mathcal{D}$ denote the set of all finite decimals.. Show that $\sup\{a \in \mathcal{D} : a^2 \le 3\} = \sqrt{3}$.
HINT: show that if $0 < d < 2$, then $(d + 10^{-n})^2 - d^2 < 5 \cdot 10^{-n}$. Consider the largest $d = 1.a_1 \dots a_n$ such that $d^2 < 3$. Hence get an estimate for $\sqrt{3} - d$.

E. A more elegant way to develop the arithmetic properties of the real numbers is to prove the results of this section first and then define addition and multiplication using suprema. Let $\mathbb{Q}$ denote the set of all rational numbers.
(a) Let $x, y \in \mathbb{R}$. Prove that $x + y = \sup\{a + b : a, b \in \mathbb{Q},\ a \le x,\ b \le y\}$.
(b) Suppose that $x, y \in \mathbb{R}$ are positive. Show that $xy = \sup\{ab : a, b \in \mathbb{Q},\ 0 \le a \le x,\ 0 \le b \le y\}$.
(c) How do we define multiplication in general?

4. Monotone Sequences

We now consider some consequences of the Least Upper Bound Principle (3.3).

4.1. Definition. A sequence (a_n) is **(strictly) monotone increasing** if $a_n \le a_{n+1}$ (or $a_n < a_{n+1}$) for all $n \ge 1$. Similarly, we define (strictly) monotone decreasing sequences.

4.2. Monotone Convergence Theorem.
A monotone increasing sequence that is bounded above converges.
A monotone decreasing sequence that is bounded below converges.

Proof. Suppose $(a_n)_{n=1}^{\infty}$ is an increasing sequence that is bounded above. Then by the Least Upper Bound Principle, there is a number

$$L = \sup\{a_n : n \in \mathbb{N}\}.$$

We will show that $\lim\limits_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$ be given. Since $L - \varepsilon$ is not an upper bound for A, there is some integer N such that $a_N > L - \varepsilon$. Then because the sequence is monotone increasing,

$$L - \varepsilon < a_N \le a_n \le L \quad \text{for all} \quad n \ge N.$$

So $|a_n - L| < \varepsilon$ for all $n \ge N$ as required. Therefore, $\lim\limits_{n\to\infty} a_n = L$.

If (a_n) is decreasing and bounded below by B, then the sequence $(-a_n)$ is increasing and bounded above by $-B$. Thus the sequence $(-a_n)_{n=1}^{\infty}$ has a limit $L = \lim\limits_{n\to\infty} -a_n$. Therefore $-L = \lim\limits_{n\to\infty} a_n$ exists. ∎

4.3. Example. Consider the sequence given recursively by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \sqrt{2 + \sqrt{a_n}} \quad \text{for all} \quad n \ge 1.$$

Evaluating $a_2, a_3, \dots, a_9$, we obtain 1.7320508076, 1.8210090645, 1.8301496356, 1.8310735189, 1.831166746, 1.8311761518, 1.8311771007, 1.8311771965. It appears that this sequence increases to some limit.

To prove this, first we show by induction that

$$1 \le a_n < a_{n+1} < 2 \quad \text{for all} \quad n \ge 1.$$

Since $1 = a_1 < \sqrt{3} = a_2 < 2$, this is valid for $n = 1$. Suppose that it holds for some n. Then

$$a_{n+2} = \sqrt{2 + \sqrt{a_{n+1}}} > \sqrt{2 + \sqrt{a_n}} = a_{n+1} \ge 1,$$

and

$$a_{n+2} = \sqrt{2 + \sqrt{a_{n+1}}} < \sqrt{2 + \sqrt{2}} < 2.$$

This verifies our claim for $n+1$. Hence by induction, it is valid for each $n \geq 1$.

