PART IV

APPENDICES

Analysis in Vector Spaces.
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APPENDIX A

CONSTRUCTION OF THE REAL NUMBERS

Introduction. Let $\mathbb{Q}$ be the set of rational numbers. The most important properties of $\mathbb{Q}$ are the properties of the arithmetic operations and the properties of the order relation on $\mathbb{Q}$. These properties are summarized by two sets of rules called the *field axioms* and the *order axioms*. The principal deficiency of $\mathbb{Q}$ is that it is not complete. The set of rational numbers x such that $x^2 < 2$ is bounded above, yet there is no rational least upper bound. We would like to extend $\mathbb{Q}$ to $\mathbb{R}$, a set of numbers that not only satisfies the field axioms and the order axioms but also has the completeness property. This section reviews a standard way to construct the real numbers that meets this objective.

Outline of the construction of $\mathbb{R}$. Prior to attempting any construction such as the present one, we freely make use of the real numbers without proving their existence.

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We know many of the properties that they must have. In particular, we know that any real number is the limit of a sequence of rational numbers. That fact suggests that we try to define real numbers as the limits of convergent sequences of rational numbers. But Definition 2.3.2 of convergent sequences depends explicitly upon the limit point. We have defined convergence to L, not convergence *per se*. To circumvent this difficulty, we base our construction on Cauchy sequences of rational numbers rather than convergent sequences. Cauchy sequences are defined purely in terms of rational numbers, and they coincide with the class of sequences that we shall ultimately regard as convergent to a limit in $\mathbb{R}$.

Let $\mathcal{C}$ be the class of all Cauchy sequences of rational numbers. $\mathcal{C}$ cannot be identified with the set of real numbers because many different Cauchy sequences may converge to the same real number. Hence we introduce an equivalence relation on $\mathcal{C}$, making two Cauchy sequences equivalent if the difference sequence converges to zero. Note that convergence to zero can be defined in terms of rational numbers only. Hence real numbers will be defined as equivalence classes of Cauchy sequences of rational numbers. We will see that the usual arithmetic operations and the order relation can be defined easily for these equivalence classes, and that the resulting system is complete.

In what follows, we only assume basic knowledge of the rational numbers. We restate some of the earlier definitions using only rational numbers, making sure that all of our arguments are formulated in terms of rational numbers only.

A.1 FIELD AND ORDER AXIOMS IN $\mathbb{Q}$

Definition A.1.1 Field axioms on $\mathbb{Q}$. There are two binary operations $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ called *addition* and *multiplication*. Addition applied to $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ gives $(a+b)$. Multiplication applied to $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ gives $(a \cdot b)$. These operations have the following properties.

Commutativity

$$a + b = b + a \text{ and } a \cdot b = b \cdot a \text{ for all } a, b \in \mathbb{Q}.$$

Associativity

$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ for all } a, b, c \in \mathbb{Q}.$$

Distributivity

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \text{ for all } a, b, c \in \mathbb{Q}.$$

Existence of the neutral elements There are two elements in $\mathbb{Q}$, 0 and 1, such that $0 \neq 1$ and such that

$$0 + a = a \text{ and } 1 \cdot a = a \text{ for all } a \in \mathbb{Q}.$$

Existence of inverses for addition (subtraction)

For each $a \in \mathbb{Q}$, there is a $(-a) \in \mathbb{Q}$ such that $a + (-a) = 0$.

Existence of the inverse for multiplication (division)

If $a \in \mathbb{Q}$ and if $a \neq 0$, then there is a $(1/a) \in \mathbb{Q}$ such that $a \cdot (1/a) = 1$.

Definition A.1.2 Order axioms on $\mathbb{Q}$. There is a set $P_{\mathbb{Q}} \subset \mathbb{Q}$, called the *positive (rational) numbers*, with the following two properties.

