Note 09 Work and Potential Energy

Sections Covered in the Text: Chapter 10. Some sections of Chapter 12.

In Note 08 we discussed the kind of mechanical energy an object may possess by virtue of its motion. This energy is called *kinetic energy*.[1] Here we examine the kind of mechanical energy an object may possess by virtue of its position relative to a reference position. This energy is called *potential energy*. We shall see that when an object undergoes a change in its state of motion (as when it falls under gravity for example), its potential energy can be transformed into kinetic energy. We also describe a force as being conservative or non-conservative. We then examine what is meant by the stability of a system.

Potential Energy of a System

We shall see that potential energy is associated with position. We therefore begin by considering an object whose position is being changed. Suppose the object is a book being lifted through a distance Δr in a vertical direction above the Earth (Figure 9-1). We suppose that as the external agent we move the book from position y_a to position y_b by exerting an upward force on the book. The force we apply is only infinitesimally greater than the gravitational force exerted downward by the Earth to ensure that the book moves very slowly and no significant portion of the work we do goes into increasing the book's kinetic energy. The system is the book itself.

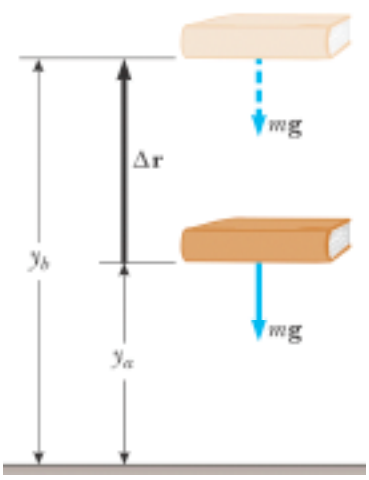

Figure 9-1. *A book is being lifted through a vertical distance Δr by an external agent (us). Shown is the force of gravity exerted by the Earth on the book. The reference position is the Earth's surface.*

[1] Strictly speaking, we mean translational kinetic energy here. We shall be discussing rotational kinetic energy in Note 10.

According to the definition of the work done by the agent of a constant force (Note 07) the work we do is

$$W_{we.do} = (-\vec{F}_g) \bullet \Delta\vec{r} = (mg\hat{j}) \bullet [(y_b - y_a)\hat{j}]$$

$$= mgy_b - mgy_a > 0. \qquad \ldots[9\text{-}1]$$

Since $y_b > y_a$, the work done is positive. By the work-energy theorem (Note 08), the work done on the book is by definition the increase in the book's mechanical energy. The energy here is not kinetic energy because the book is moving very slowly. But the position of the book relative to the Earth's surface has changed. It is logical to describe this energy acquired as *potential energy*.

According to eq[9-1] the work done equals the difference between two terms of the form mgy. We therefore define

$$U_g(y) \cong mgy, \qquad \ldots[9\text{-}2]$$

as the book's *potential energy* at position y. Automatically, this definition sets the potential energy as a function of y (there will be more on this below). $U_g(y)$ has the unit of work, namely *joules* (J). In proposing this definition, we assume that the potential energy at the position $y = 0$, the surface of the Earth, is zero.[2] Thus we can write

$$W_{we.do} = \Delta U_g = U_g(y_b) - U_g(y_a). \qquad \ldots[9\text{-}3]$$

$$> 0.$$

The work we do on the book goes into increasing the book's potential energy.

The Energy of an Isolated System

Strictly speaking, the book in the previous section is not an isolated system because it is being acted upon by an external force (the force we apply from outside the system). To consider the book as part of a larger isolated system, we now include the Earth in the system and remove our force on the book (Figure 9-2).

[2] We shall see soon enough that the zero of potential energy is arbitrary. Gravitational potential energy has a different form from eq[9-2] if the zero is defined at a position at infinity.

The book will then fall towards the Earth under the action of the force of gravity.