Therefore, (a_n) is a monotone increasing sequence. So by the Monotone Convergence Theorem (4.2), it follows that there is a limit $L = \lim_{n\to\infty} a_n$. It is not clear that there is a nice expression for L. However, once we know that the sequence converges, it is not hard to find a formula for L. Notice that

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + \sqrt{a_n}} = \sqrt{2 + \sqrt{\lim_{n\to\infty} a_n}} = \sqrt{2 + \sqrt{L}}.$$

We used the fact that the limit of square roots is the square root of the limit (see Exercise 2.C). Squaring both sides gives $L^2 - 2 = \sqrt{L}$, and further squaring yields

$$0 = L^4 - 4L^2 - L + 4 = (L-1)(L^3 + L^2 - 3L - 4).$$

Since $L > 1$, it must be a root of the cubic $p(x) = x^3 + x^2 - 3x - 4$ in the interval $(1, 2)$. There is only one such root. as graphing the curve shows (see Figure 4.1). Indeed,

$$p'(x) = 3x^2 + 2x - 3 = 3(x^2 - 1) + 2x$$

is positive on $[1, 2]$. So p is strictly increasing. Since $p(1) = -5$ and $p(2) = 2$, p has exactly one root in between.

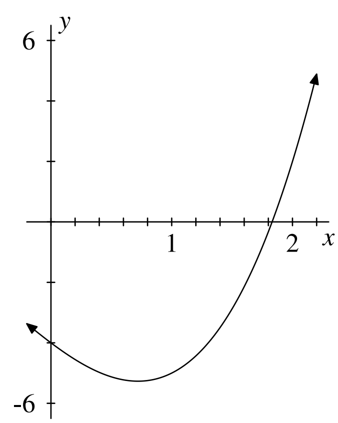

Figure 4.1. Graph of $x^3 + x^2 - 3x - 4$.

For the amusement of the reader, we give an explicit algebraic formula:

$$L = \frac{1}{3}\Big(\sqrt[3]{\tfrac{79+\sqrt{2241}}{2}} + \sqrt[3]{\tfrac{79-\sqrt{2241}}{2}} - 1\Big).$$

4.4. Example. Notice that we first prove that the sequence converges, and then evaluate the limit. This is important. Consider the sequence given by $a_1 = 2$ and $a_{n+1} = (a_n^2 + 1)/2$ for $n \geq 1$. This is a monotone increasing sequence. Suppose we let L denote the limit and compute

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (a_n^2 + 1)/2 = (L^2 + 1)/2.$$

Thus $(L-1)^2 = 0$, which means that $L = 1$. This is an absurd conclusion because this sequence is monotone increasing and greater than 2. The fault lay in assuming that the limit L actually exists, because instead it diverges to $+\infty$ (see Exercise 4.A).

Exercises for Section 4

A. Say that $\lim_{n\to\infty} a_n = +\infty$ if for every $R \in \mathbb{R}$, there is an integer N such that $a_n > R$ for all $n \geq N$. Show that a divergent monotone increasing sequence converges to $+\infty$ in this sense.

B. Let $a_1 = 0$ and $a_{n+1} = \sqrt{5 + 2a_n}$ for $n \geq 1$. Show that $\lim_{n\to\infty} a_n$ exists and find the limit.

C. Is $S = \{x \in \mathbb{R} : 0 < \sin(\frac{1}{x}) < \frac{1}{2}\}$ bounded above (below)? If so, find $\sup S$ ($\inf S$).

D. (a) Evaluate $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n}$.
(b) Show that this sequence is monotone decreasing.

E. Suppose (a_n) is a sequence of positive real numbers such that $a_{n+1} - 2a_n + a_{n-1} > 0$ for all $n \geq 1$. Prove that the sequence either converges or tends to $+\infty$.

F. Let a, b be positive real numbers. Set $x_0 = a$ and $x_{n+1} = (x_n^{-1} + b)^{-1}$ for $n \geq 0$.
(a) Prove that x_n is monotone decreasing.
(b) Prove that the limit exists and find it.

