4 Force, Work, and Potential Energy — Fall 2003

References

Young and Freedman ***University Physics*** (10th or 11th edition), Chapters 6, 7

Chabay and Sherwood ***Matter and Interactions***, Sections 4.1, 4.3, 4.4

Stewart ***Calculus -- Early Transcendentals*** (4th edition), Sections 5.3, 16.1 through 16.3

Work

In this section we review the concepts of force, work, kinetic energy, and potential energy, and their relations. Let's begin with one-dimensional motion along a straight line. A particle with mass m moves along a straight line (the x axis) under the action of a force F whose x-component depends *only* on the particle's position, not on its velocity or time. That is, F is a function only of x; we denote this relationship as $F(x)$. During a displacement of the particle from x_1 to x_2, the *work* W_{12} done by the force is defined as

$$W_{12} = \int_{x_1}^{x_2} F(x)\, dx. \tag{1}$$

Potential Energy

The work defined in Eq. (1) can be expressed in terms of a *potential-energy* function, and we can think of potential energy as a shorthand way to calculate the work done by a force. Here are two familiar examples.

Example 1: The particle moves along a vertical line in a uniform gravitational field g. The coordinate x is the vertical displacement of the particle above a reference position where $x = 0$. (That is, x is positive when the particle is above the reference position, negative when below.) The force (or, more precisely, the x-component of force) is constant and is given by $F = -mg$. (Do you understand why this is negative?) During a displacement from point x_1 to point x_2, the work W_{12} done by the gravitational force is

$$W_{12} = \int_{x_1}^{x_2} (-mg)\, dx = mgx_1 - mgx_2. \tag{2}$$

We define a potential-energy function $V(x)$ as

$$V(x) = mgx. \tag{3}$$

Then we can express Eq. (2) as

$$W_{12} = V(x_1) - V(x_2). \tag{4}$$

When $x_2 > x_1$, the particle is rising, W_{12} is negative, and $V(x_2)$ is greater than $V(x_1)$. When $x_2 < x_1$, the particle is falling, W_{12} is positive, and $V(x_2)$ is less than $V(x_1)$. So we can think of $V(x)$ as representing an *opportunity* for the force to do work. When the work W done by the gravitational force is positive, some of that opportunity is "used up," and V decreases. When W is negative, V increases.

Example 2: The particle moves along the x axis under the action of a spring that applies a force having an x component that is *directly proportional* to the particle's displacement from an equilibrium position, which we take to be $x = 0$, and always directed *toward* the equilibrium position (i.e., opposite to the displacement). That is,

$$F(x) = -kx. \tag{5}$$

We assume the spring has open coils, so it can be both stretched and compressed, with the same force constant. Then Eq. (5) is valid for both positive and negative values of x.

When the particle moves from point x_1 to point x_2, the work W_{12} done by the force is

$$W_{12} = \int_{x_1}^{x_2} (-kx)\, dx = \tfrac{1}{2} kx_1^{\,2} - \tfrac{1}{2} kx_2^{\,2}. \tag{6}$$

As in Example 1, we define a potential-energy function $V(x)$ as

$$V(x) = \tfrac{1}{2} kx^2. \tag{7}$$

Then Eq. (6) becomes

$$W_{12} = V(x_1) - V(x_2), \tag{8}$$

just as in Example 1.

Note that in each example the force function $F(x)$ and the potential-energy function $V(x)$ are related by

$$F(x) = -\frac{d}{dx} V(x). \tag{9}$$

This relation is valid for *any* force function $F(x)$ and the corresponding potential-energy function $V(x)$, as we will now prove.

A general definition of the potential-energy function $V(x)$ is

$$V(x) = \int_{x}^{x_{\text{ref}}} F(x')\, dx', \tag{10}$$

where x_{ref} is an arbitrary reference point. By definition, $V(x_{\text{ref}}) = 0$. In both of the above examples, $x_{\text{ref}} = 0$ You should verify that both examples are consistent with the general definition given by Eq. (10). This expression represents the work done by the force when the particle moves from an arbitrary initial position x to the reference position x_{ref}, and so it is the opportunity for doing work that is "used up" during this displacement.