1. For each $p \in \mathbb{Q}$, exactly one of the following three cases is true: $p = 0$, $p \in P_{\mathbb{Q}}$, or $-p \in P_{\mathbb{Q}}$.
2. If $p, q \in P_{\mathbb{Q}}$, then $p + q \in P_{\mathbb{Q}}$ and $pq \in P_{\mathbb{Q}}$.

Remarks A.1.3 Our objective. We would like to construct a set $\mathbb{R}$ with the following properties.

1. There are two binary operations $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on $\mathbb{R}$, *addition* and *multiplication*, that satisfy the field axioms on $\mathbb{R}$.
2. There is a set $P_{\mathbb{R}} \subset \mathbb{R}$, the set of *positive (real) numbers*, that satisfies the order axioms on $\mathbb{R}$.
3. There is a one-to-one mapping $\varphi : \mathbb{Q} \to \mathbb{R}$ such that $\varphi(p + q) = \varphi(p) + \varphi(q)$ and $\varphi(pq) = \varphi(p)\varphi(q)$ for all $p, q \in \mathbb{Q}$ and such that $\varphi(P_{\mathbb{Q}}) \subset P_{\mathbb{R}}$. Here $\varphi(p)+\varphi(q)$ and $\varphi(p)\varphi(q)$ are stated in terms of the addition and multiplication operations on $\mathbb{R}$.
4. Every nonempty subset of $\mathbb{R}$ with an upper bound has a least upper bound. Hence, $\mathbb{R}$ satisfies the completeness axiom, Axiom 2.2.3.

A.2 EQUIVALENCE CLASSES OF CAUCHY SEQUENCES IN $\mathbb{Q}$

Definition A.2.1 Cauchy sequences of rational numbers. A sequence $x : \mathbb{N} \to \mathbb{Q}$ is called a *Cauchy sequence* of rational numbers if for each rational number $a > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < a$ for all $m, n \geq N$. Let $\mathcal{C}$ be the set of all Cauchy sequences of rational numbers.

Definition A.2.2 Zero sequences of rational numbers. A sequence $x : \mathbb{N} \to \mathbb{Q}$ is called a *zero sequence* of rational numbers if for each rational number $a > 0$, there is an $N \in \mathbb{N}$ such that $|x_n| < a$ for all $n \geq N$. Let $\mathcal{Z}$ be the set of all zero sequences of rational numbers.

Lemma A.2.3 *Every zero sequence of rational numbers is also a Cauchy sequenceof rational numbers. Hence* $\mathcal{Z} \subset \mathcal{C}$.

Proof. This is left as an exercise. □

Definition A.2.4 Constant sequences of rational numbers. Each rational number $q \in \mathbb{Q}$ defines a constant sequence $\bar{q} : \mathbb{N} \to \mathbb{Q}$: we set $\bar{q}_n = q$ for all $n \in \mathbb{N}$. Let $\mathcal{Q}$ be the set of all constant sequences of rational numbers. It is clear that $\bar{q} \in \mathcal{C}$ for each $q \in \mathbb{Q}$. Hence $\mathcal{Q} \subset \mathcal{C}$. Note that $\bar{q} \in \mathcal{Z}$ if and only if $q = 0$.

Definition A.2.5 Sum and product of two sequences. If $x : \mathbb{N} \to \mathbb{Q}$ and $y : \mathbb{N} \to \mathbb{Q}$ are two sequences, then their sum is defined as the sequence $(x + y)_n = x_n + y_n$ and their product as the sequence $(xy)_n = x_n y_n$.

Lemma A.2.6 *Let* $x : \mathbb{N} \to \mathbb{Q}$ *and* $y : \mathbb{N} \to \mathbb{Q}$ *be two sequences.*

(1) If $x, y \in \mathcal{C}$, *then* $(x + y) \in \mathcal{C}$ *and* $(xy) \in \mathcal{C}$.

(2) If $x, y \in \mathcal{Z}$, *then* $(x + y) \in \mathcal{Z}$ *and* $(xy) \in \mathcal{Z}$.