G. Let $a_n = (\sum_{k=1}^n 1/k) - \log n$ for $n \geq 1$. **Euler's constant** is defined as $\gamma = \lim_{n\to\infty} a_n$. Show that $(a_n)_{n=1}^\infty$ is decreasing and bounded below by zero, and so this limit exists.
HINT: Prove that $1/(n+1) \leq \log(n+1) - \log n \leq 1/n$.
(It is unknown whether γ is rational or not. It is known that if γ is rational, then the denominator has more than 242,000 decimal digits [as of 2007]. So it is *suspected* to be irrational.)

H. Let $x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}$.
(a) Show that $x_n < x_{n+1} < 1 + \sqrt{2}x_n$.
(b) Hence show that x_n converges.

5. Subsequences

Given one sequence, we can build a new sequence, called a subsequence of the original, by picking out some of the entries. Perhaps surprisingly, when the original sequence does not converge, it is often possible to find a subsequence that does.

5.1. Definition. A **subsequence** of a sequence $(a_n)_{n=1}^\infty$ is a sequence

$$(a_{n_k})_{k=1}^\infty = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots),$$

where $n_1 < n_2 < n_3 < \cdots$.

For example, $(a_{2k})_{k=1}^\infty$ and $(a_{k^3})_{k=1}^\infty$ are subsequences, where $n_k = 2k$ and $n_k = k^3$, respectively. Notice that if we pick $n_k = k$ for each k, then we get the original sequence; so $(a_n)_{n=1}^\infty$ is a subsequence of itself.

It is easy to verify that if $(a_n)_{n=1}^\infty$ converges to a limit L, then $(a_{n_k})_{k=1}^\infty$ also converges to the same limit. On the other hand, the sequence $(1, 2, 3, \ldots)$ does not have a limit, nor does any subsequence, because any subsequence must diverge to $+\infty$. However, we will show that as long as a sequence remains bounded, it has subsequences that converge.

5.2. Bolzano–Weierstrass Theorem.
Every bounded sequence of real numbers has a convergent subsequence.

Proof. Let (a_n) be a sequence bounded by B. Thus the interval $[-B, B]$ contains the whole (infinite) sequence. Now if I is an interval containing infinitely many points of the sequence (a_n), and $I = J_1 \cup J_2$ is the union of two smaller intervals, then at least one of them contains infinitely many points of the sequence, too.

So let $I_1 = [-B, B]$. Split it into two closed intervals of length B, namely $[-B, 0]$ and $[0, B]$. One of these halves contains infinitely many points of (a_n); call it I_2. Similarly, divide I_2 into two closed intervals of length $B/2$. Again pick one, called I_3, that contains infinitely many points of our sequence. Recursively, we construct a decreasing sequence I_k of closed intervals of length $2^{2-k}B$ such that each contains infinitely many points of our sequence. Figure 5.1 shows the choice of I_3 and I_4, where the terms of the sequence are indicated by vertical lines.

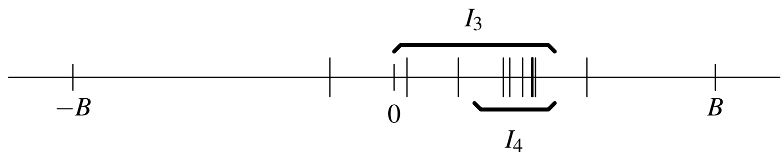

Figure 5.1. Choice of intervals I_3 and I_4.

Let the left and right endpoints of I_k be b_k and c_k. Since $I_{k+1} \subset I_k$, we have

$$b_k \le b_{k+1} < c_{k+1} \le c_k.$$

Therefore (b_k) is an increasing sequence which is bounded above by c_1 (or any c_m). So by the Monotone Convergence Theorem 4.2, $\lim_{k\to\infty} b_k = L$ exists. Similarly, (c_k) is a monotone decreasing sequence which is bounded below by b_1. So $\lim_{k\to\infty} c_k = M$ exists. Moreover, $M = L$ because

$$M - L = \lim_{k\to\infty} c_k - b_k = \lim_{k\to\infty} 2^{2-k}B = 0.$$

Finally, choose an *increasing* sequence n_k such that a_{n_k} belongs to I_k. This is possible since each I_k contains infinitely many elements of the sequence, and only finitely many have index at most n_{k-1}. Then $b_k \le a_{n_k} \le c_k$. By the Squeeze Theorem 1.5, $\lim_{k\to\infty} a_{n_k} = L$. ■

