1.2. Properties of the Real Numbers as an Ordered Field.

Note. In this section we give 8 axioms related to the definition of the real numbers, $\mathbb{R}$. All properties of sets of real numbers, limits, continuity of functions, integrals, and derivatives will follow from this definition.

Definition. A *field* $\mathbb{F}$ is a nonempty set with two operations $+$ and $\cdot$ called addition and multiplication, such that:

(1) If $a, b \in \mathbb{F}$ then $a + b$ and $a \cdot b$ are uniquely determined elements of $\mathbb{F}$ (i.e., $+$ and $\cdot$ are *binary operations*).

(2) If $a, b, c \in \mathbb{F}$ then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (i.e., $+$ and $\cdot$ are *associative*).

(3) If $a, b \in \mathbb{F}$ then $a + b = b + a$ and $a \cdot b = b \cdot a$ (i.e., $+$ and $\cdot$ are *commutative*).

(4) If $a, b, c \in \mathbb{F}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$ (i.e., $\cdot$ *distributes* over $+$).

(5) There exists $0, 1 \in \mathbb{F}$ (with $0 \neq 1$) such that $0 + a = a$ and $1 \cdot a = a$ for all $g \in \mathbb{F}$.

(6) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that $a + (-a) = 0$.

(7) If $a \in \mathbb{F}$ $a \neq 0$, then there exists a^{-1} such that $a \cdot a^{-1} = 1$.

0 is the *additive identity*, 1 is the *multiplicative identity*, $-a$ and a^{-1} are *inverses* of a.

Example. Some examples of fields include:

1. The rational numbers $\mathbb{Q}$.
2. The rationals extended by $\sqrt{2}$: $\mathbb{Q}[\sqrt{2}]$.
3. The algebraic numbers $\mathbb{A}$.
4. The real numbers $\mathbb{R}$.
5. The complex numbers $\mathbb{C}$.
6. The integers modulo p where p is prime $\mathbb{Z}_p$.

Theorem 1-3. For $\mathbb{F}$ a field, the additive and multiplicative identities are unique.

Theorem 1-4. For $\mathbb{F}$ a field and $a \in \mathbb{F}$, the additive and multiplicative inverses of a are unique.

Theorem 1-5. For $\mathbb{F}$ a field, $a \cdot 0 = 0$ for all $a \in \mathbb{F}$.

Theorem 1-6. For $\mathbb{F}$ a field and $a, b \in \mathbb{F}$:

(a) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$.

(b) $-(-a) = a$.

(c) $(-a) \cdot (-b) = a \cdot b$.

Note. We add another axiom to our development of the real numbers.

Axiom 8/Definition of Ordered Field. A field $\mathbb{F}$ is said to be *ordered* if there is $P \subset \mathbb{F}$ (called the *positive subset*) such that

(i) If $a, b \in P$ then $a + b \in P$ (closure of P under addition).

(ii) If $a, b \in P$ then $a \cdot b \in P$ (closure of P under multiplication).

(iii) If $a \in \mathbb{F}$ then exactly one of the following holds: $a \in P$, $-a \in P$, or $a = 0$ (this is *The Law of Trichotomy*).

Example. $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{A}$, and $\mathbb{R}$ is an ordered field. $\mathbb{C}$ and $\mathbb{Z}_p$ are fields that are not ordered.

Definition. Let $\mathbb{F}$ be a field and P the positive subset. We say that $a < b$ (or $b > a$) if $b - a \in P$.

Note. The above definition allows us to compare pairs of elements of $\mathbb{F}$ and to "order" the elements of the field.

Exercise 1.2.5. If $\mathbb{F}$ is an ordered field, $a, b \in \mathbb{F}$ with $a \leq b$ and $b \leq a$ then $a = b$.

Theorem 1-7. Let $\mathbb{F}$ be an ordered field. For $a, b, c \in \mathbb{F}$:

(a) If $a < b$ then $a + c < b + c$.

(b) If $a < b$ and $b < c$ then $a < c$ ("$<$" is *transitive*).

(c) If $a < b$ and $c > 0$ then $ac < bc$.

(d) If $a < b$ and $c < 0$ then $ac > bc$.

(e) If $a \neq 0$ then $a^2 = a \cdot a > 0$.

Note. Recall interval notation from Calculus 1 (see page 18).

Note. We have trouble defining exponentiation when the exponent is irrational (at least, for now).

Theorem 1-8. Let x be a positive real number and let n be a positive integer. Then there is a unique positive number y such that $y^n = x$.

Note. The proof of Theorem 1-8 depends on a result from the next section and we will consider it then.

Note. In Theorem 1-8, we say $y = x^{1/n} = \sqrt[n]{x}$. We define $x^{p/q} = (x^{1/q})^p$ where p and q are positive integers.

Theorem 1-9. Let x be a positive real number, and let s_1 and s_2 be positive rational numbers where $s_1 < s_2$. Then

(a) $x^{s_1} < x^{s_2}$ if $x > 1$.

(b) $x^{s_1} > x^{s_2}$ if $0 < x < 1$.

Theorem 1-10. Let x and y be positive real numbers with $x < y$ and let s be a positive rational number. Then $x^s < y^s$.

Exercise 1.2.7. Prove:

(a) $1 > 0$.

(b) If $0 < a < b$ then $0 < 1/b < 1/a$.

(c) If $0 < a < b$ then $a^n < b^n$ for natural number n.

(d) If $a > 0$, $b > 0$ and $a^n < b^n$ for some natural number n, then $a < b$.

(f) Prove Theorem 1-10.

Theorem 1-12. The Binomial Theorem.

Let a and b be real numbers and let m be a natural number. Then

$$(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}.$$

Note. We can use Mathematical Induction to prove the Binomial Theorem (in fact, you likely did so in Math Reasoning [MATH 2800]).

Definition. For $a \in \mathbb{R}$, the *absolute value* of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

Note. Theorem 1-13 lists several familiar properties of the absolute value function. In particular:

Theorem 1-13. For all $a, b \in \mathbb{R}$

(g) $|a| < |b|$ if and only if $-b < a < b$.

(h) $|a + b| \leq |a| + |b|$ (this is the *Triangle Inequality*).

(i) $||a| - |b|| \leq |a - b|$.

Definition. Let X be a set and d a function $d : X \times X \to \mathbb{R}$ satisfying

(i) $d(a, b) \geq 0$ for all $a, b \in X$ and $d(a, b) = 0$ if and only if $a = b$.

(ii) $d(a, b) = d(b, a)$.

(iii) $d(a, c) \leq d(a, b) + d(b, c)$ (this is the *Triangle Inequality*).

Function d is then called a *metric* on X.

Note. A metric on $\mathbb{R}$ based on absolute value is $d(x,y) = |x-y|$. This is the metric we will use throughout this course to define such fundamental things as limits.

Example. Examples of metrics on $X = \mathbb{R}^2$ include the Euclidean metric and the taxicab metric.

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