Potential energy

In one dimension, we treated the case of a force $F(x)$ which depended only on the position x. It proved very useful to write the force as the negative derivative of a function called the potential energy:

$$F(x) = -\frac{dV(x)}{dx} \ .$$

This equation says that the force is in the direction in which V decreases, and is larger, the faster V decreases.

In one dimension, there always exists an appropriate potential energy function:

$$V(x) = -\int_{x_0}^{x} F(x')\, dx' \ ,$$

where the integral starts at some arbitrary initial point x_0.

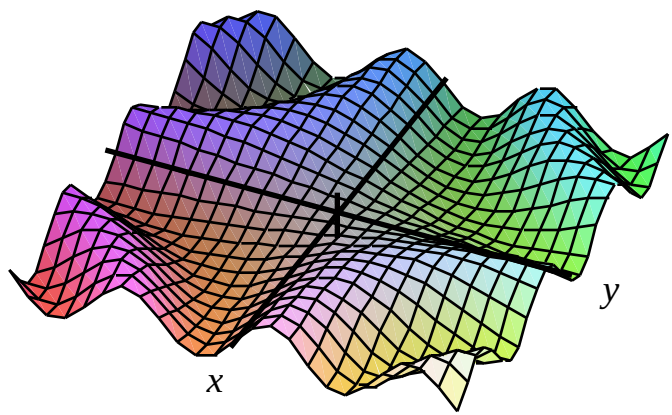
In more than one dimension, however, the situation is quite different. It turns out not always to be possible to find a suitable potential energy function. We will see why this is so.

Before beginning, however, we need to know how the force and the potential energy should be related in more than one dimension. The answer is

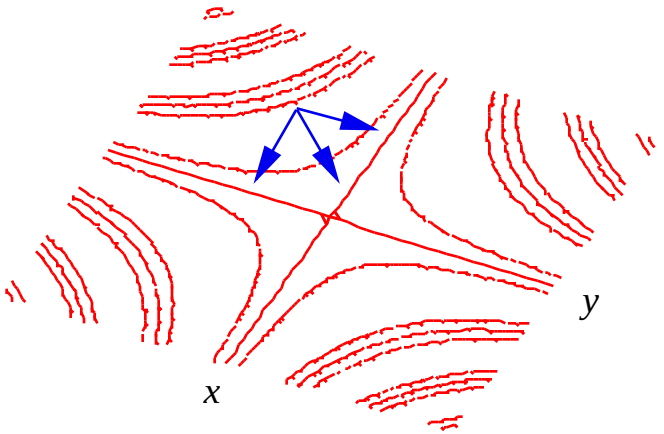
$$\vec{F}(\vec{r}) = -\left(\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}\right) \equiv -\vec{\nabla} V(\vec{r}) \ .$$

Let's motivate and explain this, bit by bit. The first thing to notice is that while the force is a vector, the potential energy is a *scalar* function. This means that its value at any point is a number (with the units of energy), and not a vector. (Scalar and vector fields were introduced in an earlier section.)

How can you picture such a function? If there are only two dimensions, you can plot the value of the potential energy on the vertical axis. For example:



Or, you can make a contour plot. Here is a contour plot of the same function:



If there are three dimensions, it's a little harder to picture a scalar function. You can imagine some smoke in a room - the density of the smoke is a scalar function. In some places the density is high and in other places not so high.

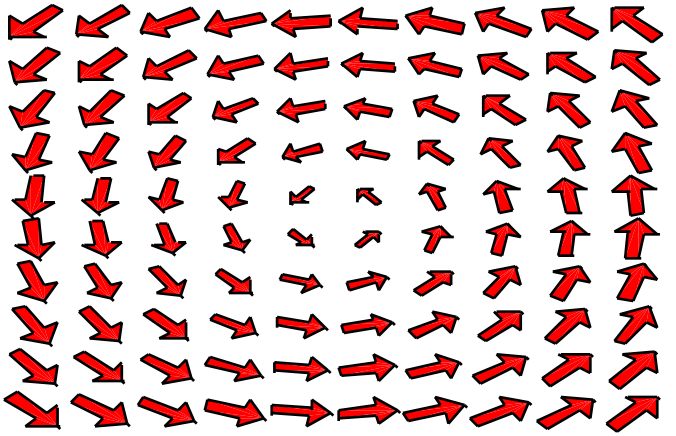
What about the direction of the force? How do we get something with a direction from a scalar field? We do this by applying the negative of the *gradient* operator onto V, as shown on the right-hand side of the last equation. This obviously gives something that is a vector field. What are its properties?