5.3. Example. Consider the sequence $(a_n) = (\operatorname{sign}(\sin n))_{n=1}^{\infty}$, where the sign function takes values ± 1 depending on the sign of x except for $\operatorname{sign} 0 = 0$. Without knowing anything about the properties of the sine function, we can observe that the sequence (a_n) takes at most three different values. At least one of these values is taken infinitely often. Thus it is possible to deduce the existence of a subsequence that is constant and therefore converges.

Using our knowledge of sine allows us to get somewhat more specific. Now $\sin x = 0$ exactly when x is an integer multiple of π. Since π is irrational, $k\pi$ is never an integer for $k > 0$. Therefore, a_n takes only the values ± 1. Note that $\sin x > 0$ if there is an integer k such that $2k\pi < x < (2k+1)\pi$; and $\sin x < 0$ if there is an integer k such that $(2k-1)\pi < x < 2k\pi$. Observe that n increases by steps of length 1, while the intervals on which $\sin x$ takes positive or negative values has length $\pi \approx 3.14$. Consequently, a_n takes the value $+1$ for three or four terms in a row, followed by three or four terms taking the value -1. Consequently, both 1 and -1 are limits of certain subsequences of (a_n).

To compute a particular sequence n_k for which $a_{n_k} = 1$ for all k requires a much more delicate analysis depending on π. One of the nice things about analysis is that one can often make significant use of such a sequence *without* knowing the details of which subsequence is used.

5.4. Example. Now consider the sequence $(a_n) = (\sin n)_{n=1}^{\infty}$. As the angles n radians for $n \ge 1$ are marked on a circle, they appear gradually to fill in a dense subset. If this can be demonstrated, we should be able to show that L is a limit of a subsequence of our sequence for every L in $[-1, 1]$.

The key is to approximate the angle 0 modulo 2π by integers. If x is a real number, let $\{x\}$ denote the number θ in $(-\pi, \pi]$ so that $x - \theta$ is an integer multiple of 2π.

Let $\varepsilon > 0$. Choose an integer N so large that $N\varepsilon > 2\pi$. Divide the circle into N arcs of length $2\pi/N$ radians each. Then consider the $N+1$ points $\{0\}, \{1\}, \{2\}, \dots, \{N\}$. Since there are $N+1$ points distributed into only N arcs, the Pigeonhole Principle implies that at least one arc contains two points, say i and j, where $i < j$. Then $n = j - i$ represents an angle with $|\{n\}| < \frac{2\pi}{N} < \varepsilon$. That is, $n = \theta + 2\pi s$ for some integer s and real number $|\theta| < \varepsilon$. Since π is not rational, n is not an exact multiple of 2π; and thus $\{n\} \ne 0$.

So given L in $[-1, 1]$, find an angle α so that $\sin\alpha = L$. Construct a subsequence as follows. Let $n_1 = 0$. Recursively we construct an increasing sequence $n_k < n_{k+1}$ so that

$$|\sin n_k - L| \le \frac{1}{k}.$$

Once n_k is defined, take $\varepsilon = \frac{1}{k+1}$. As in the previous paragraph, there is an integer n such that $n = \theta + 2\pi s$ and $|\theta| < \frac{1}{k+1}$. So multiples of n wrap around the circle with less than $\frac{1}{k+1}$ gap between each and the next. Thus there is a positive integer $t > n_k/n$ such that $|\{\alpha - tn\}| = |\{\alpha - t\theta\}| < \frac{1}{k+1}$. Therefore

$$|\sin(tn) - L| = |\sin(t\theta) - \sin(\alpha)| \le |\{\alpha - tn\}| < \frac{1}{k+1}.$$

Set $n_{k+1} = tn$. This completes the induction. The result is a subsequence such that

$$\lim_{k\to\infty} \sin(n_k) = L.$$

This used the fact that $|\sin x - \sin y| \le |x - y|$. This will be proven later using the Mean Value Theorem.

