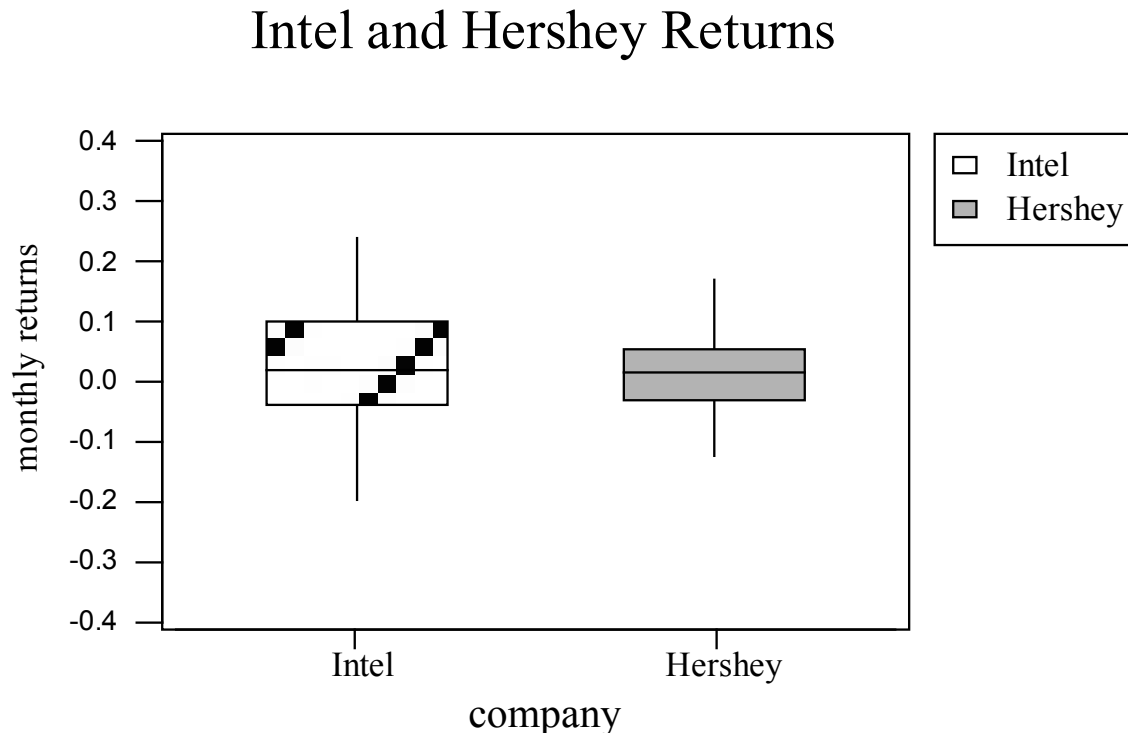


# 5. STANDARD DEVIATIONS

The **standard deviation** is the most widely used measure of dispersion (fluctuation, heterogeneity, risk, volatility).

**Eg 1:** Consider the monthly returns of Intel and Hershey Foods, over a period of 10 years.



The boxplots indicate that Intel has a higher mean return, but is much more volatile than Hershey.

Which stock is a better investment?

To answer this, we need a more refined measure of risk. The IQR is not adequate here, since it ignores the most extreme 50% of the returns.

These are the months of the largest profits and losses!

The standard deviation uses all available data, and therefore provides a more sensitive measure of volatility.

For a population  $x_1, \dots, x_N$  we define the **population standard deviation**

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2}$$

- $\sigma$  = Root Mean Squared deviation of the population values from their mean  $\mu$ .

Explanation:

- 1) For each data point  $x_i$  compute the deviation  $x_i - \mu$
- 2) Square the deviations to make them nonnegative
- 3) Average the squared deviations
- 4) Take the square root to retain the original units of measurement

- The standard deviation is a very important concept in Statistics since it is the basic tool for summarizing the amount of randomness in a situation.

The **population variance** is the square of the population standard deviation:

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2$$

- Despite its name, the variance does not measure fluctuation; the S.D. does.

To see why, consider:

- 1) If we double all values, what happens to the amount of “fluctuation” or “volatility”? To the S.D.? To the variance?
- 2) If the data are prices in dollars, what are the units of  $\mu$ ? Of  $\sigma$ ? Of  $\sigma^2$ ?

- Variance is nevertheless sometimes used in Finance for measuring risk.

Given a sample  $x_1, \dots, x_n$  from the population, we ask: How much do the data values fluctuate from their mean,  $\bar{x}$ ?