If you know how to take the derivative of a definite integral with respect to one of its limits, deriving Eq. (9) from Eq. (10) is trivial. (See Stewart, Section 5.3) If not, consider the following argument. For a very small displacement Δx, we can consider the force $F(x)$ to be nearly constant, so $W_{12} \cong F(x)\,\Delta x$. Then Eq. (4) becomes

$$W_{12} \cong F(x)\,\Delta x = V(x) - V(x + \Delta x), \qquad \text{or} \qquad F(x) = -\frac{V(x + \Delta x) - V(x)}{\Delta x}. \quad (11)$$

In the limit as $\Delta x \to 0$, this becomes Eq. (9).

Work and Kinetic Energy

The kinetic energy K of a particle with mass m is defined as

$$K = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \dot{x}^2, \quad (12)$$

where $v = \dot{x} = \dfrac{dx}{dt}$ is the particle's instantaneous velocity (or, more precisely, the x-component of its instantaneous velocity). If the particle has velocity v_1 at position x_1 and velocity v_2 at position x_2, its kinetic energies at these two points are K_1 and K_2, where

$$K_1 = \tfrac{1}{2} v_1^{\,2} \qquad \text{and} \qquad K_2 = \tfrac{1}{2} v_2^{\,2} \ .$$

The *work-energy theorem*, derived from Newton's second law, states that

$$K_2 - K_1 = W_{12}, \qquad \text{or} \qquad \tfrac{1}{2} v_2^{\,2} - \tfrac{1}{2} v_1^{\,2} = W_{12} = \int_{x_1}^{x_2} F\,dx. \quad (13)$$

If F is the total force acting on the particle, Eq. (13) is *always* true, irrespective of the nature of the force, which may be constant or may depend on position (x), velocity $(v = \dot{x},)$ time (t) or all of these.

In the special case when F depends *only* on x, W_{12} can *always* be expressed in terms of a potential energy function $V(x)$, as in Eqs. (4) and (8). Abbreviating $V(x_1)$ as V_1, and so on, we can rewrite Eq. (13) as

$$K_2 - K_1 = W_{12} = V_1 - V_2, \qquad \text{or} \qquad K_1 + V_1 = K_2 + V_2. \quad (14)$$

The points x_1 and x_2 are arbitrary, so Eq. (14) shows that the sum of kinetic and potential energies $(K + V)$ is the same at all points of the motion. We call this the *total energy* E, and Eq. (14) says that in this case $K + V = E =$ constant, an example of conservation of energy. But if the total force F depends explicitly on time or velocity, then there is no such thing as a potential-energy function, and Eq. (14) is not valid.

Equilibrium

A particle is in *equilibrium* if the net force F acting on it is zero. If it is at rest at such a point, it stays there forever because it has zero acceleration and so can never begin to move. The equilibrium may be *stable* or *unstable*. At a stable equilibrium point, when the particle is displaced slightly, the direction of the force is such as to tend to push it back toward the equilibrium point. That is, for small displacements, the force is always *opposite* in direction to the displacement.

At an unstable equilibrium point, a particle displaced slightly from equilibrium experiences a force directed away from the equilibrium position (i.e., the *same* direction as the displacement), and the particle tends to move farther and farther from the equilibrium point..

The conditions for stable and unstable equilibrium can be expressed simply in terms of the force and potential energy functions $F(x)$ and $V(x)$. Suppose x_o is an equilibrium point, so that $F(x_o) = 0$. We expand $F(x)$ in a Taylor series about this point:

$$F(x) = F(x_o) + \left[\frac{dF}{dx}\right]_{x=x_o}(x - x_o) + \frac{1}{2!}\left[\frac{d^2F}{dx^2}\right]_{x=x_o}(x - x_o)^2 + \cdots . \tag{15}$$

The first term is zero because x_o is an equilibrium point. If $x - x_o$ is very small, then the second term is the dominant one. Thus for points very near the equilibrium position,

$$F(x) \cong \left[\frac{dF}{dx}\right]_{x=x_o}(x - x_o) \quad . \tag{16}$$

We see that if dF/dx is *positive* at x_o, F is positive when x is greater than x_o and negative when x is less than x_o. In each case the force tends to push the particle farther from the equilibrium position. We conclude that when dF/dx is *positive* at x_o , this is a point of *unstable* equilibrium. A similar argument for the opposite case shows that when dF/dx is *negative* at x_o, it is a point of *stable* equilibrium.