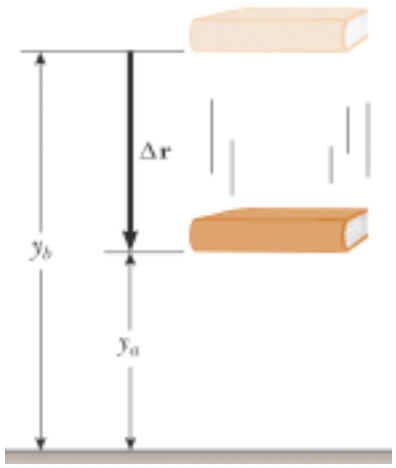

Figure 9-2. *A book falls through a vertical distance Δr under the action of the force of gravity. During the process the book and the Earth are regarded as one isolated system.*

As the book falls from y_b to y_a the work done by the force of gravity is[3]

$$W_{gravity} = (\vec{F}_g) \bullet \Delta\vec{r} = (-mg\hat{j}) \bullet (y_a - y_b)\hat{j}$$

$$= mgy_b - mgy_a > 0. \qquad \ldots[9\text{-}4]$$

Rearranging eq[9-4] we can write

$$W_{gravity} = mgy_b - mgy_a = -(mgy_a - mgy_b)$$

$$= -[U_g(y_a) - U_g(y_b)] = -\Delta U_g. \qquad \ldots[9\text{-}5]$$

where ΔU_g is as given by eq[9-3]. Thus the work done by the force of gravity on the book equals the negative of the change in the book's potential energy. This means that as the work done by gravity increases, the potential energy of the book decreases.

As the book falls, its speed increases and so does its kinetic energy. We have seen in Note 07 that the work done by gravity on the book equals the increase in the book's kinetic energy. Thus

$$W_{gravity} = \Delta K_{book}. \qquad \ldots[9\text{-}6]$$

Since the $W_{gravity}$ in eqs[9-5] and [9-6] are one and the same we have

$$\Delta K = -\Delta U_g,$$

or

$$\Delta K + \Delta U_g = 0. \qquad \ldots[9\text{-}7]$$

Thus the change in kinetic energy plus the change in potential energy of the book is zero. Or in other words, as the book's potential energy decreases, its kinetic energy increases and vice versa. If we call the sum of kinetic and potential energy the *total mechanical energy*, $E_{mechanical}$, then for the book

$$E_{mechanical} = K + U_g,$$

and

$$\Delta E_{mechanical} = 0, \qquad \ldots[9\text{-}8]$$

using eq[9-7]. Since the change in the book's mechanical energy is zero, the book's total energy remains unchanged, or is *conserved*. This is the *law of conservation of energy* for an isolated system. Thus the sum of the kinetic and potential energies at some final position y_f equals the sum of the kinetic and potential energies at some initial position y_i. Written in full:

$$\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i. \qquad \ldots[9\text{-}9]$$

We have a very useful result here that we can apply to any number of cases. Let us consider one such case, a ball in freefall.[4]

Example Problem 9-1
A Ball in Freefall

A ball of mass m is dropped from rest a height h above the surface of the Earth (Figure 9-3). Calculate (a) the speed of the ball when at a height y above the Earth, and (b) the speed of the ball at y if it is given an initial speed v_i (downwards) at the initial height h.

Solution:
This problem is similar to the falling book. We have already seen that this problem can be solved using kinematic equations alone. However, in the spirit of this note we shall take the energy approach.
(a) The ball and the Earth make up an isolated system, so we can apply the law of conservation of energy, eq[9-9]:

[3] Strictly speaking, the work is done by the source or agent of the force not by the force itself. The source of the force here is the Earth's gravitational field (see Note 05).

[4] Another important case is in the experiment "Linear Motion". As a glider moves down an inclined air track, its potential energy is converted into kinetic energy.

$$K_i + U_i = K_f + U_f.$$

The ball is dropped from rest so the initial kinetic energy is zero. The initial height is h, the final height is y and the final speed is v_f. Thus

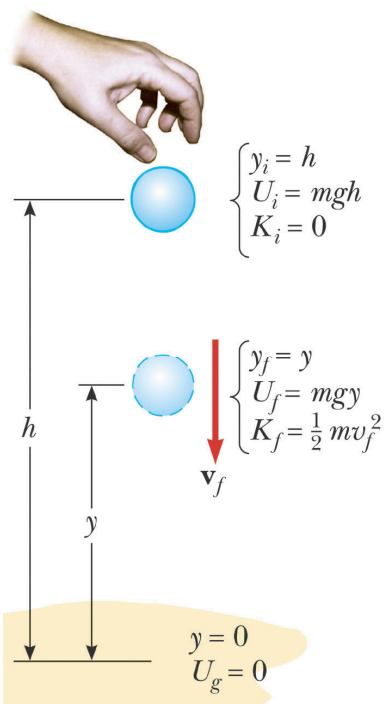

Figure 9-3. *A ball in freefall.*

$$0 + mgh = \frac{1}{2}mv_f^2 + mgy.$$

Solving for v_f we obtain:

