CrossMark

A summation due to Ramanujan revisited

DONG JUN JEE, TUO YEONG LEE*
and NATHANIEL ZHI-WEI LEON

Department of Mathematics, Statistics and Computing, NUS High School
of Mathematics and Science, 20 Clementi Avenue 1, Singapore 129957,
Republic of Singapore
*Corresponding author.
E-mail: h1130010@nushigh.edu.sg; nhsleety@nus.edu.sg; h0910083@nhshigh.edu.sg

Abstract.   We employ Hardy's regularly convergent double series to refine an argument of Nanjundiah [7]. In particular, we evaluate some alternating series.

Keywords.   Regularly convergent double series; alternating series; hyperbolic functions.

2010 Mathematics Subject Classification.   Primary: 40B05; Secondary: 40A05.

1. Introduction

The following result

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^5} \operatorname{sech} \frac{(2j+1)\pi}{2} = \frac{\pi^5}{768} \tag{1}$$

originates in a letter of Ramanujan to Hardy; see p. xxvi of [8] or p. 295 of [1]. Later, Watson [9] gave a proof of (1) using contour integration. In 1951, Nanjundiah [7] provided a real-variable proof of (1) via the partial fraction decomposition of the hyperbolic secant function; for each nonnegative integer q his proof involved the following double series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k}(2j+1)}{(2k+1)^{2q+1}((2j+1)^2 + (2k+1)^2)} \tag{2}$$

and the equality

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k}(2j+1)}{(2k+1)^{2q+1}((2j+1)^2 + (2k+1)^2)}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{j+k}(2j+1)}{(2k+1)^{2q+1}((2j+1)^2 + (2k+1)^2)}. \tag{3}$$

However, since the double series (2) is non-absolutely convergent, the proof of (3) seems to be overlooked by Nanjundiah. The main aim of this paper is to provide a proof of (3) using Hardy's regularly convergent double series.

2. Hardy's regularly convergent double series

Let $\mathbb{N}_0$ be the set of all nonnegative integers. We begin with the following definition due to Hardy [3].

DEFINITION 2.1 (Definition 8.1.4 of [5])

A double series $\sum_{(j,k)\in\mathbb{N}_0^2} u_{j,k}$ of real numbers is said to be regularly convergent if for each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}_0$ such that

$$\left| \sum_{j=p_1}^{q_1} \sum_{k=p_2}^{q_2} u_{j,k} \right| < \varepsilon$$

for all $(p_1, p_2), (q_1, q_2) \in \mathbb{N}_0^2$ with $q_i \geq p_i$ $(i = 1, 2)$ and $\max\{p_1, p_2\} \geq N(\varepsilon)$.

Obviously, every absolutely convergent double series of real numbers is regularly convergent. Morever, we infer from Theorem 8.1.9 of [5] that if the double series $\sum_{(j,k)\in\mathbb{N}_0^2} u_{j,k}$ of real numbers is regularly convergent, then all the single series $\sum_{j=0}^{\infty} u_{j,k}$ $(k = 0, 1, \dots)$ and $\sum_{k=0}^{\infty} u_{j,k}$ $(j = 0, 1, \dots)$ converge.

The following corollary shows that Fubini's theorem holds for regularly convergent double series.

COROLLARY 2.2 (Corollary 8.1.10 of [5])

If the double series $\sum_{(j,k)\in\mathbb{N}_0^2} u_{j,k}$ of real numbers converges regularly, then the iterated series $\sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} u_{j,k} \right\}, \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{\infty} u_{j,k} \right\}$ converge and

$$\sum_{(j,k)\in\mathbb{N}_0^2} u_{j,k} = \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} u_{j,k} \right\} = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{\infty} u_{j,k} \right\}.$$

For any double sequence $(v_{j,k})_{(j,k)\in\mathbb{N}_0^2}$ of real numbers, we set

$$\Delta_{\{1\}}(v_{j,k}) = v_{j,k} - v_{j+1,k}, \quad \Delta_{\{2\}}(v_{j,k}) = v_{j,k} - v_{j,k+1}$$

and

$$\Delta_{\{1,2\}}(v_{j,k}) := \Delta_{\{1\}}(\Delta_{\{2\}}(v_{j,k})) = v_{j,k} - v_{j,k+1} + v_{j+1,k+1} - v_{j+1,k}.$$

We are now ready to state a generalized Dirichlet's test for double series.