(3) If $x \in \mathcal{Z}$ *and* $y \in \mathcal{C}$, *then* $(xy) \in \mathcal{Z}$.

Proof. This is left as an exercise. □

Definition A.2.7 Addition and multiplication on $\mathcal{C}$. *Addition* on $\mathcal{C}$ is the operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that takes the pair $(x, y) \in \mathcal{C} \times \mathcal{C}$ to $(x+y) \in \mathcal{C}$. Similarly, *multiplication* on $\mathcal{C}$ is the operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that takes the pair $(x, y) \in \mathcal{C} \times \mathcal{C}$ to $(xy) \in \mathcal{C}$. Note that if $p, q \in \mathbb{Q}$, then $\overline{(p+q)} = \bar{p} + \bar{q}$ and $\overline{(pq)} = \bar{p}\,\bar{q}$, with the notations of Definition A.2.4.

Definition A.2.8 An equivalence relation on $\mathcal{C}$. Define a relation on $\mathcal{C} \times \mathcal{C}$ as follows. If $x, y \in \mathcal{C}$, then x is related to y if and only if $(x - y) \in \mathcal{Z}$. It is easy to check that this is an equivalence relation on $\mathcal{C}$ in the sense of Definition 1.1.9. We write $x \sim y$ for $(x - y) \in \mathcal{Z}$. Note that if $p, q \in \mathbb{Q}$, then $\bar{p} \sim \bar{q}$ if and only if $p = q$, with the notations of Definition A.2.4.

Definition A.2.9 Real numbers. Equivalence classes were defined in Definition 1.1.11. The equivalence class represented by $x \in \mathcal{C}$ is the set

$$E_x = \{ x' \in \mathcal{C} \mid x' \sim x \} \subset \mathcal{C}.$$

Any two equivalence classes are either identical or disjoint. In fact, if $x \sim y$, then $E_x = E_y$, and if $x \not\sim y$, then $E_x \cap E_y = \emptyset$. The union of all equivalence classes is $\mathcal{C}$. Each equivalence class is also called a *real number*. The equivalence class E_x is the real number represented by the sequence x. The same real number can be represented by any sequence in E_x, i.e., by any sequence $x' \sim x$. Let $\mathbb{R}$ denote the collection of all equivalence classes. This collection is called the set of *real numbers*.

Definition A.2.10 Operations on real numbers. We will define addition and multiplication operations $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Let E_x, $E_y \in \mathbb{R}$. Hence E_x and E_y are two real numbers and also two equivalence classes represented by the sequences x and y. Their sum $E_x + E_y$ and their product $E_x E_y$ are defined as

$$E_x + E_y = E_{x+y} \quad \text{and} \quad E_x E_y = E_{xy}.$$

This definition has to be justified. If $E_x = E_{x'}$ and $E_y = E_{y'}$, must we have $E_{x+y} = E_{x'+y'}$ and $E_{xy} = E_{x'y'}$? If not, then the definitions above are meaningless. The required justification is provided by Lemma A.2.11 below.

These definitions are reasonable. They depend on the fact that the limit of the sum of two sequences is the sum of their limits. The same is also true for products. Theorem A.2.15 below shows that the field axioms are satisfied on $\mathbb{R}$ with these operations.

Lemma A.2.11 *Let x, x', y, $y' \in \mathcal{C}$. If $x \sim x'$ and $y \sim y'$, then $(x+y) \sim (x'+y')$ and $(xy) \sim (x'y')$.*

Proof. Let $x' = x + p$ and $y' = y + q$ with p, $q \in \mathcal{Z}$. Then

$$(x' + y') - (x + y) = p + q \in \mathcal{Z}$$

by the second part of Lemma A.2.6. Hence $(x + y) \sim (x' + y')$. Also,

$$(x'y') - (xy) = (py) + (xq) + (pq) \in \mathcal{Z},$$

by the second and third parts of Lemma A.2.6. Hence $(xy) \sim (x'y')$. $\square$

