Math 2505 - Introductory Analysis

Representing Real Numbers With Respect to a Base

In times past, various notational systems were developed for representing numbers. Roman numerals are still used in many special situations where one knows that only positive integers, and relatively small ones at that, will arise. One of the most significant advances in the history of science and technology was the development of the place-value, base ten, system of representing numbers. With the rise of modern digital computation, base two and base sixteen systems have been widely used. There is a special subset of the real numbers called the Cantor ternary set whose properties can be explored by representing numbers in base three. It makes sense to explain how real numbers can be represented using an arbitrary base. To start with, we only have afixed symbol for two real numbers, 0 and 1.

Our first step is to introduce exponential notation. Let $a \in \mathbb{R}, a \neq 0$. We define a^0 to be equal to 1 and $a^1 = a$. For $n \in \mathbb{N}$, if a^n has been defined, then $a^{n+1} = (a^n)a$. By induction then, a^n is defined for all $n \in \mathbb{N}$. Since $a \neq 0$, an induction argument shows that $a^n \neq 0$, for $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, define $a^{-n} = (a^n)^{-1}$. An inductive argument shows that $(a^{-1})^n = a^{-n}$, for all $n \in \mathbb{N}$. Thus, if $a \neq 0$, then a^j is defined for all $j \in \mathbb{Z}$. The reader should prove the properties of exponents below. In most cases, an induction argument works.

- $a^{j+k} = (a^j)(a^k), \qquad \forall j, k \in \mathbb{Z}.$
- $(a^j)^k = a^{jk}, \qquad \forall j, k \in \mathbb{Z}.$
- $(ab)^j = a^j b^j, \qquad \forall\, j \in \mathbb{Z}, a \neq 0, b \neq 0.$
- If $1 + 1 \leq a$, then $n < a^n, \qquad \forall\, n \in \mathbb{N}.$

We will describe base beta representation, where beta refers to some fixed $\beta \in \mathbb{N}$, $\beta > 1$. The set of *base beta digits* is

$$D_\beta = \{d \in \mathbb{N} \cup \{0\} \,:\, d < \beta\}.$$

Each base beta digit larger than 1 is given its own symbol. Since 0 and 1 are already named real numbers, we stick with those symbols for these first members of every digit set. The symbols for the base two digits are just 0 and 1. The base three digits are traditionally written $\{0, 1, 2\}$, where $2 = 1 + 1$. The base ten digits are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Base sixteen is used for specifying levels of Red, Green, and Blue in the RGB system for colours. The base sixteen digits are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$

Before showing how to represent integers in base beta, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set of non-negative integers. We also note that any nonempty subset of $\mathbb{N}$ has a smallest element. That is, if $A \subseteq \mathbb{N}$ and $A \neq \emptyset$, then there exists $m \in A$ satisfying $m \leq a$, for all $a \in A$. The point

m is called the *minimum* of A and is denoted $\min(A)$.This is known as the *well-ordered* property of $\mathbb{N}$ and is easily proved by induction. Try it!

Now, let $M \in \mathbb{N}$.

Let $A = \{k \in \mathbb{N} : M < \beta^k\}$. Since $M < \beta^M$, we have $M \in A$. So $A \neq \emptyset$. Let $k_0 = \min(A)$ and let $n = k_0 - 1 \in \mathbb{N}_0$. Then n is the unique non-negative integer satisfying

$$\beta^n \leq M < \beta^{n+1}.$$

Since $1 \cdot \beta^n = \beta^n$ and $\beta \cdot \beta^n = \beta^{n+1}$, there is a unique digit $a_n \in D_\beta$ such that

$$a_n\beta^n \leq M < (a_n + 1)\beta^n.$$

If $n = 0$, then $M = a_0 = a_0\beta^0$ and we stop. If $1 \leq n$, consider β^{n-1}. As a runs through D_β, $a_n\beta^n + a\beta^{n-1}$ steps from $a_n\beta^n + 0 \cdot \beta^{n-1}$ up to $a_n\beta^n + (\beta - 1)\beta^{n-1}$. There is a unique $a_{n-1} \in D_\beta$ such that

$$a_n\beta^n + a_{n-1}\beta^{n-1} \leq M < a_n\beta^n + (a_{n-1} + 1)\beta^{n-1}$$