Theorem 2.3 (cf. Theorem 2.3 of [4] or Theorem 8.1.13 of [5]). *Let $(u_{j,k})_{(j,k)\in\mathbb{N}_0^2}$ and $(v_{j,k})_{(j,k)\in\mathbb{N}_0^2}$ be two double sequences of real numbers such that $\lim_{\max\{n_1,n_2\}\to\infty} v_{n_1,n_2} = 0$ and*

$$\sum_{(j,k)\in\mathbb{N}_0^2} \left|\Delta_{\{1,2\}}(v_{j,k})\right| \max_{\substack{r_0=0,\dots,j \\ s_0=0,\dots,k}} \left|\sum_{r=0}^{r_0}\sum_{s=0}^{s_0} u_{r,s}\right| < \infty. \tag{4}$$

Then the double series $\sum_{(j,k)\in\mathbb{N}_0^2} \Delta_{\{1,2\}}(v_{j,k})\sum_{r=0}^{j}\sum_{s=0}^{k} u_{r,s}$ *is regularly convergent, the double series* $\sum_{(j,k)\in\mathbb{N}_0^2} u_{j,k}v_{j,k}$ *is absolutely convergent, and*

$$\sum_{(j,k)\in\mathbb{N}_0^2} u_{j,k}v_{j,k} = \sum_{(j,k)\in\mathbb{N}_0^2} \Delta_{\{1,2\}}(v_{j,k})\sum_{r=0}^{j}\sum_{s=0}^{k} u_{r,s}. \tag{5}$$

3. Main results

We set $\eta(s) := \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^s}$ $(s>0)$. Using the idea of Nanjundiah (p. 216 of [7]), Theorem 2.3 and Corollary 2.2, we obtain the following result.

Theorem 3.1. *If* $q \in \mathbb{N}_0$, $a>0$ *and* $b>0$, *then*

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{2q+1}} \left\{ a^{2q}\operatorname{sech}\frac{(2j+1)\pi b}{2a} + (-1)^q b^{2q}\operatorname{sech}\frac{(2j+1)\pi a}{2b} \right\}$$

$$= \frac{4}{\pi}\sum_{r=0}^{q}(-1)^r\eta(2r+1)\eta(2q-2r+1)a^{2q-2r}b^{2r}.$$

Proof. First, we use the following partial fraction

$$\operatorname{sech}\frac{(2j+1)\pi x}{2} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{(2j+1)^2x^2+(2k+1)^2}$$

to write

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{2q+1}} \left\{ a^{2q}\operatorname{sech}\frac{(2j+1)\pi b}{2a} + (-1)^q b^{2q}\operatorname{sech}\frac{(2j+1)\pi a}{2b} \right\}$$

$$= \frac{4a^{2q+2}}{\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2q+1}}\left\{\frac{(-1)^k(2k+1)}{(2j+1)^2b^2+(2k+1)^2a^2}\right\}$$

$$+(-1)^q\frac{4b^{2q+2}}{\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2q+1}}\left\{\frac{(-1)^k(2k+1)}{(2j+1)^2a^2+(2k+1)^2b^2}\right\}. \tag{6}$$

Next, we consider the first iterated series appearing on the right-hand side of (6). Clearly, this particular sum satisfies the following equality

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2q+1}}\left\{\frac{(-1)^k(2k+1)}{(2j+1)^2b^2+(2k+1)^2a^2}\right\}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2q+1}} \left\{ \frac{(-1)^j(2j+1)}{(2k+1)^2 b^2 + (2j+1)^2 a^2} \right\}. \tag{7}$$