$$v_f = \pm\sqrt{2g(h-y)}.$$

The positive sign should be taken here to conform with the sign convention of Figure 9-3.
(b) In this case the ball is thrown downwards so the initial kinetic energy is not zero. Applying the same law of conservation of energy we have

$$\frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2 + mgy.$$

Rearranging we can solve for v_f:

$$v_f^2 = v_i^2 + 2g(h-y).$$

so $$v_f = \pm\sqrt{v_i^2 + 2g(h-y)}. \qquad \text{...[9-10]}$$

Again we take the positive sign here to conform with Figure 9-3. It is useful to note that this result would apply even if the initial velocity were at an angle to the horizontal.

Question. Eq[9-10] is the same as what equation in kinematics?

Conservative and NonConservative Forces

In physics, a force is commonly classified as being *conservative* or *non-conservative*. A conservative force is defined in at least three ways:

1 *A conservative force is a force that acts between members of a system that causes no transformation of mechanical energy to internal energy within the system.*

2 *The work done by the agent of a conservative force does not depend on the path followed by the members of the system; the force depends only on the initial and final states of the system.*

3 *The work done by the agent of a conservative force when a member of the system is moved through a closed path is zero.*

Examples of conservative forces are the forces of gravity and electromagnetism. A well-known non-conservative force is the force of friction. These statements require a few words of explanation.

The idea of a conservative force can be understood with reference to the falling book (Figure 9-2). The work done by gravity is given by eq[9-4]. This value depends on the endpoints of the book's travel (y_a and y_b) and not on the path taken. So according to description 2, the force is a conservative one. Moreover, none of the book's initial potential energy is transformed into internal energy; in fact, the potential energy initially possessed by the book is transformed completely into kinetic energy, eq[9-7]. So the force is conservative according to description 1 as well.

On the other hand, the work done by the agent of a force exerted against friction is partially transformed into heat. This work depends on the path taken, the longer the path the greater the work. The force of friction is therefore a non-conservative force.

One characteristic of a conservative force is that a relationship exists between it and a potential energy function. This mathematical fact is very useful. We explore this in more detail in the next section.

Math Unit 7
Relationship Between a Conservative Force and a Potential Energy Function

If a force is conservative then a relationship exists between the force and a potential energy function. To see this we return to the case of the falling book discussed above. We found that the work done on the book by the force of gravity can be expressed as the negative of the difference between the initial and final potential energies of the system of book and Earth:

$$W_{gravity} = mgy_b - mgy_a = -\Delta U_g . \qquad \text{…[9-11]}$$

This expression is characteric of a conservative force. A force is conservative if a potential energy function can be found such that the work done by the force on a member of the system in which the force acts depends only on the difference between the initial and final values of the function. This cannot be done for a non-conservative force because the work done by the agent of the force depends on the path followed between the initial and final positions.

We can therefore derive a very general relationship between a conservative force and its potential energy function. Using the definition of work done by the agent of a force in the x-direction, we can rewrite eq[9-1] as[5]

$$W = \int_{x_i}^{x_f} F_s(x)\,dx = -\Delta U(x)$$

$$= -\left[U(x_f) - U(x_i)\right] = -U(x_f) + U(x_i). \quad \text{…[9-12]}$$

Thus the potential energy function can be written as

$$U(x_f) = -\int_{x_i}^{x_f} F_s(x)\,dx + U(x_i). \qquad \text{…[9-13]}$$

This result makes clear that if we know what the force function *F(x)* is then we can calculate the potential energy function $U(x_f)$ by integration. The value of $U(x_i)$ can be arbitrarily set to zero (or to any other value). This value is not important because it simply shifts the value of $U(x_f)$ by a constant, and it is the change in the potential energy that is meaningful.