These conditions can also be expressed in terms of the potential-energy function $V(x)$. Expanding it in a Taylor series about x_o, we get

$$V(x) = V(x_o) + \left[\frac{dV}{dx}\right]_{x=x_o}(x - x_o) + \frac{1}{2!}\left[\frac{d^2V}{dx^2}\right]_{x=x_o}(x - x_o)^2 + \cdots . \tag{17}$$

From Eq. (9), the second term is zero because $F(x_o) = 0$. In the third term the derivative is equal to $(-dF/dx)_{x = xo}$. We conclude that the above conditions can be written in terms of the *sign* of the value of d^2V/dx^2 at the equilibrium point $x = x_o$:

$$\left[\frac{d^2V}{dx^2}\right]_{x=x_o} > 0 \Leftrightarrow \text{stable equilibrium}, \quad \left[\frac{d^2V}{dx^2}\right]_{x=x_o} < 0 \Leftrightarrow \text{unstable equilibrium}. \tag{18}$$

In exceptional cases, one or more derivatives of $F(x)$ may be zero at x_o. It turns out that if the first non-vanishing derivative of $F(x)$ is of *even* order, the equilibrium is *unstable*; if it is of *odd* order, the equilibrium is *stable*.

These relationships between $F(x)$ and $V(x)$ have a simple graphical interpretation. At each point, the force F is the negative of the slope of the V curve. At a point of stable equilibrium, $F(x)$ is zero and $V(x)$ has a *minimum*. Such a situation is often called a *potential well*.

Equations (15), (16), and (17) show that when x is very close to x_o, the force is approximately *directly proportional* to the displacement $(x - x_o)$ from equilibrium, with proportionality constant k, where

$$k = -\left[\frac{dF}{dx}\right]_{x=x_o} = \left[\frac{d^2V}{dx^2}\right]_{x=x_o}. \tag{19}$$

(The negative sign is included so that k is a positive quantity.) This direct proportionality of force and displacement is characteristic of simple harmonic motion; therefore the resulting motion is approximately *simple harmonic*, with angular frequency

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{-\left[dF/dx\right]_{x=x_o}}{m}} = \sqrt{\frac{\left[d^2V/dx^2\right]_{x=x_o}}{m}}. \tag{20}$$

Caution

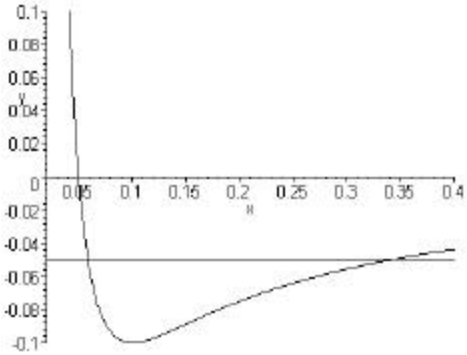
This discussion of motion near a stable equilibrium point x_o depends on the derivatives in Eq. (19) being non-zero at $x = x_o$. In exceptional cases, they may be zero at $x = x_o$. Then the first nonvanishing term in the Taylor expansion of $F(x)$ in Eq. (15) contains $(x - x_o)^2$ or even a higher power of $(x - x_o)$. In this case, there is *no* approximation in which F is directly proportional to $(x - x_o)$, and the motion is not even approximately simple harmonic, even for the smallest displacements from equilibrium.

Example

As an example of some of the foregoing discussion, consider a potential-energy function

$$V(x) = 0.1\left(\frac{0.01}{x^2} - \frac{0.2}{x}\right),$$

with x measured in nanometers and V in electronvolts. This might be an approximate potential-energy function for the atoms in a diatomic molecule. A potential-energy function with this general shape is often called a *potential well*. We would like to draw graphs of $V(x)$ and the corresponding force $F(x)$, show that there is a stable equilibrium point x_o, determine its location, and find the value of V at the equilibrium point x_o. Here is an outline of the calculation; you should try to fill in the details, using Maple.



Use Maple to obtain a graph similar to the one at the left. The force acting on the particle is given by Eq. (9). Calculate the derivative and set it equal to zero to find $x_o = 0.10$ nm. Substitute this back into $V(x)$ to find that at the equilibrium point, $V = -0.10$ eV. Also take the second derivative of V, to show that at $x = x_o$ this derivative is positive (showing that x_o is a point of *stable* equilibrium).

If the *total* energy E is greater than zero, x can become indefinitely large, but when it is less than zero, x is limited to a finite range.