Continuing this process exactly $n+1$ times, we obtain unique digits $a_n, a_{n-1}, \cdots, a_1, a_0$ such that

$$a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 \leq M < a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 + 1$$

Since $M \in \mathbb{N}$, we conclude that $M = a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 = \sum_{j=0}^{n} a_j\beta^j$. Using *place-value* to represent the respective powers of β, one can denote M in the compact form $a_na_{n-1}\cdots a_1a_0$. For example, the natural numbers written in base two start out

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001, \cdots$$

In base three, we count

$$1, 2, 10, 11, 12, 20, 21, 22, 100, \cdots$$

We write $-M = -a_na_{n-1}\cdots a_1a_0$ when $M = a_na_{n-1}\cdots a_1a_0$. Of course, 0 is just 0. So all members of $\mathbb{Z}$ have an unambiguous representation in base beta. How does this extend to representing non-integer real numbers?

Fix $x \in \mathbb{R}, x > 0$. Let $A = \{n \in \mathbb{N} : x < n\}$. By the Archimedean Property, $A \neq \emptyset$. Let $n_0 = \min(A)$. Define $\lfloor x \rfloor = n_0 - 1$. Then $\lfloor x \rfloor \in \mathbb{N}_0$ and it is the largest integer less than or equal to x. It is sometimes called the *floor of* x. Let $\langle\!\langle x \rangle\!\rangle = x - \lfloor x \rfloor$. Then $\langle\!\langle x \rangle\!\rangle \in [0, 1)$ and $x = \lfloor x \rfloor + \langle\!\langle x \rangle\!\rangle$. We have a way of representing $\lfloor x \rfloor$ in base beta. If $\lfloor x \rfloor = 0$, we take $n = 0$ and $a_0 = 0$. If $\lfloor x \rfloor > 0$, we have some $n \in \mathbb{N}$ and digits $a_0, a_1, \cdots, a_n$ so that in base beta place-value representation, we have

$$\lfloor x \rfloor = a_na_{n-1}\cdots a_1a_0. \qquad (*)$$

The representation $(*)$ works whether $\lfloor x \rfloor$ is 0 or a larger integer.

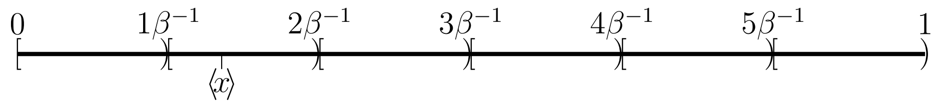
Since $\beta > 1$, we have $0 < \beta^{-1} < 1$. Indeed,

$$0 = \frac{0}{\beta} < \frac{1}{\beta} < \cdots < \frac{\beta - 1}{\beta} < \frac{\beta}{\beta} = 1.$$

Therefore, the interval $[0, 1)$ is partitioned into β non-overlapping intervals as follows:

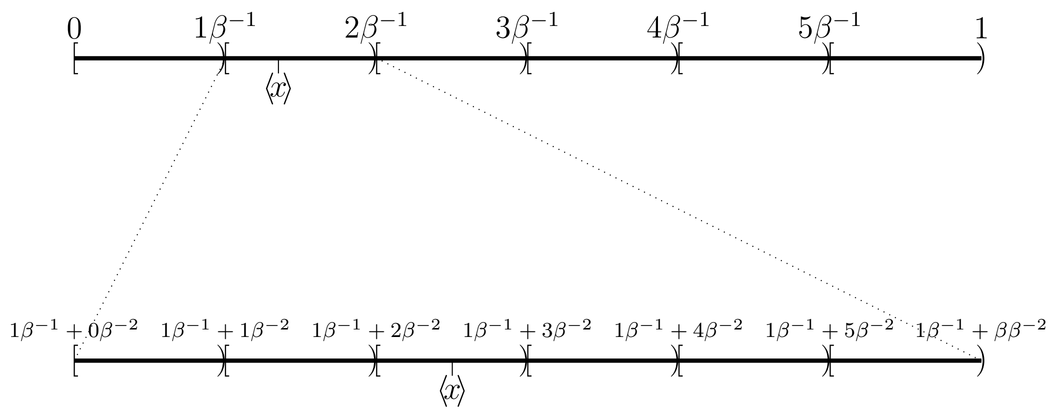
$$[0, 1) = [0, 1\beta^{-1}) \cup [1\beta^{-1}, 2\beta^{-1}) \cup \cdots \cup [(\beta - 1)\beta^{-1}, 1).$$

On the section of the number line from 0 to 1 with beta being six, it looks like the following:



Sample location of $\langle x \rangle$ illustrated in base six.