Conversely, if we know the potential energy function then the corresponding force function can be found by differentiation. It follows from eq[9-13] that

$$F(x) = -\frac{dU(x)}{dx}. \qquad \text{…[9-14]}$$

This relationship is of great value in solving for either *F(x)* or *U(x)* for a conservative system. Let us consider two examples.

Example Problem 9-2
The Potential Energy Function of the Force of Gravity

Starting from the general expression for a potential energy function. eq[9-13], find the potential energy function for the force of gravity, *mg*, near the surface of the Earth.

Solution:
Writing eq[9-13] in terms of the variable *y* and taking the initial and final positions as y_a and y_b we have

$$U(y_b) = -\int_{y_a}^{y_b} (-mg)\,dy + U(y_a)$$
$$= -[(-mg)y_b - (-mg)y_a] + U(y_a).$$

Now if we let $y_a = 0$ (for the surface of the Earth) and $y_b = y$ and put U(0) = 0 as we did earlier, we have

$$U(y) = mgy .$$

This is obviously the same as eq[9-2].

This expression can also be obtained directly from eq[9-14]. Changing the variable from *x* to *y*, taking y_a = 0 for the surface of the Earth and rearranging,

$$U(y) = -\int_0^y F(y)\,dy = -\int_0^y (-mg)\,dy$$
$$= mgy .$$

Example Problem 9-3
The Potential Energy Function of a Spring

Starting from the general expression for a potential energy function associated with a conservative force, eq[9-13], find the potential energy function for the spring force, $F(x) = -kx$.

Solution:
A spring obeys Hooke's law so $F(x) = -kx$ (Note 08). The potential energy stored in a block-spring system is

[5] This could just as well have been written for the y- or vertical direction of Figure 9-2.

$$U(x_f) = -\int_{x_i}^{x_f} (-kx)dx + U(x_i)$$

$$= \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 + U(x_i).$$

If we let $x_i = 0$ and put $x_f = x$ and $U(0) = 0$ when the block is at the equilibrium position we are left with

$$U(x) = \frac{1}{2}kx^2. \qquad \text{...[9-15]}$$

Notice that by applying eq[9-14] we obtain in reverse the spring force $F(x) = -kx$. Eq[9-15] can be obtained directly from eq[9-14]:

$$U(x) = -\int_0^x F(x)dx = -\int_0^x (-kx)dx$$

$$= \frac{1}{2}kx^2 - 0 = \frac{1}{2}kx^2$$

Gravitational Potential Energy

We have already examined the concept of gravitational potential energy when the system is in essence a single particle moving near the Earth's surface. Here the force of gravity is constant to a good approximation (Figures 9-1 and 9-2). Here too the potential energy function takes a particularly simple form, eq[9-2].

However, $F_g = -mg$ and $U_g(y) = mgy$ are not the most general expressions for the force of gravity or for the corresponding potential energy function. We now broaden our perspective to include the Earth in the system. In the process we shall change the reference or zero position of potential energy of the system <u>from a point on the surface of the Earth to a point infinitely distant from the system</u>.

Consider a particle of mass m moving between two arbitrary positions [A] and [B] above the surface of the Earth (Figure 9-4). We begin by writing the gravitational force in its most general vector form:

$$\vec{F}_g(r) = -\frac{GM_E m}{r^2}\hat{r}. \qquad \text{...[9-16]}$$

$\hat{r}$ is a unit vector directed from the Earth toward the particle. The negative sign indicates that the force is attractive and points downwards toward the center of the Earth. Since the gravitational force is a conservative force we can write it in terms of a potential energy function using eq[9-13]:

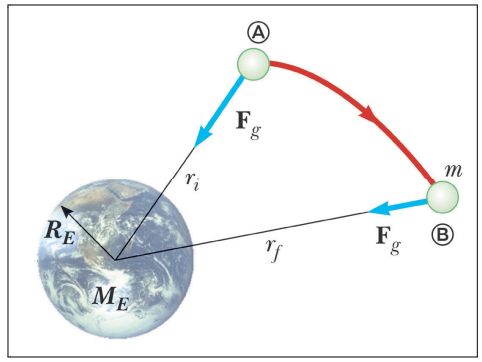

Figure 9-4. *A mass m moves between two positions above the center of the Earth.*

$$U_g(r_f) = -\int_{r_i}^{r_f} F(r)dr + U_g(r_i)$$

$$= GM_E m\int_{r_i}^{r_f} \frac{dr}{r^2} + U_g(r_i)$$

or $$U_g(r_f) = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) + U_g(r_i) \text{....[9-17]}$$