Exercises for Section 5

A. Show that $(a_n) = \left(\frac{n\cos^n(n)}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$ has a convergent subsequence.

B. Does the sequence $(b_n) = \left(n + \cos(n\pi)\sqrt{n^2+1}\right)_{n=1}^{\infty}$ have a convergent subsequence?

C. Does the sequence $(a_n) = (\cos\log n)_{n=1}^{\infty}$ converge?

D. Show that every sequence has a monotone subsequence.

E. Use trig identities to show that $|\sin x - \sin y| \le |x - y|$.
HINT: Let $a = (x + y)/2$ and $b = (x - y)/2$. Use the addition formula for $\sin(a \pm b)$.

F. Let $(x_n)_{n=1}^\infty$ be a sequence of real numbers. Suppose that there is a real number L such that $L = \lim_{n\to\infty} x_{3n-1} = \lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} x_{3n}$. Show that $\lim_{n\to\infty} x_n$ exists and equals L.

G. Suppose that $(a_n)_{n=1}^\infty$ is a sequence such that $\lim_{n\to\infty} a_n = L$. Let $(a_{n_k})_{k=1}^\infty$ be any subsequence of $(a_n)_{n=1}^\infty$. Prove that $\lim_{k\to\infty} a_{n+k} = L$.

H. Define $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$ for $n \geq 1$.
(a) Find a formula for $x_{n+1}^2 - 5$ in terms of $x_n^2 - 5$.
(b) Hence evaluate $\lim_{n\to\infty} x_n$.
(c) Compute the first ten terms on a computer or a calculator.
(d) Show that the tenth term approximates the limit to over 600 decimal places.

I. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. Suppose there is a number L such that every subsequence $(x_{n_k})_{k=1}^\infty$ has a subsubsequence $\left(x_{n_{k(l)}}\right)_{l=1}^\infty$ with $\lim_{l\to\infty} x_{n_{k(l)}} = L$. Show that the whole sequence converges to L. HINT: If not, you could find a subsequence bounded away from L.

J. Suppose $(x_n)_{n=1}^\infty$ is a sequence in $\mathbb{R}$, and that L_k are real numbers with $\lim_{k\to\infty} L_k = L$. If for each $k \geq 1$, there is a subsequence of $(x_n)_{n=1}^\infty$ converging to L_k, show that some subsequence converges to L. HINT: Find an increasing sequence n_k such that $|x_{n_k} - L| < 1/k$.

K. Let $(x_n)_{n=1}^\infty$ be an arbitrary sequence. Prove that there is a subsequence $(x_{n_k})_{k=1}^\infty$ which converges or $\lim_{k\to\infty} x_{n_k} = \infty$ or $\lim_{k\to\infty} x_{n_k} = -\infty$.

L. Construct a sequence $(x_n)_{n=1}^\infty$ such that for every real number L, there is a subsequence $(x_{n_k})_{k=1}^\infty$ with $\lim_{k\to\infty} x_{n_k} = L$.

6. Cauchy Sequences

Can we decide whether a sequence converges *without* first finding the value of the limit? To do this, we need an intrinsic property of a sequence which is equivalent to convergence that does not make use of the value of the limit. This intrinsic property shows which sequences are 'supposed' to converge. This leads us to the notion of a subset of $\mathbb{R}$ being *complete* if all sequences in the subset that are 'supposed' to converge actually do. As we shall see, this completeness property has been built into the real numbers by our construction of infinite decimals.

To obtain an appropriate condition, notice that if a sequence (a_n) converges to L, then as the terms get close to the limit, they are getting close to each other.