If we have

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2q+1}} \left\{ \frac{(-1)^j(2j+1)}{(2k+1)^2 b^2 + (2j+1)^2 a^2} \right\}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2q+1}} \left\{ \frac{(-1)^j(2j+1)}{(2k+1)^2 b^2 + (2j+1)^2 a^2} \right\}, \tag{8}$$

then the desired conclusion follows from the right-hand side of (6):

$$\frac{4a^{2q+2}}{\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2q+1}} \left\{ \frac{(-1)^j(2j+1)}{(2k+1)^2 b^2 + (2j+1)^2 a^2} \right\}$$

$$+ (-1)^q \frac{4b^{2q+2}}{\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j}{(2j+1)^{2q+1}} \left\{ \frac{(-1)^k(2k+1)}{(2j+1)^2 a^2 + (2k+1)^2 b^2} \right\}$$

$$= \frac{4}{\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k}}{(2j+1)^{2q+1}(2k+1)^{2q+1}}$$

$$\times \left\{ \frac{(a^2(2j+1)^2)^{q+1} - (-b^2(2k+1)^2)^{q+1}}{(2j+1)^2 a^2 + (2k+1)^2 b^2} \right\}$$

$$= \frac{4}{\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \sum_{r=0}^{q} \frac{(-1)^r a^{2q-2r} b^{2r}}{(2j+1)^{2r+1}(2k+1)^{2q-2r+1}}$$

$$= \frac{4}{\pi} \sum_{r=0}^{q} (-1)^r \eta(2r+1)\eta(2q-2r+1) a^{2q-2r} b^{2r}.$$

It remains to establish (8). In view of Corollary 2.2, it suffices to prove that the following non-absolutely convergent double series

$$\sum_{(j,k)\in\mathbb{N}_0^2} \frac{(-1)^k}{(2k+1)^{2q+1}} \left\{ \frac{(-1)^j(2j+1)}{(2k+1)^2 b^2 + (2j+1)^2 a^2} \right\} \tag{9}$$

is regularly convergent. Since we have

$$\Delta_{\{1,2\}} \left(\frac{1}{a^2(2j+1)^2 + b^2(2k+1)^2} \right)$$

$$= \frac{128a^2 b^2 (j+1)(k+1)\left(a^2(2j+1)^2 + b^2(2k+1)^2 + a^2 + b^2\right)}{\prod_{r=0}^{1}\prod_{s=0}^{1}\left(a^2(2j+2r+1)^2 + b^2(2k+2s+1)^2\right)},$$

we deduce that the following double series

$$\sum_{(j,k)\in\mathbb{N}_0^2} \Delta_{\{1,2\}} \left(\frac{1}{a^2(2j+1)^2 + b^2(2k+1)^2} \right) \max_{\substack{r_0 = 0,\dots,j \\ s_0 = 0,\dots,k}} \left| \sum_{r=0}^{r_0} \sum_{s=0}^{s_0} \frac{(-1)^{r+s}(2r+1)}{(2s+1)^{2q+1}} \right|$$

is absolutely convergent. Finally, since

$$\lim_{\max\{m,n\}\to\infty} \frac{1}{a^2(2m+1)^2 + b^2(2n+1)^2} = 0,$$

an application of Theorem 2.3 completes the argument. □

Theorem 3.2 [8]. $\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^5} \operatorname{sech} \frac{(2j+1)\pi}{2} = \frac{\pi^5}{768}$

Proof. The proof is similar to that of Nanjundiah [7]. Using Theorem 3.1 with $q = 2$, we have the result. □

The proof of Theorem 3.1 yields a generalization of p. 295, Entry 25(vii) of [1].