If we take $r_i = \infty$, $r_f = r$ (some arbitrary position) and put $U(\infty) = 0$, eq[9-17] reduces to

$$U_g(r) = -\frac{GM_E m}{r}. \qquad \text{...[9-18]}$$

How $U(r)$ varies with r is illustrated in Figure 9-5.[6] $U(r)$ is seen to be always negative. This is an attribute of a system in which the force is attractive. As $r \to \infty$ $U(r) \to -0$ (where the –ve sign means it approaches zero from negative values). A mass m released above the surface of the Earth will move naturally toward the Earth (downwards in potential energy). As the mass moves toward the Earth the potential energy of the system becomes more and more negative. Because energy is conserved, the kinetic energy of the system becomes more and more positive. At the Earth's surface $r = R_E$ and the potential energy function takes the value

[6] A quick comparison of eqs[9-2] and [9-18] shows them to be quite different. You should be able to explain this difference based on the points of view of system, form of the gravitational force and position of the zero of potential energy.

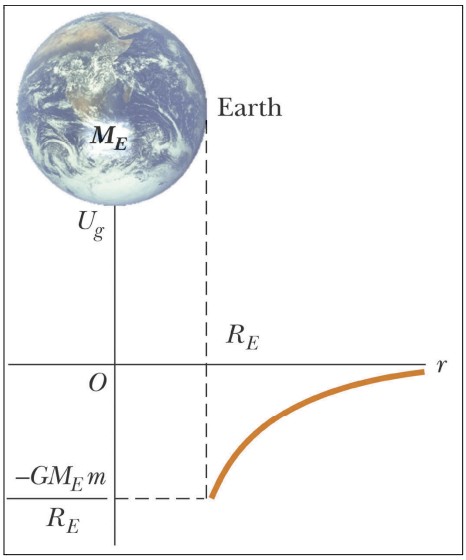

Figure 9-5. *U(r) vs r for gravitational potential energy.*

$$U_g(R_E) = -\frac{GM_E m}{R_E}$$

This treatment can also be followed for the electric force. It will be covered in PHYA21.

Energy Diagrams and Stability of Equilibrium

Information about the large-scale motion of a system can often be deduced from a study of the graph of the system's potential energy function. A case in point is the graph for the block-spring system (Figure 9-6a).

The spring force $F(x)$ is related to a potential energy function $U(x)$ as follows:

$$F(x) = -\frac{dU(x)}{dx} = -kx.$$

This means that the force $F(x)$ is equal to the negative of the slope of the $U(x)$ curve at x. If the block is placed at rest at the equilibrium position, $x = 0$, where $F(x) = 0$, then it will remain there unless some external force acts on it. If the spring is stretched to the right from equilibrium, then x is positive and the slope $dU(x)/dx$ is positive; $F(x)$ is therefore negative and the block will accelerate back toward $x = 0$. If the spring is compressed, then x is negative and the slope is negative; $F(x)$ is therefore positive and again the block will accelerate toward $x = 0$.

These facts indicate that the $x = 0$ position is a special position—a position of *stable equilibrium*. By this is meant that any movement away from this position results in a force directed back toward this position. This is characteristic of a *restoring force*. In general, positions of stable equilibrium correspond to those values for which $U(x)$ has a relative minimum value on an energy diagram.

