

Notation for sums. In Notes §4.1, we define the integral $\int_a^b f(x) dx$ as a limit of approximations. That is, we split the interval $x \in [a, b]$ into n increments of size $\Delta x = \frac{b-a}{n}$, we choose sample points x_1, x_2, \dots, x_n , and we take:

$$\int_a^b f(x) dx = \lim_{\Delta x \rightarrow 0} f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

The sum which appears on the right is called a *Riemann sum*. Similar sums appear frequently in mathematics, and we define a special notation to handle them.

In the most general situation, we have a sequence of numbers $q_0, q_1, q_2, q_3, \dots$ so that for any $i = 0, 1, 2, \dots$ we have a number q_i . We consider an interval of integers $i = m, m+1, m+2, \dots, n$, and we introduce a notation for the sum of all the q_i for $i = m$ to n :

$$\sum_{i=m}^n q_i = q_m + q_{m+1} + q_{m+2} + \cdots + q_n.$$

The summation symbol Σ is capital sigma, the Greek letter S, standing for “sum”. The variable i is called the index of summation.

Note: In the WebWork problems, a sequence is denoted $f(i)$ instead of q_i . This is because we can consider the sequence of q_i 's as a function with input i (an integer) and output q_i (a specified number).

Examples

- Letting $q_i = \sqrt{i}$, we have $q_0 = 0$, $q_1 = 1$, $q_2 = \sqrt{2}$, $q_3 = \sqrt{3}$, etc., and taking the interval of integers $i = 2, 3, 4, 5$, we have:

$$\sum_{i=2}^5 \sqrt{i} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} \approx 7.38.$$

- Letting $q_i = 1$, we have: $\sum_{i=1}^{10} 1 = \underbrace{1 + 1 + \cdots + 1}_{10 \text{ terms}} = 10$.
- Given the sum of the first ten square numbers $1 + 4 + 9 + 16 + \cdots + 100$, we wish to write this compactly in sigma notation. Considering the terms as a sequence $q_i = i^2$, we get:

$$1 + 4 + 9 + \cdots + 100 = 1^2 + 2^2 + 3^2 + \cdots + 10^2 = \sum_{i=1}^{10} i^2.$$

- Given the sum of the first five odd numbers $1 + 3 + 5 + 7 + 9$, we can write this in sigma notation by considering the terms as $q_i = 2i-1$:

$$1 + 3 + 5 + 7 + 9 = (2(1)-1) + (2(2)-1) + \cdots + (2(5)-1) = \sum_{i=1}^5 (2i-1).$$

Another way would be to consider the terms as $q_i = 2i+1$:

$$1 + 3 + 5 + 7 + 9 = (2(0)+1) + (2(1)+1) + \cdots + (2(4)+1) = \sum_{i=0}^4 (2i+1).$$

- The sum of the first n odd numbers, where n is an unspecified whole number, can be written as:

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{i=1}^n (2i-1).$$

- We can write a Riemann sum as:

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x.$$

Summation Rules. As for limits and derivatives, we can sometimes compute summations by starting with known Basic Summations, and combining them by Summation Rules.

- *Sum:* $\sum_{i=m}^n (q_i+p_i) = \sum_{i=m}^n q_i + \sum_{i=m}^n p_i.$
- *Difference:* $\sum_{i=m}^n (q_i-p_i) = \sum_{i=m}^n q_i - \sum_{i=m}^n p_i.$
- *Constant Multiple:* $\sum_{i=m}^n C q_i = C \cdot \sum_{i=m}^n q_i$, where C does not depend on i .

Like all facts about summations, these formulas can be understood by writing out the terms in dot-dot-dot (ellipsis) notation:

$$\begin{aligned} \sum_{i=m}^n (q_i+p_i) &= (q_m+p_m) + (q_{m+1}+p_{m+1}) + \cdots + (q_n+p_n) \\ &= (q_m + q_{m+1} + \cdots + q_n) + (p_m + p_{m+1} + \cdots + p_n) \\ &= \sum_{i=m}^n q_i + \sum_{i=m}^n p_i. \end{aligned}$$

Similarly for the other two rules.