As an additional exercise, suppose that the total energy of the system is $E = -0.05$ eV, as shown by the horizontal line. The points where this line intersects the $V(x)$ curve are the points where the potential energy equals the total energy and hence the *kinetic* energy is zero. (I.e., the particle stops and reverses direction.) Hence these points represent the limits of the motion. Use Maple to solve for the values of x at these points and show that $x_{\min} = 0.059$ nm and $x_{\max} = 0.341$ nm.

Motion in a Potential Well

When a particle moves in a potential well, as in the above example, the motion is confined to a finite range of values of x if the energy is sufficiently small. In this case the particle's motion is *periodic*; it moves from $x_{\min}$ to $x_{\max}$ and back, with a definite period T (time for one complete cycle). We can use energy considerations to derive a general expression for the period. First, the time dt required to move a distance dx is given by $dt = dx/v$. From the energy relation, $E = K + V$, we get

$$\tfrac{1}{2}mv^2 + V(x) = E. \tag{21}$$

Solving this for v, we get

$$v = \sqrt{\frac{2}{m}\left[E - V(x)\right]}, \qquad \text{so} \tag{22}$$

$$dt = \frac{dx}{v} = \sqrt{\frac{m}{2}}\,\frac{dx}{\sqrt{E - V(x)}}. \tag{23}$$

The time for the particle to go from $x_{\min}$ to $x_{\max}$ is the integral of this expression, and the period (the total time for the round trip) is twice that. Thus the period is given by

$$T = \sqrt{2m}\int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{E - V(x)}}. \tag{24}$$

Note that this is a general result, not limited to small oscillations. The small-oscillation approximation was not used in its derivation. In general the period T of the motion depends on the total energy E.

Three Dimensions

When a particle moves in two or three dimensions, the generalized definition of work, corresponding to Eq. (1), involves a *line integral*. Consider a particle acted on by a force that depends only on the particle's position (position vector $\vec{\boldsymbol{r}}$). We denote this force as $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$. When the particle moves from a point with position vector $\vec{\boldsymbol{r}}_1$ to a point with position vector $\vec{\boldsymbol{r}}_2$, the work W_{12} done by the force is defined as

$$W_{12} = \int_{\vec{r}_2}^{\vec{r}_1} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \cdot d\vec{\boldsymbol{r}}. \tag{25}$$

With this more general definition of work, Eq. (13) is still valid. That is,

$$\tfrac{1}{2} v_2^{\,2} - \tfrac{1}{2} v_1^{\,2} = W_{12} = \int_{\vec{r}_2}^{\vec{r}_1} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \cdot d\vec{\boldsymbol{r}}. \tag{26}$$

There is a significant added complication in this case because there are many different *paths* leading from point $\vec{\boldsymbol{r}}_1$ to point $\vec{\boldsymbol{r}}_2$. In general the integral in Eq. (26) is different for different paths.

However, there is a class of position-dependent forces for which Eq. (25) *is* independent of path. Such a force is called a *conservative force field*. In this case we can define a potential energy function in analogy to Eq. (10). We choose a reference point with position vector $\vec{\boldsymbol{r}}_{\text{ref}}$, and we define

$$V(\vec{\boldsymbol{r}}) = \int_{\vec{r}}^{\vec{r}_{\text{ref}}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,') \cdot d\vec{\boldsymbol{r}}\,'. \tag{27}$$

This definition is unambiguous if the integral is independent of path. The line integral of a conservative force around a *closed* path is always zero. Can you prove this?

From this definition of $V(\vec{\boldsymbol{r}})$, it can be shown that

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = -\text{grad}\, V(\vec{\boldsymbol{r}}) = -\nabla V(\vec{\boldsymbol{r}}). \tag{28}$$

This equation is the three-dimensional generalization of Eq. (9).

Finally, Stokes' theorem can be used to show that if $\nabla \times \vec{\boldsymbol{F}} = \mathbf{0}$ everywhere in a region, the force is conservative for all points in the region.

Thus there are four equivalent definitions of a conservative force in three dimensions:

(1) The line integral of $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ is independent of path, for all paths in a region.

(2) The line integral of $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ around every closed path in a region is zero.

(3) There is a potential-energy function $V(\vec{\boldsymbol{r}})$ such that

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = -\text{grad}\, V(\vec{\boldsymbol{r}}) = -\nabla V(\vec{\boldsymbol{r}}).$$

(4) $\nabla \times \vec{\boldsymbol{F}} = \mathbf{0}$ everywhere in the region.