THE SUPREMUM AXIOM FOR THE REAL NUMBERS

Definitions. A nonempty subset $A \subset \mathbb{R}$ is bounded above if:

$$(\exists M \in \mathbb{R})(\forall x \in A)(x \leq M).$$

Any such M is an *upper bound* for A. A real number L is a *least upper bound* for A (or a *supremum* for A) if: L is an upper bound for A and $(\forall M \in \mathbb{R})(M$ is an upper bound for $A \to L \leq M)$. Notation: $L = \sup(A)$.

Similar definitions apply to subsets of $\mathbb{R}$ which are bounded from below: any real number M with the property $(\forall x \in A)(x \geq M)$ is a *lower bound* for A and a real number L which is a lower bound for A and, in addition, satisfies: $(\forall M \in \mathbb{R})(M$ is a lower bound for $A \to L \geq M)$ is a *greatest lower bound* for A, or an *infimum* for A. Notation: $L = \inf(A)$.

SUPREMUM AXIOM: Any nonempty subset $A \subset \mathbb{R}$ which is bounded above has a supremum $L \in \mathbb{R}$.

It is not hard to see that the supremum of A is *unique*. $\sup(A)$ is not necessarily an element of A; when it is, we say it is the *maximum* of A.

It follows from the supremum axiom that any nonempty subset $A \subset \mathbb{R}$ which is bounded from below has an infimum. The infimum of a set is also unique, and $\inf(A)$ may fail to be an element of A. When it is in A, we say it is the *minimum* of A.

Remark: AXIOMS FOR THE REAL NUMBERS. There are three groups of axioms:

1) *Algebraic axioms*: $(\mathbb{R}, +, \cdot, 0, 1)$ is a *field*. This means:

1a) $(\mathbb{R}, +, 0)$: addition $(+)$ is an associative, commutative operation with neutral element 0; any $x \in \mathbb{R}$ has a unique additive inverse $-x$ (meaning $x + (-x) = 0$).

1b) $(\mathbb{R}, \cdot, 1)$: multiplication $(\cdot)$ is an associative, commutative operation, with neutral element 1. Any $x \in \mathbb{R}, x \neq 0$ has a unique multiplicative inverse x^{-1} (meaning $x \cdot x^{-1} = 1$).

1c) multiplication distributes over addition: $x \cdot (y + z) = x \cdot y + x \cdot z$.

1d) Nontriviality: $1 \neq 0$.

2) *Order axioms.* There exists a subset $\mathbb{R}_+ \subset \mathbb{R}$ with the following properties:

2a) defining $x \leq y \leftrightarrow y - x \in \mathbb{R}_+$ establishes a *total order* relation in $\mathbb{R}$.

2b) $x \leq y$ and $z \leq w \to x + z \leq y + w$.
2c) $x \leq y$ and $z \geq 0 \to x \cdot z \leq y \cdot z$.

3) The supremum axiom.

Note that the set of rational numbers $\mathbb{Q}$ also satisfies the algebraic and order axioms. What distinguishes $\mathbb{R}$ from $\mathbb{Q}$ is the fact that $\mathbb{Q}$ *does not* satisfy the supremum axiom. That is, a non-empty subset of $\mathbb{Q}$ which is bounded above may fail to have a supremum in $\mathbb{Q}$.

Example. Let $A = \{x \in \mathbb{Q} | x^2 < 2\}$. Clearly A is nonempty and bounded above (say, 2 is an upper bound, since $x > 2 \to x^2 > 4 \to x \neq A$). However, we'll see shortly that if $L = \sup(A)$ exists (and, regarding A as a subset of $\mathbb{R}$, it does) we must have $L^2 = 2$, and hence L can't be an element of $\mathbb{Q}$.

Theorem (Characterization of the supremum). Let $A \subset \mathbb{R}$ be nonempty and bounded above. Then $L = \sup(A)$ if and only if L is an upper bound for A and:

$$(\forall \epsilon > 0)(\exists a \in A)(L - \epsilon < a \leq L).$$

Analogously, if $A \subset \mathbb{R}$ is nonempty and bounded below, $L = \inf(A)$ if, and only if, L is a lower bound for A and:

$$(\forall \epsilon > 0)(\exists a \in A)(L - \epsilon > a \geq L).$$

This isn't hard to prove.

