Chapter1 Notes

Axioms of the real numbers: The Field Axioms, the Order Axiom, and the Axiom of completeness.

Axioms of a Field: A field F is a nonempty set together with two operations + and * called addition and multiplication, which satisfy the following axioms.

1. The operations + and * are binary operations: that is, if $a, b \in F$, then $a + b$ and $a * b$ are uniquely determined elements of F.
2. The operations + and * are associative. That is for $a, b, c \in F$ $(a + b) + c = a + (b + c)$& $(a * b) * c = a * (b * c)$
3. The operations + and * are associative. That is for $a, b \in F$ $a + b = b + a$ & $a * b = b * a$
4. The distributive property holds. That is for $a, b, c \in F$ $a * (b + c) = (a * b) + (a * c)$
5. There exist additive and multiplicative identities. That is there are elements 0 and 1 in F for which $0 + a = a$ & $1 * a = a$ for $\forall a \in F$
6. There exists an additive inverse for $\forall a \in F$. That is, if $a \in F$, there is an element in F denoted $-a$ for which $a + (-a) = 0$
7. For $\forall a \in F$ for which $a \neq 0$, there is a multiplicative inverse. That is if $a \in F$ & $a \neq 0$, there is an element in F denoted a^{-1} or $\frac{1}{a}$ for which $a * a^{-1} = 1$

The Order Axiom: Let F be a Field. Then F is an order field if it satisfies the additional axiom:

8. There is a nonempty subset P of F (called the positive subset) for which
   i. If $a, b \in P$, then $a + b \in P$ (closure under addition)
   ii. If $a, b \in P$, then $ab \in P$ (closure under multiplication)
   iii. For $\forall a \in F$ exactly one of the following holds: $a \in P, -a \in P, or\ a = 0$ (law of trichotomy)

The Axiom of Completeness:

9. The real numbers are complete. Which is equivalent to saying: Let S be an ordered field. Then S is complete if for any nonempty subset A of S that is bounded below, the greatest lower bound of A is in S.

Note:

Greatest Lower Bound: Let S be a nonempty set of real numbers that has a lower bound. A number c is the called the greatest lower bound (or the <u>infimum</u>) for S <u>iff</u> it satisfies the following properties:

1. $c \leq x$ for all $x \in S$.

2. For all real numbers k, if k is a lower bound for S, then $k \leq c$. Least Upper Bound: Let S be a nonempty set of real numbers that has an upper bound. Then a number c is called the least upper bound (or the <u>supremum</u>) for S <u>iff</u> it satisfies the following properties:

1. $c \geq x$ for all $x \in S$.

2. For all real numbers k, if k is an upper bound for S, then $k \geq c$.

The Archimedean Principle: If $a \,\&\, b \in \mathbb{R}$: $a > 0$, then $\exists n \in \mathbb{Z}^+ : na > b$

Cardinality: We can classify sets by the number of elements they contain or their cardinality. Intuitively, sets have the same cardinality if they have the same number of elements.

Def: Two sets A and B are said to have the same cardinality if there is a 1-1 onto function from the set A to the set B. In case we say that A is equivalent to B.

Def: A set S is said to be finite with cardinality n if S is equivalent to {1,2,..,n} where n is some positive integer. The empty set is also finite with cardinality 0. Sets that do not have finite cardinality are infinite sets. A set S is called countable if it has the same cardinality as some subset of the positive integers. Sets that are not countable are said to be uncountable. Saying that S is countable is equivalent to saying there is a 1-1 (not necessarily onto) function $f: S \to \mathbb{N}$. In particular, finite sets are countable.

Cantor's Theorem: For any set X, let $\wp(X)$ denote the set of subsets of X. Then the cardinal number of $\wp(X)$ is strictly larger than the cardinal number of X.