In the one-dimensional case, the force at any point is in the direction in which the potential energy decreases away from that point. Now it's a little more complicated, because the potential energy can decrease in several different directions away from a given point. The three blue arrows in the previous figure all point in directions in which the potential energy decreases.

So, we should make a rule that says that the force is in the direction in which the potential energy decreases the *fastest* . It turns out that this is exactly what the negative of the gradient operator does! (You might learn this later in a mathematics course. Just accept it, for now.)

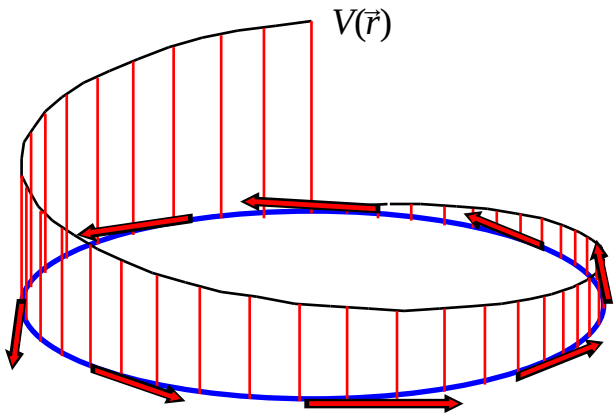
Now we can return to explaining why not every force in more than one dimension can be described by the negative gradient of a potential energy function. Here is the reason:

in more than one dimension, it is possible to have a force which *circles around* like this:



Suppose we try to find a potential energy

function V such that this particular force always points in the direction of fastest decrease in V. Then we would have a problem, because V would have to decrease all the way around a closed path like this:



There's no way for V to do this, and still remain single-valued. At each point, V would have to take on many values, so it would not be an ordinary function. A potential energy function that takes on more than one value is not of much use to us.

(One is reminded of a painting by M.C. Escher in which he manages to distort reality enough so that a staircase appears *always* to be descending!)

Under what conditions *will* it be possible to find a potential energy function? Obviously, the force can't curl around anywhere. It turns out that there is a mathematical theorem that says that this is the *only* requirement on the force.

Theorem

If the *curl* of the force is zero everywhere, then there exists a potential energy function.

Believe it or not, the *curl* of a force is a well-defined mathematical quantity. We'll start by giving the full-blown definition in three dimensions, and then show how it makes sense in a special two-dimensional case. Don't worry if you don't understand the next bit of mathematics - you already know the physical ideas.

Suppose that the force, written in components, is

$$\vec{F}(\vec{r}) = \hat{x}F_x + \hat{y}F_y + \hat{z}F_z \ .$$

Then the definition of the curl, in components, is

$$\text{curl}\,\vec{F} =$$

$$\hat{x}\left(\partial_y F_z - \partial_z F_y\right) + \hat{y}\left(\partial_z F_x - \partial_x F_z\right) + \hat{z}\left(\partial_x F_y - \partial_y F_x\right)$$

where we have used a shorthand for the partial derivative

$$\partial_x = \frac{\partial}{\partial x} \quad \text{etc.}$$

The curl is easiest to remember if you notice the *cyclic* nature of the first terms of each pair: $(x,y,z)\rightarrow(y,z,x)\rightarrow(z,x,y)$.