Theorem 3.3. *Let d be any positive integer. Then*

$$\sum_{(k_1,\dots,k_d)\in\mathbb{N}_0^d} \left\{\prod_{r=1}^{d} \frac{(-1)^{k_r}}{2k_r+1}\right\} \operatorname{sech}\left(\frac{\pi}{2}\sqrt{\sum_{r=1}^{d}(2k_r+1)^2}\right) = \frac{1}{d+1}\left(\frac{\pi}{4}\right)^d.$$

Proof. We consider the following multiple series

$$\sum_{(k_0,\dots,k_d)\in\mathbb{N}_0^{d+1}} \left\{\prod_{r=0}^{d} \frac{(-1)^{k_r}}{2k_r+1}\right\} \cdot \frac{(2k_0+1)^2}{\sum_{r=0}^{d}(2k_r+1)^2}. \tag{10}$$

Following the proof of Theorem 3.1, we conclude that the multiple series (10) is regularly convergent. Thus,

$$\sum_{(k_1,\dots,k_d)\in\mathbb{N}_0^d} \left\{\prod_{r=1}^{d} \frac{(-1)^{k_r}}{2k_r+1}\right\} \operatorname{sech}\left(\frac{\pi}{2}\sqrt{\sum_{r=1}^{d}(2k_r+1)^2}\right)$$

$$= \frac{4}{\pi} \sum_{(k_0,\dots,k_d)\in\mathbb{N}_0^{d+1}} \left\{\prod_{r=0}^{d} \frac{(-1)^{k_r}}{2k_r+1}\right\} \cdot \frac{(2k_0+1)^2}{\sum_{r=0}^{d}(2k_r+1)^2}.$$

Since

$$\sum_{(k_0,\dots,k_d)\in\mathbb{N}_0^{d+1}} \left\{\prod_{r=0}^{d} \frac{(-1)^{k_r}}{2k_r+1}\right\} \cdot \frac{(2k_q+1)^2}{\sum_{r=0}^{d}(2k_r+1)^2}$$

$$= \sum_{(k_0,\dots,k_d)\in\mathbb{N}_0^{d+1}} \left\{\prod_{r=0}^{d} \frac{(-1)^{k_r}}{2k_r+1}\right\} \cdot \frac{(2k_0+1)^2}{\sum_{r=0}^{d}(2k_r+1)^2} \qquad (q = 1,\dots,d)$$

and $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} = \frac{\pi}{4}$, the result follows. □

4. Conclusion

By showing that the double series form of alternating hyperbolic series is regularly convergent, we have given a modest application of Hardy's regularly convergent double series to Ramanujan's work. Although the notion of Hardy's regularly convergent double series seems to be commonly overlooked by some researchers, we believe that such double series can be applied to the evaluation of many series as we have demonstrated in this paper. For instance, we deduce from the partial fraction decomposition of the hyperbolic cosecant function (cf. [2]) and Remark 1.4 of [6] that if x is a nonzero real number, then the double series $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(xm)^2+(an+b)^2}$ is regularly convergent and

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2} = \frac{\pi}{x} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \operatorname{csch}\left(\frac{\pi(an+b)}{x}\right)}{an+b}.$$

Acknowledgement

References

[1] Berndt B C, Ramanujan's Notebooks Part II (1989) (Springer)
[2] Berndt B C, Lamb G and Rogers M, Two-dimensional series evaluations via the elliptic functions of Ramanujan and Jacobi, *Ramanujan J.* **29** (2012) 185–198
[3] Hardy G H, On the convergence of certain multiple series, *Proc. Cambridge Philos. Soc.* **19** (1916–1919) 86–95
[4] Lee T Y, Some convergence theorems for Lebesgue integrals, *Analysis (Munich)* **28** (2008) 263–268
[5] Lee T Y, Henstock–Kurzweil integration on Euclidean spaces, Series in Real Analysis vol. 12 (2011) (Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd.)
[6] Móricz F, On the regular convergence of multiple series of numbers and multiple integrals of locally integrable functions over $\bar{\mathbb{R}}^m$, *Anal. Math.* **39** (2013) 139–152
[7] Nanjundiah T S, Certain summations due to Ramanujan and their generalisations, *Proc. Indian Acad. Sci. Sec. A* **34** (1951) 215–228
[8] Ramanujan S, Collected Papers (1962) (New York: Chelsea)
[9] Watson G N, Theorems stated by Ramanujan II, *J. London Math. Soc.* **3** (1928) 216–225