Definition A.2.12 Positive real numbers. Call a sequence $x : \mathbb{N} \to \mathbb{Q}$ an *eventually positive sequence* if there is an $n \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq N$. Call a real number $E_x \in \mathbb{R}$ a *positive real number* if each $y \in E_x$ is an eventually positive

sequence. Let $P_{\mathbb{R}} \subset \mathbb{R}$ be the set of all positive real numbers. These definitions are reasonable as well. In fact, the limit of a sequence is positive if and only if all the sequences converging to the same limit are eventually positive. Theorem A.2.16 below shows that the order axioms are satisfied on $\mathbb{R}$ with this definition of positivity.

Lemma A.2.13 *Let $x \in \mathcal{C}$. Assume that for each rational number $a > 0$ and for each $N \in \mathbb{N}$, there is an $n \geq N$ such that $|x_n| \leq a$. Then $x \in \mathcal{Z}$.*

Proof. Given a rational $a > 0$, find an $N \in \mathbb{N}$ such that $|x_n - x_m| \leq a/2$ for all $m, n \geq N$. This can be done since $x \in \mathcal{C}$. By assumption, we can find an $m \in \mathbb{N}$ such that $m \geq N$ and such that $|x_m| \leq a/2$. Let $n \geq N$. Then

$$|x_n| = |x_m + (x_n - x_m)| \leq |x_m| + |x_n - x_m| \leq (a/2) + (a/2) = a.$$

Hence $x \in \mathcal{Z}$. □

Lemma A.2.14 *Let $x \in \mathcal{C}$ and $x \notin \mathcal{Z}$. Then there is an $a > 0$, $a \in \mathbb{Q}$, such that the following are true.*

(1) There is an $M \in \mathbb{N}$ such that $|x_m| > a$ for all $m \geq M$.

(2) There is an $N \in \mathbb{N}$ such that either $x_n > a$ for all $n \geq N$ or $-x_n > a$ for all $n \geq N$.

(3) Let $y \sim x$. Then there is a $K \in \mathbb{N}$ such that x_k and y_k are both nonzero and have the same sign for all $k \geq K$.

Proof. Since $x \notin \mathcal{Z}$, the hypothesis of Lemma A.2.13 cannot be true. Hence there is a rational number $a > 0$ and an $M \in \mathbb{N}$ such that $|x_m| > a$ for all $m \geq M$. This proves the first part.

Now find an $N \geq M$, $N \in \mathbb{N}$, such that $|x_m - x_n| \leq a$ for all $m, n \geq N$. This can be done since x is a Cauchy sequence. Let $m, n \geq N$. If x_m and x_n have opposite signs, then $|x_n - x_m| > 2a$, since $|x_m| > a$ and $|x_n| > a$. This violates the condition that $|x_m - x_n| \leq a$ for all $m, n \geq N$. Hence either $x_n > a$ for all $n \geq N$ or $x_n < -a$ for all $n \geq N$. This proves the second part.

Let $y \sim x$. Then $(x - y) \in \mathcal{Z}$. Find a $K \geq N$, $K \in \mathbb{N}$, such that $|y_k - x_k| < a/2$ for all $k \geq K$. Then $-a/2 \leq y_k - x_k \leq a/2$ shows that

$$x_k - a/2 \leq y_k \leq x_k + a/2$$

for all $k \geq K$. Hence we see that if $a < x_k$, then $a/2 < y_k$, and if $x_k < -a$, then $y_k < -a/2$ for all $k \geq K$. This proves the last part. □

Theorem A.2.15 *Let $\mathbb{R}$ be the collection of all equivalence classes in $\mathcal{C}$. Let the addition and multiplication operations $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on $\mathbb{R}$ be defined by*

$$E_x + E_y = E_{x+y} \quad \textit{and} \quad E_x E_y = E_{xy},$$

as in Definition A.2.10. Then these operations satisfy the field axioms, Axiom A.1.1, on $\mathbb{R}$.