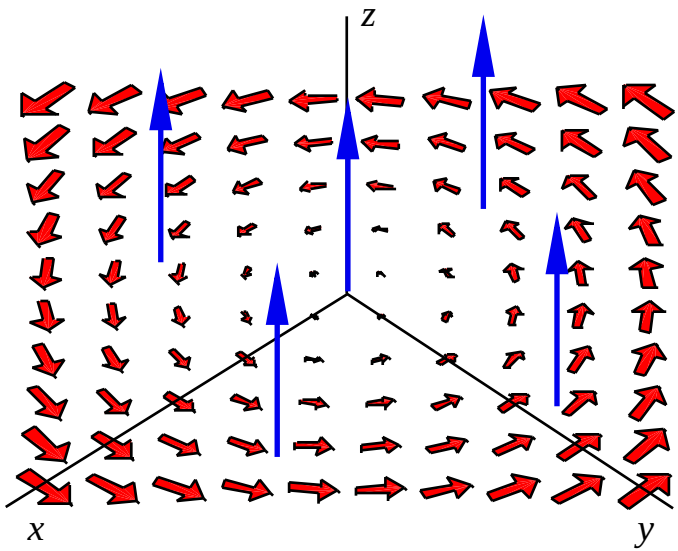
The definition of the curl looks pretty complicated, but if you are given the force, you can easily work out what the curl is. Here is an example which is worth remembering:

$$\vec{F} = -\hat{x}y + \hat{y}x \ .$$

This is the prototype vector field with nonzero curl:

$$\text{curl}\,\vec{F} = \hat{z}\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right) = 2\hat{z} \ .$$

Here is a plot of this field in red, with its curl shown in blue:



Notice that the curl points *perpendicular* to the plane in which the force curls around. Its direction is given by the **right-hand rule**:

You wrap the fingers of your right hand around in the same sense as the vector field, and stick out your thumb. Then the curl of the vector field is in the direction of your thumb.

The big equation defining the curl in three dimensions shown above contains some magic. It takes any vector field and gives its curl in accordance with the right-hand rule, no matter what direction the curl points in or how strongly it curls.

Example: the gravitational potential

What about the gravitational force? Can it be described by a potential energy function?

Let's put a body of mass M at the origin of our coordinate system. Then the force on a body of mass m located at position

$$\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z$$

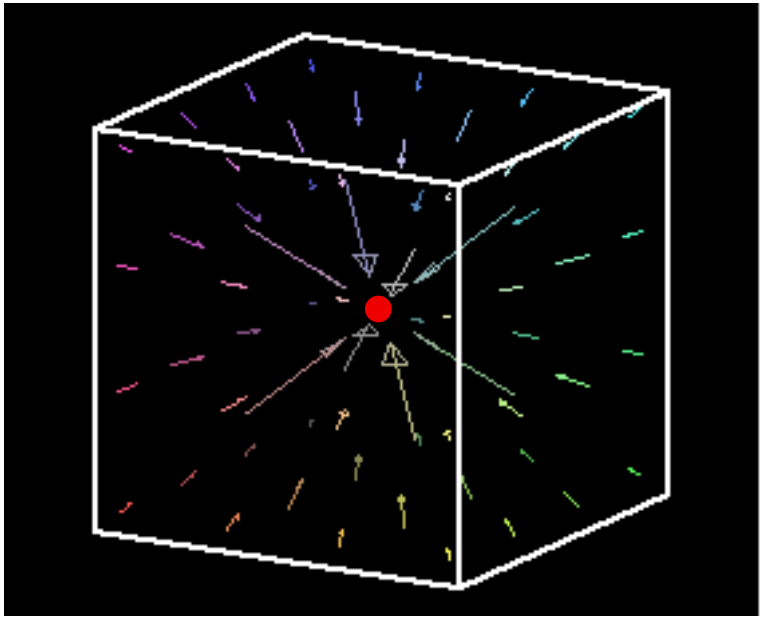
is

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\hat{r} \ , \quad \text{where} \quad \hat{r} \equiv \frac{\vec{r}}{|\vec{r}|} \ .$$

This force points inwards towards M, and decreases with the square of the distance between the bodies. It is spherically symmetric about M, and there's no apparent "curly" nature about this

force. Hence, we might expect it to be described by a potential energy function.