Example/Theorem. Let $A = \{x \in \mathbb{R} | x > 0 \text{ and } x^2 < 2\}$. Then $L = \sup(A)$ satisfies $L^2 = 2$. (Note that A is nonempty and bounded above, and that clearly $L > 0$; thus this theorem establishes the *existence* of the positive square root of 2, as a real number.)

Proof. By contradiction. If $L^2 \neq 2$, either $L^2 < 2$ or $L^2 > 2$. Assume first $L^2 < 2$. Let:

$$\epsilon = \min\{\frac{1}{2}, \frac{2 - L^2}{2L + 1}\}.$$

Note $0 < \epsilon < 1$, so $\epsilon^2 < \epsilon$. Let $a = L + \epsilon > L$. Note:

$$a^2 = L^2 + 2L\epsilon + \epsilon^2 < L^2 + (2L + 1)\epsilon \leq L^2 + (2 - L^2) = 2.$$

So $a^2 < 2$. This means $a \in A$, and yet $a > L$, contradicting the fact L is an upper bound for A.

Now suppose $L^2 > 2$. Then $x = L - \frac{L^2 - 2}{2L} < L$, and:

$$x^2 = L^2 - (L^2 - 2) + \frac{(L^2 - 2)^2}{4L^2} > 2.$$

Let $a \in A$ be arbitrary. Since $a > 0$, $a^2 < 2$ and $x^2 > 2$, it follows that $a^2 < x^2$, so $a < x$. This shows x is an upper bound for A. Since $x < L$, this contradicts the fact that L is the least upper bound for A. Thus we must have $L^2 = 2$, ending the proof.

Functions. If $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ is a function and $E \subset A, E \neq \emptyset$, the following are standard definitions:

If $f(E) \subset \mathbb{R}$ is bounded above: $\sup_E f = \sup f(E)$; $\max_E f = \max f(E)$.

If $f(E) \subset \mathbb{R}$ is bounded below: $\inf_E f = \inf f(E)$; $\min_E f = \min f(E)$.

PROBLEMS.

1. Find the supremum and the infimum of the following sets (when they exist.)

(a) $A = \{x \in \mathbb{R} | 0 < x^2 - 3x - 5 < 1\}$.

(b) $A = \{x \in \mathbb{R} | (\exists n \in \mathbb{N})(x = \frac{1}{n} - (-1)^n)\}$.

(c)$A = \{1 + \frac{(-1)^n}{2n}; n \in \mathbb{N}\}$.

2. Let A, B be nonempty subsets of $\mathbb{R}$, both bounded above.

(a) Prove that if $A \subset B$, then $\sup(A) \leq \sup(B)$.

(b) Prove that if $A \cap B \neq \emptyset$, then $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$.

(c) Give an example where the inequality in (b) is strict: $\sup(A \cap B) < \min\{\sup(A), \sup(B)\}$

(d) Prove that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

Remark: analogous results apply to the infimum.

3. Let $A \subset \mathbb{R}, A \neq \emptyset$. Let $f, g : A \to \mathbb{R}$.

(a) Show that if $f(A), g(A)$ are bounded above, $\sup(f+g)(A) \leq \sup f(A) + \sup g(A)$.

(b) Show that if $f(A), g(A)$ are bounded below, $\inf(f+g)(A) \geq \inf f(A) + \inf g(A)$.

4. For each of the following functions f and sets A, find $\sup_A f$ and $\inf_A f$ (if they exist), and indicate if they are also a maximum (resp. a minimum.)

(a) $f(x) = \frac{1+x2}{x^2}, A = [-1, 1] \setminus \{0\}$.

(b) $f(x) = \frac{x}{2+x^2}, A = (-1, 1)$.

(c) $f(x) = \frac{2}{2+\sin x}, A = \mathbb{R}$.

(d) $f(x) = \frac{x}{1+x}, A = \mathbb{N}$.

(e) $f(x) = \frac{x+1}{x+2}; A = \{x \in \mathbb{R}; x \geq 0\}$