Proof. Let $\bar{0}$ and $\bar{1}$ be the constant sequences consisting of all 0s and all 1s. Note that $E_{\bar{0}} = \mathcal{Z}$. We see that $E_{\bar{0}}$ is the neutral element for addition in $\mathbb{R}$ and $E_{\bar{1}}$ is the neutral element for multiplication in $\mathbb{R}$.

First, we verify the division axiom. Let $E_x \neq E_{\bar{0}}$. We will show that there is a $y \in \mathcal{C}$ such that $E_x E_y = E_1$. Now $x \in \mathcal{C}$ and $x \notin \mathcal{Z}$. Then the first part of Lemma A.2.14 shows that there is a rational number $a > 0$ and an $N \in \mathbb{N}$ such that $|x_n| > a$ for all $n \geq N$. Define a sequence $y : \mathbb{N} \to \mathbb{Q}$ by $y_n = 0$ if $n < N$ and $y_n = 1/x_n$ if $n \geq N$. We claim that $y \in \mathcal{C}$ and $E_x E_y = E_1$. If $m, n \geq N$, then

$$\begin{aligned} |y_n - y_m| &= |(1/x_n) - (1/x_m)| = |x_n - x_m|/|x_n x_m| \\ &\leq (1/a^2)|x_n - x_m|. \end{aligned}$$

Given a rational number $b > 0$, find an $M \geq N$, $M \in \mathbb{N}$, such that $|x_n - x_m| \leq a^2 b$ for all $m, n \geq M$. Then we see that $|y_n - y_m| \leq b$ for all $m, n \geq M$. Hence $y \in \mathcal{C}$. We see that $x_n y_n = 1$ for all $n \geq N$. Therefore $(x_n y_n - 1) = 0$ for all $n \geq N$. Then we see that $(xy - 1) \sim 0$ or that $xy \sim 1$. Hence $E_x E_y = E_1$.

The verification of the other axioms is quite routine. We verify only distributivity as an example. Addition and multiplication in $\mathcal{C}$ have the distributivity property. In fact, if $a, b, c \in \mathcal{C}$, then

$$\begin{aligned} (a(b+c))_n &= a_n(b+c)_n = a_n b_n + a_n b_n \\ &= (ab)_n + (ac)_n = ((ab) + (ac))_n. \end{aligned}$$

Now let $E_a, E_b, E_c \in \mathbb{R}$ with $a, b, c \in \mathcal{C}$. Then

$$\begin{aligned} E_a(E_b + E_c) &= E_a E_{b+c} = E_{a(b+c)} \\ &= E_{(ab)+(ac)} = E_{ab} + E_{ac} \\ &= E_a E_b + E_a E_b. \end{aligned}$$

Therefore the distributivity axiom is satisfied. □

Theorem A.2.16 *Let $\mathbb{R}$ be the set of real numbers with the arithmetic operations as defined in Definition A.2.10. Let $P_{\mathbb{R}}$ be the set of all positive real numbers as defined in Definition A.2.12. Then the order axioms, in Definition A.1.2, are satisfied on $\mathbb{R}$.*

Proof. First, note that $E_{\bar{0}} = -E_{\bar{0}}$ is not positive. For example, the sequence $x_n = (-1)^n(1/n)$ is a zero sequence which is not eventually positive. Now let $E_x \neq E_{\bar{0}}$. We will show that either E_x or $-E_x$ is positive.

We have $x \notin \mathcal{Z}$. Hence the second part of Lemma A.2.14 shows that either x or $-x$ is eventually positive. Assume that x is eventually positive. Then the last part of Lemma A.2.14 shows that if $y \sim x$, that is, if $y \in E_x$, then y is also eventually positive. Hence E_x is a positive real number. Similarly, if $-x$ is eventually positive and if $y \sim x$, then $-y$ is also eventually positive. In this case $-E_x = E_{-x}$ is a positive real number. This proves the first part of the order axioms.