Here is a picture of the gravitational force, with the body of mass M shown in red:



To check that the curl is zero, we must re-write the force in terms of the coordinates x,y,z . We find

$$\vec{F} = \frac{-GMm}{(x^2+y^2+z^2)^{3/2}}\left(\hat{x}x + \hat{y}y + \hat{z}z\right) \ .$$

Let's just work out the z-component of the curl (the others are similar):

$$\left(\text{curl } \vec{F}\right)_z \propto \frac{\partial}{\partial x}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial y}\left(\frac{-x}{(\ldots)^{3/2}}\right)$$

$$= \frac{3yx}{(\ldots)^{5/2}} - \frac{3xy}{(\ldots)^{5/2}} = 0 \ .$$

Hence, there will be a potential energy function. It is called

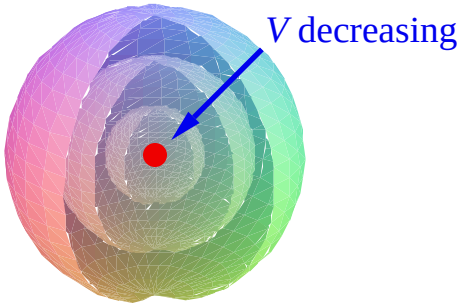
the gravitational potential energy

$$\boxed{V(\vec{r}) = -\frac{GMm}{r}} \ .$$

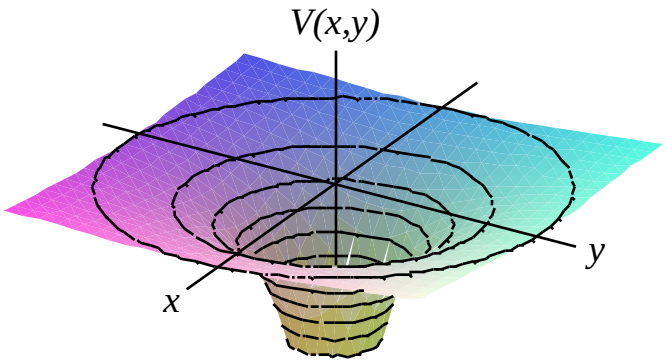
(Note that a constant can always be added. We have set it to zero by choosing $V \to 0$ as $r \to \infty$.)

We will see how to actually derive this expression later. For now, you should just check that the negative gradient of this function does indeed give the gravitational force.

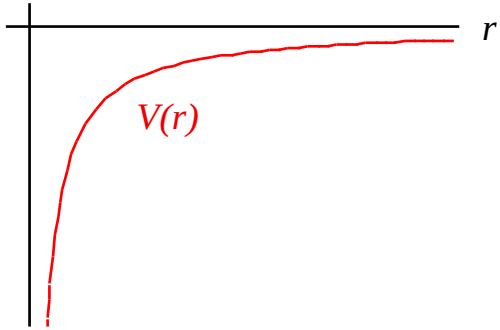
V is a function which decreases as the distance between the bodies decreases, and its value depends *only* on the distance. Surfaces of constant V are concentric spheres:



Another way to view the potential V is to plot its values on a slice through the origin in the x-y plane, for example. The result shows why one often talks about a "potential well":



Another representation is simply a plot of V versus r :



We will consider the problem of motion in this potential in our section on gravity.

Path-independence

What about the expression we had in one dimension giving the potential energy explicitly in terms of the force

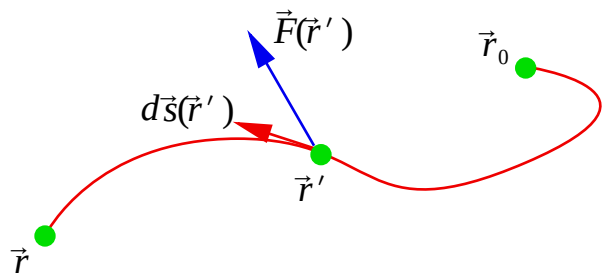
$$V(x) = -\int_{x_0}^{x} F(x')\,dx' \quad ?$$

How does it get generalized in more than one dimension?