Measure the dispersion of a sample by the **sample standard deviation**:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

The square of the sample standard deviation,

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

is called the **sample variance**.

Again,  $s^2$  does not measure dispersion;  $s$  does.

- Reason for using  $n - 1$ : To ensure that  $s^2$  neither underestimates nor overestimates  $\sigma^2$ , on the average.

### **[R Demo: SampleVariance]**

- Aside from the  $n - 1$ , there is a strong similarity between the definitions for  $s$  and  $\sigma$ .

**Eg 2:** The earnings per share (EPS) for FedEx in the first quarter were \$2.66, failing to meet analysts' expectations of \$2.70 (consensus forecast). The earnings surprise for FedEx was  $2.66 - 2.70 = -.04$ . Is this bad compared to the earnings surprise of five other NYSE companies observed in a similar time period?

<b>Company</b>	<b>Surprise</b>
Kroger	.04
Red Hat	.04
Rite Aid	.02
Pier 1	.00
Oracle	−.08



**Preliminary Calculations:** The other earnings surprise values are .04, .04, .02, 0,  $-.08$ . The sample mean is

$$\bar{x} = (1/5)(.04 + .04 + .02 + 0 - .08) = .02/5 = .004.$$

The sample variance is  $s^2 = (1/4)[(.04 - .004)^2 + (.04 - .004)^2 + (.02 - .004)^2 + (0 - .004)^2 + (-.08 - .004)^2]$

$$= .00992/4 = .00248.$$

The sample standard deviation is  $s = \sqrt{.00248} = .0498$ .

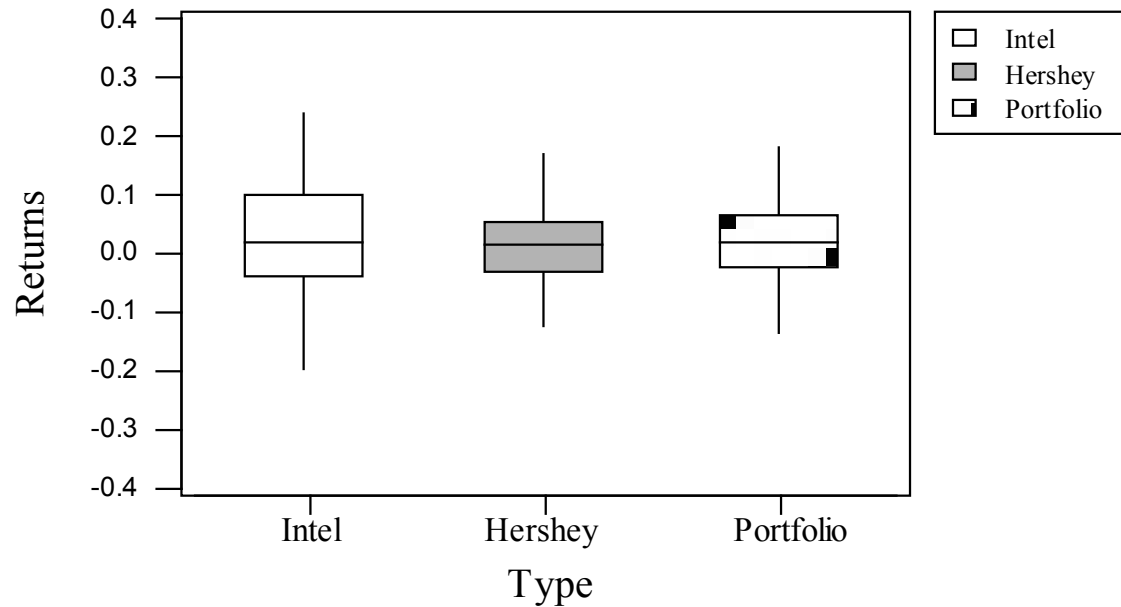
How does this help us put the surprise for FedEx ( $-.04$ ) into context? We'll return to this later, after we define the Z-score.

# **Risk-Adjusted Returns and Portfolio Analysis**

In Finance, the risk-adjusted return for an investment is defined as the ratio of the mean to the standard deviation. This can be useful for selecting a portfolio.

In Example 1, we considered the monthly returns for Intel Corp and Hershey Foods. Let's also consider a portfolio consisting of an equal dollar amount of the stocks.

