The Real Numbers

This paper gives you the kinds of real numbers that there are, the rules which govern them and a few kind of weird facts you might find interesting.

1. The Kinds of Real Numbers:

   (a) The natural numbers, $\mathcal{N}$, the ones we count with $1, 2, 3, \ldots$ and so forth.

   (b) The integers, $\mathcal{Z}$ consisting of $\mathcal{N}$ plus 0 and the negative numbers.

   (c) The fractions, or quotients $\mathcal{Q}$ of form p/q

   (d) The algebraic numbers, which are zeros of polynomials. Things like $\sqrt{2}$ are algebraic, since $\sqrt{2}$ is a zero of $x^2 - 2$.

   (e) Note that each set, up to here, contains the ones before it. All the integers are in $\mathcal{Q}$ and everything is in $\mathcal{R}$.

   (f) The transcendental numbers, which include things like π and e, the base for natural logarithms. These are not algebraic. The entire collection is called $\mathcal{R}$, the real numbers.

2. How Many of Each are There?

   (a) The $\mathcal{N}, \mathcal{Z}, \mathcal{Q}$ are all countable, that is they may be put in one to one correspondence with the natural numbers.

   (b) $\mathcal{R}$ is larger. It cannot be put into one-to-one correspondence with $\mathcal{N}$.

3. The Rules for the Real Number System:

   (a) The real numbers are closed under two operations, addition and multiplication, that is if a and b are real numbers so are $a + b$ and $a \times b$.

   (b) The Associative Law for Addition, $a + (b + c) = (a + b) + c$.

   (c) The Existence of an Additive Identity, $a + 0 = 0 + a = a$.

   (d) The Existence of an Additive Inverse, $a + (-a) = 0 = (-a) + a$.

   (e) The Commutative Law of Addition, $a + b = b + a$.

   (f) The Associative Law for Multiplication, $a \times (b \times c) = (a \times b) \times c$.

(g) The Existence of a Multiplicative Identity, $a \times 1 = 1 \times a = a$.

(h) The Existence of a Multiplicative Inverse, $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$.

(i) Commutative Law for multiplication, $a \times b = b \times a$.

(j) Distributive Law, $a \times (b + c) = a \times b + a \times c$.

(k) The Law of Tricotomy, every real number is either positive, negative or zero. They are all on the number line, there are no stray sheep.

(l) A very odd law, variously called by unusual names. I'll call it the Dedekind Law, since he worked on it. It says there are no holes in the real number line. Mathematicians would wince at this, but it is as close as I can get without a lot of mathematical machinery up with which you would not put.

4. Weird Facts

(a) Between any two reals there is a rational.

(b) Between any two reals there is an irrational.

(c) If you take all of $\mathcal{Q}$ away from $\mathcal{R}$ the remaining set is the same size as $\mathcal{R}$.

(d) There are as many real numbers between 0 and 1 as there are between 0 and 100.

(e) There are more real numbers than we can name.

(f) There is nothing in the real world like the real line.