In words, the above equation says "add up all the little pieces of (force times distance) in the direction of motion of a body which starts at some arbitrary initial position and ends up at x . Take the negative of the result."

We should try to copy this procedure in the

present situation. But we'll have to do some modifications to take the directions into account. Let's take a path $\vec{s}(\vec{r}\,')$ between some arbitrary initial position $\vec{r}_0$ and $\vec{r}$. The first complication that occurs is that the path need not be a straight line:



The second complication is that the force at some intermediate point $\vec{r}\,'$ need not be in the same direction as the path element $d\vec{s}(\vec{r}\,')$ at that point.

It is pretty obvious that we should take the *component* of the force in the direction of the path element, and add *that* up along the path:

$$\boxed{V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{s}(\vec{r}\,')} \quad .$$

This kind of integral is called a *line integral* or a *path integral*. You will study them in detail in a later mathematics course.

You can see that there's a problem, however. The potential energy V is supposed to depend only on position, not on how the position was reached. But what happens if the integral depended on the path? Then, the potential would not be well-defined at all.

There's a mathematical theorem that tells us under what conditions the line integral will be independent of the path:

Theorem:

If the curl of a vector field is zero, then the line integral of the vector field is independent of the path.

We began this section by talking about the conditions under which a potential energy exists. The result was that it exists only if the curl of the force is zero. Here we see the same result, stated using slightly different language.

So now suppose a body is acted upon by a force which depends on the position $\vec{r}$ in a known way:

$$\vec{F} = \vec{F}(\vec{r}) \ ,$$

and whose curl is zero. The differential equation which determines the path $\vec{r}(t)$ is

$$m\frac{d^2\vec{r}(t)}{dt^2} = \vec{F}(\vec{r}(t)) \ .$$

Consider the quantity $V(\vec{r}(t))$, which is well

defined because the curl of the force is zero. How does it change as time goes on?

Differentiating, we find

$$\begin{aligned}\frac{d}{dt}V(\vec{r}) &= -\frac{d\vec{r}}{dt}\cdot\vec{\nabla}V(\vec{r}) \\ &= -\vec{v}\cdot\vec{F} \\ &= -\vec{v}\cdot m\vec{a} \\ &= -\frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) .\end{aligned}$$

(The dependence on t has been suppressed in the above calculation.) So, we have shown that the quantity

$$\boxed{E = \frac{1}{2}m\vec{v}^2 + V(\vec{r})}$$

is constant in time. We recognize the first term as the

kinetic energy

$$T = \frac{1}{2}m\vec{v}^2 ,$$

and the second is the **potential energy**. Their sum E is called the **total energy**.

A force which does not depend on velocity or time and which is represented by a potential is called **conservative**, and the total energy of a body of constant mass is *conserved* in an inertial frame. This means that the energy does not change in time.

Example: the gravitational potential energy

Let's see how to calculate the gravitational potential energy at $\vec{r}$, given the expression for the force and the fact that its curl is zero. We can take any initial point we want, so for simplicity we will take a point at infinity in the direction of $\vec{r}$. We can also take any path we want, so for simplicity we will take a straight line.

The line element is

$$d\vec{s}(\vec{r}\,') = \hat{r}\,'\,dr\,' .$$

(The sign is tricky: for the path we have chosen, dr' is negative, and the line element points opposite to $\hat{r}\,'$.)

We can now calculate the potential energy:

$$
\begin{aligned}
V(\vec{r}) &= -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{s}(\vec{r}\,') \\
&= -\int_{\infty}^{r} \left(-\frac{GMm}{r'^{\,2}} \hat{r}\,' \right) \cdot \left(\hat{r}\,' \, dr' \right) \\
&= GMm \int_{\infty}^{r} \frac{dr'}{r'^{\,2}} \\
&= GMm \left(-\frac{1}{r'} \right)_{\infty}^{r} \\
&= -\frac{GMm}{r} \quad .
\end{aligned}
$$

We thus arrive at the expression for the gravitational potential energy stated before.