## Intel, Hershey, and Portfolio Returns



The investments performed as follows:

	<u>Intel</u>	<u>Hershey</u>	<u>Portfolio</u>
Mean Return, $\bar{x}$	0.0169	0.0089	0.0129
Standard Deviation, $s$	0.1349	0.0735	0.0826
Risk-Adjusted Return $\bar{x}/s$ ,	0.1253	0.1211	0.1562

Intel had a higher mean return (1.7% per month) than Hershey (0.9% per month), but Intel was much more volatile. In terms of risk-adjusted return, the two stocks are roughly equivalent, and both are inferior to the portfolio.

Note that the portfolio provides a mean return which is the average of that of the two stocks, but achieves (through *diversification*) a standard deviation close to that of the less risky stock, Hershey.

The main point of diversification is to reduce risk, and here the portfolio achieved this goal.

# Z-Scores

Suppose  $x$  is a value, or “score”, which we wish to compare to a data set with mean  $\bar{x}$  (or  $\mu$ ) and standard deviation  $s$  (or  $\sigma$ ).

$x$  may or may not be a part of the data set.

Define the **z-score** for  $x$  as  $z = \frac{x - \bar{x}}{s}$  or  $z = \frac{x - \mu}{\sigma}$

- The z-score tells us how many standard deviations  $x$  is from the mean.
- The further  $z$  is from zero, the more “atypical”  $x$  is, relative to the given data set. In fact, the “**empirical rule**” states that for normal distributions:
  - about 68% of the data values will have z-scores between  $\pm 1$ ,
  - about 95% between  $\pm 2$ , and
  - about 99.7% (i.e., almost all) between  $\pm 3$ .

Thus, for example, roughly 95% of the data should lie within two standard deviations of the mean, that is, in the interval

$$(\bar{x} - 2s, \bar{x} + 2s)$$

The empirical rule is only exactly correct for normal distributions (using known population means and variances), but in some cases it provides a useful approximation, even for non-normal distributions.

It is worth memorizing the empirical rule.

**Eg 2, contd.:** FedEx had an earnings surprise of  $-.04$ . Relative to the other 5 firms, having  $\bar{x} = .004$ ,  $s = .0498$ , the  $z$ -score for FedEx is

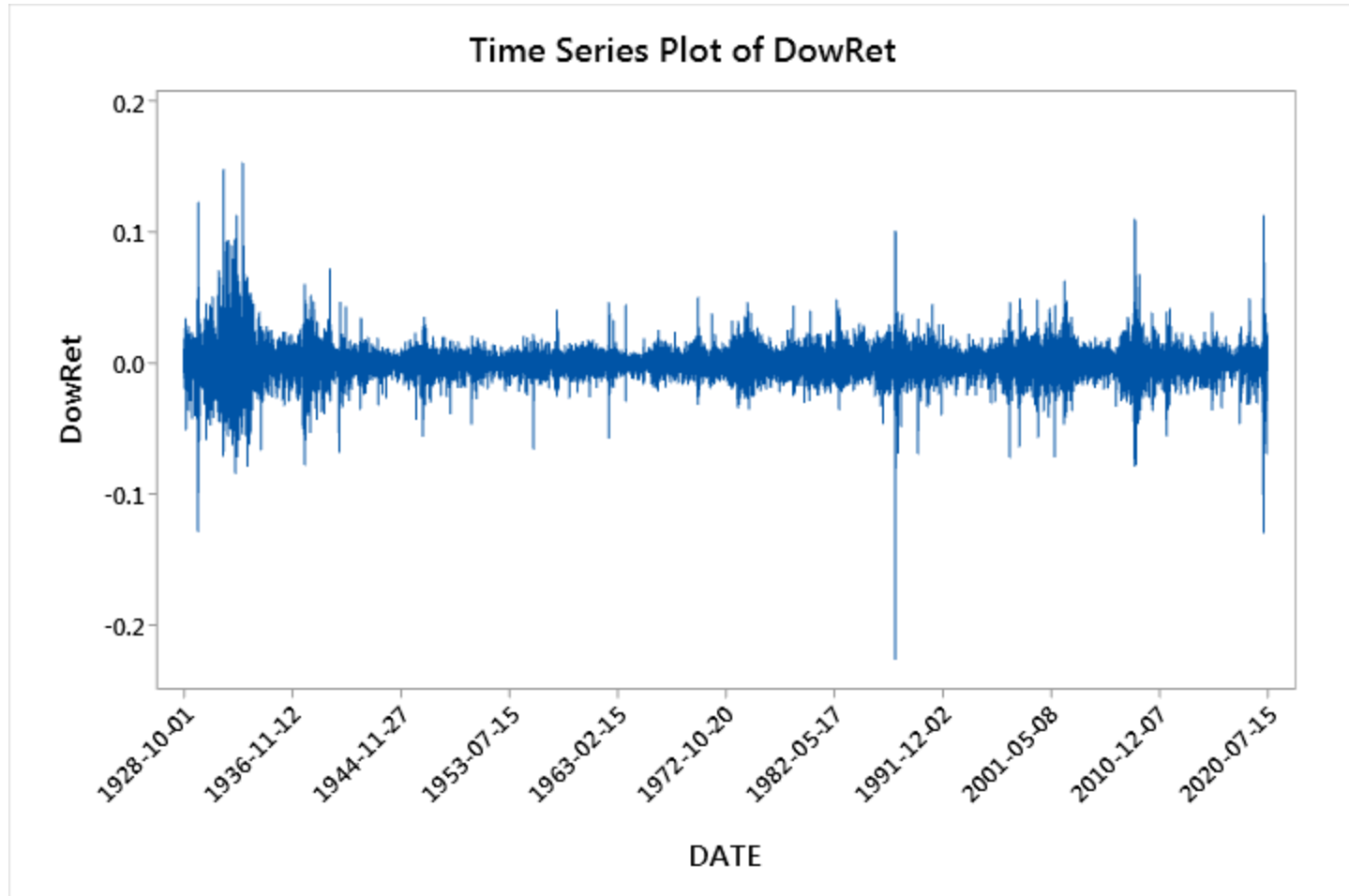
$$z = \frac{x - \bar{x}}{s} = \frac{-.04 - .004}{.0498} = -.88 \quad .$$