There is a unique digit $b_1 \in D_\beta$ so that $b_1\beta^{-1} \leq \langle x \rangle < (b_1 + 1)\beta^{-1}$. Let's zoom in on the interval $[b_1\beta^{-1}, (b_1 + 1)\beta^{-1})$ by a factor of β as illustrated below with beta being six.



There is a unique digit $b_2 \in D_\beta$ satisfying $b_1\beta^{-1} + b_2\beta^{-2} \leq \langle x \rangle < b_1\beta^{-1} + (b_2 + 1)\beta^{-2}$. We continue inductively.

Suppose $k \in \mathbb{N}$ and digits $b_1, \cdots, b_k \in D_\beta$ have been found so that

$$\sum_{j=1}^{k} b_j\beta^{-j} \leq \langle x \rangle < \sum_{j=1}^{k} b_j\beta^{-j} + \beta^{-k}.$$

Then the interval $\left[\sum_{j=1}^{k} b_j\beta^{-j}, \sum_{j=1}^{k} b_j\beta^{-j} + \beta^{-k}\right)$ is the disjoint union

$$\bigcup_{b=0}^{\beta-1}\left[\sum_{j=1}^{k} b_j\beta^{-j} + b\beta^{-(k+1)}, \sum_{j=1}^{k} b_j\beta^{-j} + (b+1)\beta^{-(k+1)}\right).$$

The point $\langle x\rangle$ lies in exactly one of these intervals. Thus, there exists a unique digit $b_{k+1} \in D_\beta$ such that

$$\sum_{j=1}^{k+1} b_j\beta^{-j} \le \langle x\rangle < \sum_{j=1}^{k+1} b_j\beta^{-j} + \beta^{-(k+1)}.$$

This inductively defines a sequence of digits $(b_j)_{j=1}^{\infty}$ so that

$$\sum_{j=1}^{k} b_j\beta^{-j} \le \langle x\rangle < \sum_{j=1}^{k} b_j\beta^{-j} + \beta^{-k}, \qquad \forall\, k \in \mathbb{N} \qquad (**)$$

When $(**)$ holds, we use the place-value notation again and write

$$\langle x\rangle = ._{\beta} b_1 b_2 b_3 \cdots .$$

The $._{\beta}$ reminds us that it is a base beta representation. Combining this with $(*)$, it is reasonable to write

$$x = \lfloor x\rfloor + \langle x\rangle = a_n a_{n-1}\cdots a_1 a_0 ._{\beta} b_1 b_2 b_3 \cdots .$$

Finally, if $y \in \mathbb{R}, y < 0$, let $x = -y$ and write $x = a_n a_{n-1}\cdots a_1 a_0 ._{\beta} b_1 b_2 b_3 \cdots$. Then denote $y = -x$ by $-a_n a_{n-1}\cdots a_1 a_0 ._{\beta} b_1 b_2 b_3 \cdots$. This gives the base beta place-value method of expressing any real number.

As an example, consider one of the most important real numbers, the ratio of the circumference of a circle to its diameter. This number is denoted π. Here are the first few digits of π represented in various bases.

Base beta = two: $\pi = 11._{\beta}001001000011111101101010 1\cdots$

Base beta = three: $\pi = 10._{\beta}010211012222010211002111 1\cdots$

Base beta = ten: $\pi = 3._{\beta}14159265358979323846264 33\cdots$

Base beta = sixteen: $\pi = 3._{\beta}243\text{F6A8885A308D313198A2E0}\cdots$

"Counting on fingers" must have been done by *homo sapiens* for at least one hundred millenia. It is not surprising that the most common base for representing numbers is ten. We usually call this the *decimal* representation of numbers.