For the second part, assume that E_x and E_y are both positive. Then, by the second part of Lemma A.2.14, there are $a, b > 0$, $a, b \in \mathbb{Q}$, and an $N \in \mathbb{N}$ such that $x_n > a$ and $y_n > b$ for all $n \geq N$. Then $x_n + y_n > a + b > 0$ and $x_n y_n > ab > 0$ for all $n \geq N$. Hence $E_x + E_y$ and $E_x E_y$ are both nonzero, and they are both positive numbers. □

Definition A.2.17 Canonical embedding of $\mathbb{Q}$ into $\mathbb{R}$. Let $\varphi : \mathbb{Q} \to \mathbb{R}$ be the function defined as $\varphi(p) = E_{\bar{p}}$. Here $\bar{p} \in \mathcal{C}$ is the constant sequence with all terms equal to $p \in \mathbb{Q}$. We will call $\varphi : \mathbb{Q} \to \mathbb{R}$ the *canonical embedding of* $\mathbb{Q}$ *into* $\mathbb{R}$.

Theorem A.2.18 *The canonical imbedding of $\mathbb{Q}$ into $\mathbb{R}$ is a one-to-one function $\varphi : \mathbb{Q} \to \mathbb{R}$. It preserves addition and multiplication in the sense that*

$$\varphi(p+q) = \varphi(p) + \varphi(q) \text{ and } \varphi(pq) = \varphi(p)\varphi(q) \text{ for all } p, q \in \mathbb{Q}.$$

Also, $\varphi : \mathbb{Q} \to \mathbb{R}$ preserves the order in the sense that $\varphi(P_{\mathbb{Q}}) \subset P_{\mathbb{R}}$.

Proof. If $E_{\bar{p}} = E_{\bar{q}}$, then $\bar{p} \sim \bar{q}$ and $(\bar{p} - \bar{q})$ is a zero sequence. But a constant sequence is a zero sequence only if the constant term is zero. This is obvious, as already mentioned in Definition A.2.4. Hence $E_p = E_q$ in $\mathbb{R}$ only if $p = q$ in $\mathbb{Q}$. This shows that $\varphi : \mathbb{Q} \to \mathbb{R}$ is one-to-one. For the second part,

$$\varphi(p+q) = E_{(p+q)} = E_p + E_q = \varphi(p) + \varphi(q).$$

Here the second equality follows from the definition of addition in $\mathbb{R}$. The proof of $\varphi(pq) = \varphi(p)\varphi(q)$ is similar. For the last part, note that if p is a positive rational number, then $\bar{p}$ is not a zero sequence and eventually (in fact, always) is positive. Hence $\varphi(p) = E_{\bar{p}}$ is a positive real number. □

Remarks A.2.19 Identification of $\mathbb{Q}$ with $\varphi(\mathbb{Q})$. We ignore the differences between $p \in \mathbb{Q}$ and $\bar{p} \in \mathcal{C}$ and $\varphi(p) = E_{\bar{p}} \in \mathbb{R}$. The meaning will be clear from the context. Hence we consider $\mathbb{Q}$ as a subset of $\mathbb{R}$. We also denote the real numbers with single letters, as is customary.

A.3 COMPLETENESS OF $\mathbb{R}$

The addition and multiplication operations and the order relation on $\mathbb{R}$ satisfy the field and order axioms. Hence we can develop all the related concepts, inequalities between real numbers, absolute values of real numbers, sequences, convergent sequences, and Cauchy sequences of real numbers as done before. We omit the repetition of these definitions and the related results. Instead, as an example, we give a detailed proof (perhaps more detailed than necessary) of the completeness of $\mathbb{R}$.

Note that any real number r is an equivalence class of Cauchy sequences of rational numbers. Any sequence x in this equivalence class will be called a *representative* (sequence) for r. The choice of representatives is arbitrary and not important. If $r = \varphi(p) = p$ is a rational number, however, then the constant sequence $\bar{p} = p$ is taken as the standard representative of $r = p$. The identifications used here are explained in Remarks A.2.19.

