Properties of Real Numbers[1]

Theorem: For an arbitrary real number x, there is exactly one interger n which satisfies the inequalities $n \leq x < n + 1$.

Proof:

1. Define: $S = \{m | m \in \mathbb{N}, m \leq x\}$
2. S is non empty b/c for any real number x there exists $m \in \mathbb{N}$ such that $m < x$. (I 3.12 2. pg 28)
3. S is bounded by x $\Rightarrow supS$ exist. (A.10 pg. 25)
4. By Thm 1.32 (pg.27) with $h = 1$ for some $a \in S$ we have $a > supS - 1 \Rightarrow supS - a < 1$, and $supS < a + 1$, b/c the integers are unique, and differ by 1, there is only one such a.
5. 4. $\Rightarrow a + 1 \notin S$ b/c $x \leq supS$ for all $x \in S$
6. By construction of S, $a \leq x$
7. b/c $a + 1 \notin S$, $a + 1 > x$, by definition of S

Which gives $a < x < a + 1$ with $a \in S$

Theorem: For arbitrary real numbers x and y such that $x < y$, there exists at least one rational number r such that $x < r < y$, and thus infinitely many . Density of the rational numbers.

Proof:

1. $x < y \Rightarrow y - x > 0$
2. From A.6 there is a real number $\frac{1}{y-x}$
3. Thm I.29 $\Rightarrow$ there is a n such that $\frac{1}{y-x} < n \Rightarrow nx + 1 < ny$

4 The first theorem states there is a m such that $m < nx < m + 1 \Rightarrow m + 1 < nx + 1 < m + 2$

5. 3. and 4. give $nx < m + 1 < nx + 1 < ny$

Thus, $nx < m + 1 < nx + 1 < ny$ *or* $x < \frac{m+1}{n} < y$, *with* $n, m \in S$

[1]References are from Calculus Vol. 1 Apostol

Theorem: For arbitrary real numbers x and y, we have: $|x+y| \leq |x|+|y|$ (The triangle inequality)

Proof:

For and real number y we have $-|r| \leq r \leq |r|$.

Thus, $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$.

Adding the two gives, $-(|x|+|y|) \leq x+y \leq |x|+|y|$

For $a \leq 0$, then $|r| \leq a$ if and only if $-a \leq r \leq a$

Applying this to $x+y$, gives $|x+y| \leq |x|+|y|$

Theorem: For arbitrary real numbers $a_1, a_2, \ldots, a_n$ we have: $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ (Generalization of the triangle inequality)

Proof:

By induction.

$n=1$ case is a triviality.

Assume holds for n.

For $n+1$: $|\sum_{k=1}^{n+1} a_k| = |\sum_{k=1}^{n} a_k + a_{n+1}| \leq |\sum_{k=1}^{n} a_k| + |a_{n+1}|$, by the previous theorem.

And $|\sum_{k=1}^{n} a_k| + |a_{n+1}| \leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k|$, by the inductive step.

Thus $|\sum_{k=1}^{n+1} a_k| \leq \sum_{k=1}^{n+1} |a_k|$.

By induction, it hold for every integer n.

Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. (The binomial theorem)

Proof:

By induction.

For $n=1$: $\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = \binom{1}{0} b + \binom{1}{1} a = a+b$

Assume holds for n. For $n+1$: $(a+b)^{n+1} = (a+b)^n(a+b) = a(a+b)^n + b(a+b)^n = a\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} + b\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{(n+1)-k}$ by the inductive step.

In the first term let $\hat{k} = k+1$

then above $= \sum_{\hat{k}=1}^{n+1} \binom{n}{\hat{k}-1} a^{\hat{k}} b^{(n+1)-\hat{k}} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{(n+1)-k} = \sum_{\hat{k}=1}^{n} \binom{n}{\hat{k}-1} a^{\hat{k}} b^{(n+1)-\hat{k}} + \binom{n}{n} a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{(n+1)-k} + \binom{n}{0} b^{n+1}$

or, removing the hat on the dummy index in the first sum: $\sum_{k=1}^{n}\left(\binom{n}{k-1}+\binom{n}{k}\right)a^k b^{(n+1)-k}+$ $a^{n+1}+b^{n+1}$

But, $\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$ and $\binom{m}{m}=\binom{p}{0}=1$ for all integer m and p.

Thus: above $=\sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n-1}$, and by induction the theorem hold for all n.