The earnings surprise for FedEx was less than one standard deviation below the mean, and therefore quite typical.

The empirical rule helps us to put variation in context. But variation cannot be completely tamed.

Overadjustment Increases Variation: Deming's "Funnel" Experiment. (In-Class Demo).

**Eg:** Consider the daily returns on the Dow.





Minitab's descriptive statistics gives the following output.

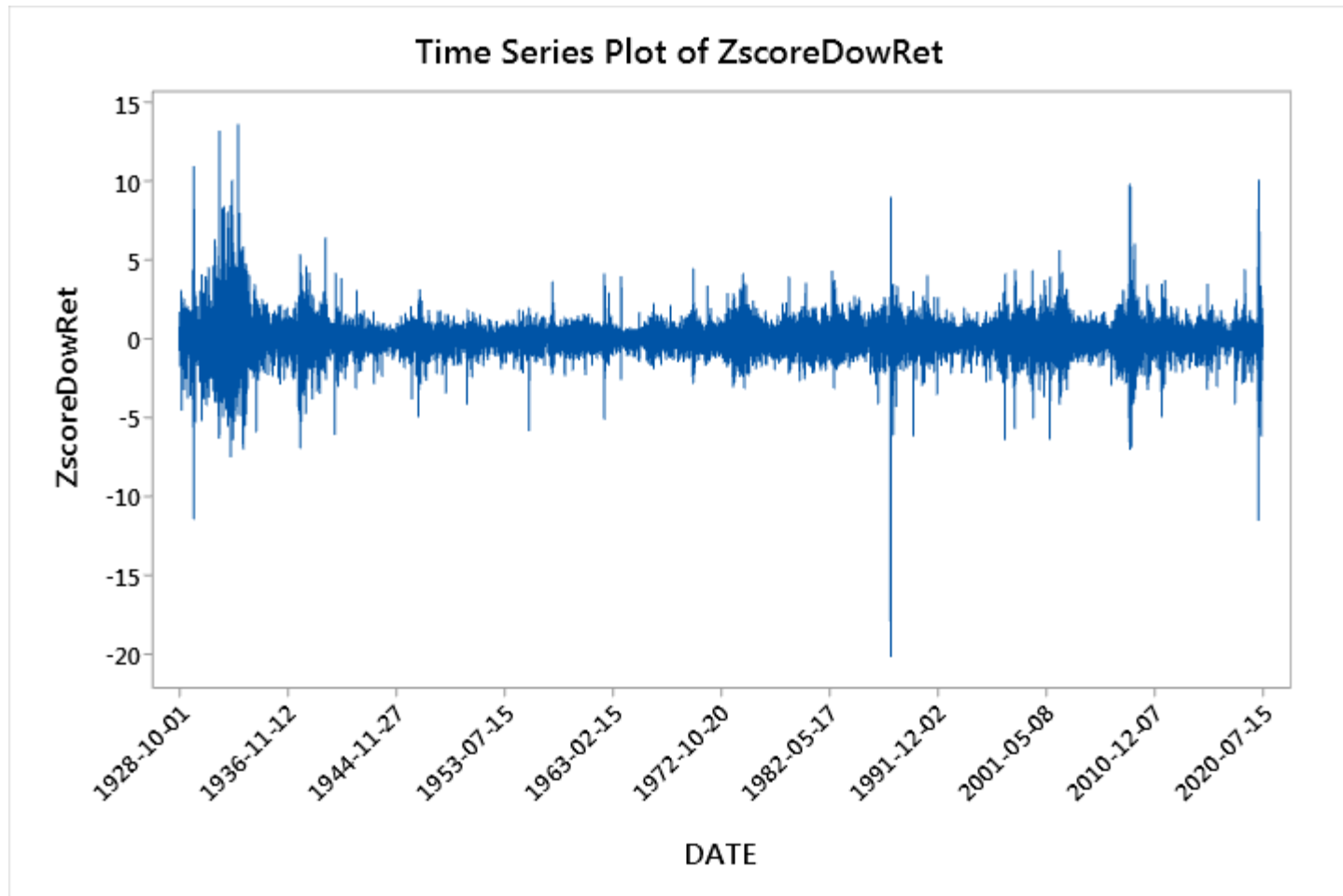
### Statistics

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3
DowRet	24144	1	0.000259	0.000072	0.011254	-0.226105	-0.004380	0.000415	0.005175
Variable		Maximum							
DowRet		0.153418							

Using Calculator in Minitab, we find that the proportion of all observations within 1, 2, 3 standard deviations of the mean is 0.814, 0.955, and 0.982. Compared with Empirical Rule, there is too much data in the middle and too much data in the tails.