Lemma A.3.1 *Let x be a representative for $r \in \mathbb{R}$. Assume that there is an $N \in \mathbb{N}$ such that $0 \leq x_n$. Then $0 \leq r$.*

Proof. Any real number must satisfy exactly one of the following conditions: $r = 0$ or $r \in P$ or $-r \in P$. Our hypothesis rules out the last condition. In fact, this condition means that all representatives of $-r$ must eventually be positive. But $-r$ has at least one representative $-x$ which is not eventually positive. Hence $-r \notin P$. Therefore either $r = 0$ or $r \in P$. □

Remarks A.3.2 Let x be a representative for $r \in \mathbb{R}$. Assume that there is an $N \in \mathbb{N}$ such that $0 < x_n$. Then $0 \leq r$ by Lemma A.3.1, but we cannot conclude that $0 < r$. In fact, $0 < r$ means that not one but all representatives of r must eventually be positive. A simple counterexample is $x_n = 1/n$, a positive sequence that represents 0.

Lemma A.3.3 *Let $r > 0$, $r \in \mathbb{R}$. Then there is a $p \in \mathbb{Q}$ such that $0 < p < r$.*

Proof. Let x be a representative sequence for r. Hence x is eventually positive, since $r > 0$. Lemma A.2.14 shows that there is an $a > 0$, $a \in \mathbb{Q}$, and an $N \in \mathbb{N}$ such that either $x_n > a$ for all $n \geq N$ or $x_n < -a$ for all $n \geq N$. The second case cannot be true, since x must eventually be positive. Hence $x_n > a$ for all $n \geq N$. Then, by Lemma A.3.1 above, $0 < a \leq r$. Let $p = a/2$. Then $0 < (a/2) < a \leq r$ shows that $0 < p < r$. □

Lemma A.3.4 *Let r, $s \in \mathbb{R}$ and $r < s$. Then there is a $q \in \mathbb{Q}$ such that $r < q < s$.*

Proof. We have $0 < (s - r)$. Use Lemma A.3.3 to find $p \in \mathbb{Q}$ such that $0 < p < (s - r)$. Let x and y be representatives for r and s. Then $0 < (s - r) - p$ shows that the sequence $(y_n - x_n - p)$ is eventually positive. Find $M \in \mathbb{N}$ such that $y_m > x_m + p$ for all $m \geq M$. Since x is a Cauchy sequence, there is an $N \geq M$, $N \in \mathbb{N}$, such that $|x_N - x_n| \leq p/4$ for all $n \geq N$.

Let $q_1 = x_N + (p/4)$ and $q_2 = x_N + (3p/4)$. If $n \geq N$, then

$$q_1 - x_n = (p/4) + (x_N - x_n) \geq (p/4) + |x_N - x_n| \geq 0$$

shows that that $r \leq q_1$, and

$$q_2 = x_N + (3p/4) = x_n + (x_N - x_n) + (3p/4) \leq x_n + p \leq y_n$$

shows that $q_2 \leq s$. If $q = (q_1 + q_2)/2 = x_N + (p/2)$, then $x < q < s$. □

Remarks A.3.5 Upper bounds. Completeness is formulated in terms of upper bounds. They were introduced in Definition 2.2.1. $M \in \mathbb{R}$ is called an *upper bound* for a set $A \subset \mathbb{R}$ if $a \leq M$ for all $a \in A$. Note that if A has an upper bound $M \in \mathbb{R}$, then it also has an upper bound $M' \in \mathbb{Q}$. This follows from Lemma A.3.4 or just from the fact that any Cauchy sequence of rational numbers is bounded by a rational number.