6.1. Proposition. *Let $(a_n)_{n=1}^\infty$ be a sequence converging to L. For every $\varepsilon > 0$, there is an integer N such that*

$$|a_n - a_m| < \varepsilon \quad \textit{for all} \quad m, n \geq N.$$

Proof. Fix $\varepsilon > 0$ and use the value $\varepsilon/2$ in the definition of limit. Then there is an integer N such that $|a_n - L| < \varepsilon/2$ for all $n \geq N$. Thus if $m, n \geq N$, we obtain

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \blacksquare$$

In order for N to work in the conclusion, for every $m \geq N$, a_m must be within ε of a_N. It is not enough to just have a_N and a_{N+1} close (see Exercise 6.B).

We make the conclusion of this proposition into a definition. This definition retains the flavour of the definition of a limit, in that it has the same logical structure: *For all* $\varepsilon > 0$*, there is an integer* N *. . . .*

6.2. Definition. A sequence $(a_n)_{n=1}^\infty$ of real numbers is called a **Cauchy sequence** provided that for *every* $\varepsilon > 0$, there is an integer N such that

$$|a_m - a_n| < \varepsilon \quad \text{for all} \quad m, n \geq N.$$

6.3. Proposition. *Every Cauchy sequence is bounded.*

Proof. The proof is basically the same as that for convergent sequences in Proposition 2.2. Let $(a_n)_{n=1}^\infty$ be a Cauchy sequence. Taking $\varepsilon = 1$, find N so large that

$$|a_n - a_N| < 1 \quad \text{for all} \quad n \geq N.$$

It follows that the sequence is bounded by $\max\{|a_1|, \dots, |a_{N-1}|, |a_N| + 1\}$. ■

Since the definition of a Cauchy sequence does not require the use of a potential limit L, it permits the following definition.

6.4. Definition. A subset S of $\mathbb{R}$ is said to be **complete** if every Cauchy sequence (a_n) in S (that is, $a_n \in S$) converges to a point in S.

This brings us to an important conclusion about the real numbers themselves, another property that distinguishes the real numbers from the rational numbers.

6.5. Completeness Theorem.
Every Cauchy sequence of real numbers converges. So $\mathbb{R}$ *is complete.*

Proof. Suppose that $(a_n)_{n=1}^\infty$ is a Cauchy sequence. By Proposition 6.3, $\{a_n : n \geq 1\}$ is bounded. By the Bolzano–Weierstrass Theorem (5.2), this sequence has a convergent subsequence, say

$$\lim_{k\to\infty} a_{n_k} = L.$$

Let $\varepsilon > 0$. From the definition of Cauchy sequence for $\varepsilon/2$, there is an integer N such that

$$|a_m - a_n| < \frac{\varepsilon}{2} \quad \text{for all} \quad m, n \geq N.$$

And from the definition of limit using $\varepsilon/2$, there is an integer K such that

$$|a_{n_k} - L| < \frac{\varepsilon}{2} \quad \text{for all} \quad k \geq K.$$

Pick any $k \geq K$ such that $n_k \geq N$. Then for every $n \geq N$,

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\lim\limits_{n\to\infty} a_n = L$. ■

6.6. Remark. This theorem is not true for the rational numbers. Here is an example of a Cauchy sequence of rational numbers that does not converge to a rational number. Define the sequence $(a_n)_{n=1}^{\infty}$ by

$$a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \quad a_4 = 1.4142, \quad a_5 = 1.41421, \ldots$$

and in general, a_n is the first $n+1$ digits in the decimal expansion of $\sqrt{2}$. If n and m are greater than N, then a_n and a_m agree for at least first $N+1$ digits. Thus

$$|a_n - a_m| < 10^{-N} \quad \text{for all} \quad m, n \geq N.$$

This shows that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. (Why?)

However, this sequence has no limit *in the rationals.* In our terminology, $\mathbb{Q}$ is not complete. Of course, this sequence does converge to a real number, namely $\sqrt{2}$. This is one way to see the essential difference between $\mathbb{R}$ and $\mathbb{Q}$: the set of real numbers is complete and $\mathbb{Q}$ is not.