The daily return on the Dow for the day of March 16, 2020 was  $-0.129$ . The corresponding z-score for that day was  $-11.5$ . This is an amazingly large z-score for supposedly normally distributed stock returns, as we will discuss later.

# Z-Scores of the Dow Returns



# Variability in Practice

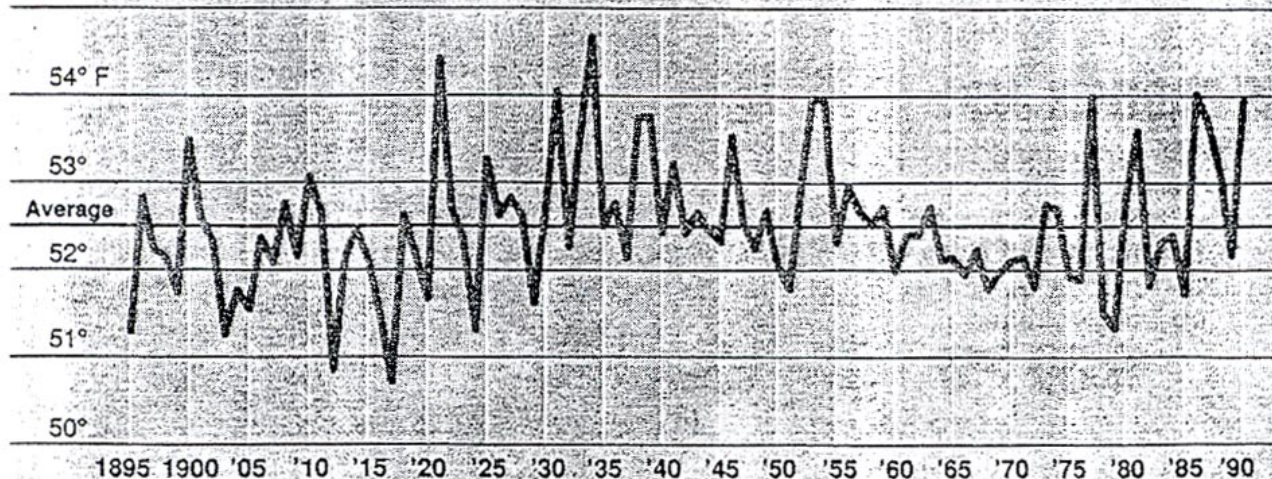
Eg: The natural variability in temperatures.

## *In the U.S. Last Year, the Heat Was On*

The average temperature of the contiguous 48 states, although relatively high, was well within the range of natural variability as expressed in the 96-year record.

### **1990: Warm, but Not the Warmest**

Annual mean temperature in the contiguous 48 states.



Source: National Climate Data Center, National Oceanographic and Atmospheric Administration

The New York Times, January 3, 1991