Lemma A.3.6 *Let A be a nonempty set of real numbers. Assume that A has an upper bound. Then there is a $p \in \mathbb{Q}$ such that p is not an upper bound for A but $p + 1$ is an upper bound for A.*

Proof. Let $a \in A$. Then $r = (a - 1)$ is not an upper bound. Let $s < r$, $s \in \mathbb{Q}$. Then s is not an upper bound. The sequence $s_n = s + n$ is an unbounded sequence. Hence there are $n \in \mathbb{N}$ for which $s + n$ is an upper bound for A. Use the induction principle to find the least $m \in \mathbb{N}$ such that $q = s + m$ is an upper bound. Then $p = q - 1 = s + (m - 1)$ is not an upper bound. (Note that this is obviously true if $m \geq 2$, but it is also true even if $m = 1$.) □

Lemma A.3.7 *Let A be a nonempty set of real numbers. Assume that A has an upper bound. Then there are two sequences of rational numbers p_n and q_n such that if $n \in \mathbb{N}$, then p_n is not an upper bound for A, q_n is an upper bound for A, and $(q_n - p_n) = (1/2)^{n-1}$.*

Proof. Let p, $q \in \mathbb{Q}$ be the two numbers obtained in Lemma A.3.6. Let $p_1 = p$ and $q_1 = q$. Our requirements are satisfied for $n = 1$. Assume that p_n, q_n are obtained such that p_n is not an upper bound, q_n is an upper bound, and $(q_n - p_n) = (1/2)^{n-1}$.

Let $s_n = (p_n + q_n)/2$. If s_n is not an upper bound, then let $p_{n+1} = s_n$ and $q_{n+1} = q_n$. If s_n is an upper bound, then let $p_{n+1} = p_n$ and $q_{n+1} = s_n$. We see that our requirements are also satisfied for $n+1$. Hence the sequences p_n and q_n are defined by the induction principle. □

Lemma A.3.8 *The sequences p_n and q_n obtained in Lemma A.3.7 are equivalent Cauchy sequences.*

Proof. We see that $p_n \leq p_{n+1} \leq q_{n+1} \leq q_n$. By induction it follows that

$$p_n \leq p_{n+k} \leq q_{n+k} \leq q_n \text{ for all } k \in \mathbb{N}.$$

Therefore, if $m \geq n$, then $|p_n - p_m| \leq |p_n - q_n| = (1/2)^{n-1}$. But $(1/2)^{n-1}$ is a zero sequence. Hence p_n is a Cauchy sequence. Similarly, q_n is also a Cauchy sequence. The equivalence of p_n and q_n follows from the fact that $(q_n - p_n) = (1/2)^{n-1}$ is a zero sequence. □

Theorem A.3.9 *Let A be a nonempty set of real numbers. Assume that A has an upper bound. Then A has a least upper bound. More explicitly, there is an $r \in \mathbb{R}$ such that r is an upper bound for A, but if $s < r$, $s \in \mathbb{R}$, then s is not an upper bound for A.*

Proof. Let p_n and q_n be the sequences obtained in Lemma A.3.7. Lemma A.3.8 shows that they are equivalent Cauchy sequences. Hence they represent the same $r \in \mathbb{R}$. We claim that r is a least upper bound for A.

Assume that r is not an upper bound for A. Then there is an $a \in A$ such that $r < a$. Use Lemma A.3.4 to find $q \in \mathbb{Q}$ such that $r < q < a$. Then the sequence $q - q_n$ represents the positive number $(q - r) \in \mathbb{R}$. Hence it must eventually be positive. Hence there is an $N \in \mathbb{N}$ such that $q > q_n$ for all $n \geq N$. This means that $q_n < q < a$ and q_n is not an upper bound for A. This is a contradiction. Hence r is an upper bound for A.

Now suppose that A has an upper bound $s \in \mathbb{R}$, $s < r$. As before, find a $p \in \mathbb{Q}$ such that $s < p < r$. Then $(p_n - p)$ must eventually be positive, as it represents $(r - p) > 0$. Therefore $p_n > p > s$ for some $n \in \mathbb{N}$. This means that p_n is an upper bound for A. This is a contradiction. Hence A cannot have upper bounds less than r. Therefore r is a least upper bound for A. □