6.7. Example. Let α be an arbitrary real number. Define $a_n = [n\alpha]/n$, where $[x]$ is the nearest integer to x. Then $\big|[n\alpha] - n\alpha\big| \leq 1/2$. So

$$|a_n - \alpha| = \frac{\big|[n\alpha] - n\alpha\big|}{n} \leq \frac{1}{2n}.$$

We claim $\lim\limits_{n\to\infty} a_n = \alpha$. Indeed, given $\varepsilon > 0$, choose N so large that $\frac{1}{N} < \varepsilon$. Then for $n \geq N$, $|a_n - \alpha| < \varepsilon/2$. Moreover, if $m, n \geq N$,

$$|a_n - a_m| \leq |a_n - \alpha| + |\alpha - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus this sequence is Cauchy.

6.8. Example. Consider the infinite **continued fraction**

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

To make sense of this, it has to be interpreted as the limit of the finite fractions

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{2 + \frac{1}{2}} = \frac{2}{5}, \qquad a_3 = \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{5}{12}, \qquad \cdots.$$

We need a better way of defining the general term. In this case, there is a recursive formula for obtaining one term from the preceding one:

$$a_1 = \frac{1}{2}, \qquad a_{n+1} = \frac{1}{2 + a_n} \quad \text{for} \quad n \geq 1.$$

In order to establish convergence, we will show that (a_n) is Cauchy. Consider

$$a_{n+1} - a_{n+2} = \frac{1}{2 + a_n} - \frac{1}{2 + a_{n+1}} = \frac{a_{n+1} - a_n}{(2 + a_n)(2 + a_{n+1})}.$$

Now $a_1 > 0$, and it is readily follows that $a_n > 0$ for all $n \geq 2$ by induction. Hence the denominator $(2+a_n)(2+a_{n+1})$ is greater than 4. So we obtain

$$|a_{n+1} - a_{n+2}| < \frac{|a_n - a_{n+1}|}{4} \quad \text{for all} \quad n \geq 1.$$

Since $|a_1 - a_2| = 1/10$, we may iterate this inequality to estimate

$$\begin{aligned} |a_2 - a_3| &< \frac{1}{10 \cdot 4} \\ |a_3 - a_4| &< \frac{1}{10 \cdot 4^2} \\ |a_n - a_{n+1}| &< \frac{1}{10 \cdot 4^{n-1}} = \tfrac{2}{5}(4^{-n}). \end{aligned}$$

The general formula estimating the difference may be verified by induction.

Now it is straightforward to estimate the difference between arbitrary terms a_m and a_n for $m < n$:

$$\begin{aligned} |a_m - a_n| &= \big|(a_m - a_{m+1}) + (a_{m+1} - a_{m+2}) + \cdots + (a_{n-1} - a_n)\big| \\ &\leq |a_m - a_{m+1}| + |a_{m+1} - a_{m+2}| + \cdots + |a_{n-1} - a_n| \\ &< \tfrac{2}{5}(4^{-m} + 4^{-m-1} + \cdots + 4^{1-n}) < \frac{2 \cdot 4^{-m}}{5(1-\frac{1}{4})} = \frac{8}{15}4^{-m} < 4^{-m}. \end{aligned}$$

This tells us that our sequence is Cauchy. Indeed, if $\varepsilon > 0$, choose N such that $4^{-N} < \varepsilon$. Then

$$|a_m - a_n| < 4^{-m} \leq 4^{-N} < \varepsilon \quad \text{for all} \quad m, n \geq N.$$

Therefore by the Completeness Theorem 6.5, it follows that $(a_n)_{n=1}^\infty$ converges. Let us write $\lim\limits_{n\to\infty} a_n = L$. To calculate L, use the recurrence relation

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2+a_n} = \frac{1}{2+L}.$$

It follows that $L^2 + 2L - 1 = 0$. Solving yields $L = \pm\sqrt{2} - 1$. Since $L > 0$, we see that $L = \sqrt{2} - 1$.