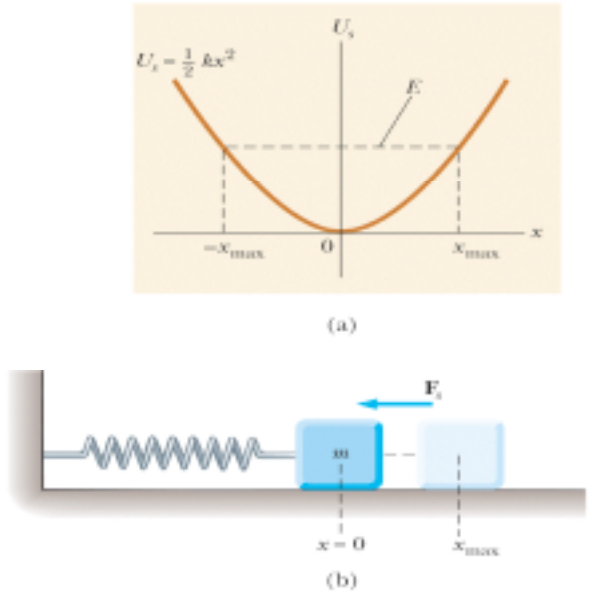

Figure 9-6. *(a) How the potential energy of a system of a mass on a spring varies with x. The equilibrium position, x = 0 is a minimum in the potential energy function and therefore a position of stable equilibrium.*

More details of the motion of the mass on the end of the spring can be deduced from the energy diagram. If we begin by releasing the mass from rest at a position $x = x_{max}$, the potential energy of the system is $(1/2)kx_{max}^2$ and the kinetic energy is zero. Thus when the mass is placed at this position the total energy E of the system is composed of potential energy only:

$$E(x_{max}) = \frac{1}{2}kx_{max}^2.$$

If friction is zero then this value of mechanical energy doesn't change; it is represented by a horizontal line in Figure 9-6a. When the mass is released it moves back towards the equilibrium position. As it passes through the equilibrium position it is moving with maximum speed and the energy is entirely kinetic. The mass moves through this position and comes to rest briefly at the position $x = -x_{max}$. Because E is

constant the mass moves no further to the left than the position $x = -x_{max}$. Subsequently the mass moves back and forth through the equilibrium position with constant energy. This motion is known as an *oscillation* or *vibration*. We shall continue this study in Note 11.

Unstable Equilibrium

Figure 9-7 shows a hypothetical potential energy function. This function goes through a maximum at x = 0. The position x = 0 is a position of equilibrium in the sense that the force on a particle placed there is zero. You should be able to reason using arguments put down in previous sections that if a mass is displaced on either side from x = 0, then it will tend to move away from the position x = 0, not towards. For this reason this position is called a position of *unstable equilibrium*.

A valley and a hill are simple examples of positions of stable and unstable equilibrium, respectively. A round boulder if released at the top of a hill will tend to roll downwards. If it does roll down it will eventually come to rest at a point in the valley.

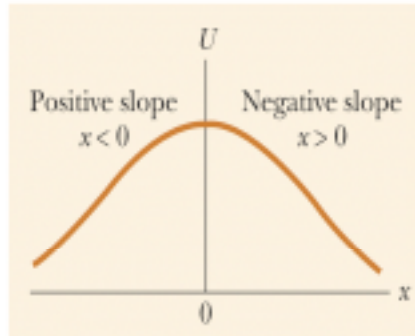

Figure 9-7. *An example of a potential energy function that goes through a maximum at x = 0. The position x = 0 is a position of unstable equilibrium.*

To Be Mastered

- Definitions: *potential energy, isolated system, conservation of mechanical energy*
- Definitions: *conservative force, non-conservative force*
- relationship between a conservative force *F(x)* and a potential energy function *U(x)* (to be committed to memory:

$$F(x) = -\frac{dU(x)}{dx}$$

- Definitions: *position of stable equilibrium, position of unstable equilibrium*

Typical Quiz/Test/Exam Questions

1. (a) Explain what is meant by a conservative force.
   (b) Give one example of a conservative force.
   (c) An object of mass m is dropped from a height *h* above the surface of the Earth. Starting from the principle of conservation of energy, derive an expression for its speed when it hits the ground (neglect air resistance).

2. Give a single definition of the following terms:
   (a) isolated system
   (b) conservative force

3. A force function is given by

$$F(x) = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \frac{x^{10}}{10!} - \frac{x^{12}}{12!}$$

Answer the following questions:

(a) Find the corresponding potential energy function $U(x)$. Take $U(0) = 0$.
(b) Using a spreadsheet program (like Microsoft *Excel*, Microsoft *Works*, *AppleWorks*, etc) plot the potential energy function between x = 0 and x = 6 (approximately). Use a convenient increment, say $\Delta x = 0.1$.
(c) Can you identify positions of stable and unstable equilibrium? If so, at what values of x?

4. Solve problem 3 taking $U(0) = 2$.