We have accumulated four different results for $\mathbb{R}$ that distinguish it from $\mathbb{Q}$.

(1) the Least Upper Bound Principle (3.3),
(2) the Monotone Convergence Theorem (4.2),
(3) the Bolzano–Weierstrass Theorem (5.2),
(4) the Completeness Theorem (6.5).

It turns out that they are all equivalent. Indeed, each of the proofs of items (2) to (4) relies only on the previous item in our list. To show how the Completeness Theorem implies the Least Upper Bound Principle, go through our proof to obtain an increasing sequence of lower bounds, y_k, and a decreasing sequence of elements $x_k \in S$ with $x_k < y_k + 10^{-k}$. Show that the sequence $x_1, y_1, x_2, y_2, \ldots$ is Cauchy. The limit L will be the greatest lower bound. Fill in the details yourself (Exercise 6.G).

Exercises for Section 6

A. Let (x_n) be Cauchy with a subsequence (x_{n_k}) such that $\lim\limits_{k\to\infty} x_{n_k} = a$. Show that $\lim\limits_{n\to\infty} x_n = a$.

B. Give a sequence (a_n) such that $\lim\limits_{n\to\infty} |a_n - a_{n+1}| = 0$, but the sequence does not converge.

C. Let (a_n) be a sequence such that $\lim\limits_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| < \infty$. Show that (a_n) is Cauchy.

D. If $(x_n)_{n=1}^{\infty}$ is Cauchy, show that it has a subsequence (x_{n_k}) such that $\sum_{k=1}^{\infty} |x_{n_k} - x_{n_{k+1}}| < \infty$.

E. Suppose that (a_n) is a sequence such that $a_{2n} \le a_{2n+2} \le a_{2n+3} \le a_{2n+1}$ for all $n \ge 0$. Show that this sequence is Cauchy if and only if $\lim\limits_{n\to\infty} |a_n - a_{n+1}| = 0$.

F. Give an example of a sequence (a_n) such that $a_{2n} \le a_{2n+2} \le a_{2n+3} \le a_{2n+1}$ for all $n \ge 0$ which does not converge.

G. Fill in the details of how the Completeness Theorem implies the Least Upper Bound Principle.

H. Let $a_0 = 0$ and set $a_{n+1} = \cos(a_n)$ for $n \ge 0$. Try this on your calculator (use radian mode!).
 (a) Show that $a_{2n} \le a_{2n+2} \le a_{2n+3} \le a_{2n+1}$ for all $n \ge 0$.
 (b) Use the Mean Value Theorem to find an explicit number $r < 1$ such that $|a_{n+2} - a_{n+1}| \le r|a_n - a_{n+1}|$ for all $n \ge 0$. Hence show that this sequence is Cauchy.
 (c) Describe the limit geometrically as the intersection point of two curves.

I. Evaluate the continued fraction:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}\ .$$

J. Let $x_0 = 0$ and $x_{n+1} = \sqrt{5 - 2x_n}$ for $n \ge 0$. Show that this sequence converges and compute the limit. Hint: Show that the even terms increase and the odd terms decrease.

K. Consider an infinite binary expansion $(0.e_1e_2e_3\dots)_{\text{base }2}$, where each $e_i \in \{0,1\}$. Show that $a_n = \sum_{i=1}^{n} 2^{-i} e_i$ is Cauchy for every choice of zeros and ones.

L. One base-independent construction of the real numbers uses Cauchy sequences of rational numbers. This exercise asks for the definitions that go into such a proof.
 (a) Find a way to decide when two Cauchy sequences should determine the same real number without using their limits. Hint: Combine the two sequences into one.
 (b) Your definition in (a) should be an equivalence relation. Is it? (See the Appendix in the first handout.)
 (c) How are addition and multiplication defined?
 (d) How is the order defined?