Note that n is a constant not depending on i , so we may factor it out of a summation: $\sum_{i=1}^n ni^2 = n \sum_{i=1}^n i^2$. This gives a separate formula for each n : for $n = 3$ it means $3(1^2)+3(2^2)+3(3^2) = 3(1^2+2^2+3^2)$. However, the variable i has no meaning outside the summation, and cannot be factored out: $\sum_{i=1}^3 i2^i \stackrel{??}{=} i \sum_{i=1}^3 2^i$ is nonsense, because the left side means $1(2^1) + 2(2^2) + 3(2^3)$, but the right side would mean some constant “ i ” times $2^1+2^2+2^3$, but i is *not* a constant.

Warning: the summation of a product $\sum q_i p_i$ is NOT equal to the product of summations $(\sum q_i)(\sum p_i)$. For example: $1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \neq (1+2+3)(1+2+3)$.

Basic Summations. We can get some surprisingly neat formulas for certain summations:

$$(a) \quad \sum_{i=1}^n 1 = n.$$

$$(b) \quad \sum_{i=1}^n i = \frac{1}{2}n(n+1).$$

$$(c) \quad \sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1).$$

Proof. (a) $\sum_{i=1}^n 1 = 1 + \dots + 1$ with n terms, which indeed equals n .

(b) Taking two copies of $\sum_{i=1}^n i$, we can pair each term with its complement:

$$\begin{array}{r} 2 \cdot \sum_{i=1}^n i = \quad 1 \quad + \quad 2 \quad + \quad \dots \quad + \quad n-1 \quad + \quad n \\ \quad \quad \quad + \quad n \quad + \quad n-1 \quad + \quad \dots \quad + \quad 2 \quad + \quad 1 \\ \hline = n+1 \quad + \quad n+1 \quad + \quad \dots \quad + \quad n+1 \quad + \quad n+1 = n(n+1). \end{array}$$

The equation $2 \cdot \sum_{i=1}^n i = n(n+1)$, divided by 2, gives the desired formula.

(c) Consider that $(i+1)^3 = i^3 + 3i^2 + 3i + 1$, so that:

$$\begin{aligned} \sum_{i=1}^n (i+1)^3 - i^3 &= \sum_{i=1}^n (3i^2 + 3i + 1) \\ &= 3 \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\ &= 3 \sum_{i=1}^n i^2 + \frac{3}{2}n(n+1) + n. \end{aligned}$$

On the other hand, we have a “collapsing sum”:

$$\begin{aligned} \sum_{i=1}^n (i+1)^3 - i^3 &= (n+1)^3 - n^3 + n^3 - (n-1)^3 + \dots + 3^3 - 2^3 + 2^3 - 1^3 \\ &= (n+1)^3 - 1^3. \end{aligned}$$

Solving the equation:

$$3 \cdot \sum_{i=1}^n i^2 + \frac{3}{2}n(n+1) + n = (n+1)^3 - 1$$

gives, as desired:

$$\sum_{i=1}^n i^2 = \frac{1}{3}((n+1)^3 - \frac{3}{2}n(n+1) - (n+1)) = \frac{1}{6}n(n+1)(2n+1).$$

A similar computation will produce a formula for $\sum_{i=1}^n i^3$, etc.

Direct Evaluation of Integrals. We can use the above rules to simplify Riemann sums and find integrals exactly. For example, consider:

$$\int_1^3 5x \, dx = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n 5x_i \Delta x.$$

On the right side, we divide the interval $[1, 3]$ into n increments of length $\Delta x = \frac{3-1}{n} = \frac{2}{n}$, with dividing points:

$$1 < 1+\Delta x < 1+2\Delta x < 1+3\Delta x < \dots < 1+n\Delta x = 3.$$

In the i^{th} increment, we arbitrarily choose the sample point x_i to be the right endpoint, that is $x_i = 1 + i \Delta x = 1 + \frac{2}{n}i$. Thus:

$$\begin{aligned} \sum_{i=1}^n 5x_i \Delta x &= \sum_{i=1}^n 5 \left(1 + \frac{2}{n}i \right) \frac{2}{n} \\ &= \frac{10}{n} \sum_{i=1}^n 1 + \frac{20}{n^2} \sum_{i=1}^n i \\ &= \frac{10}{n} \cdot n + \frac{20}{n^2} \cdot \frac{1}{2}n(n+1) \\ &= 20 + \frac{10}{n}. \end{aligned}$$

(Here n is a fixed number not depending on i , such as $n = 100$ or $n = 1000$, and we can factor it out of the \sum .) Finally, we let $\Delta x \rightarrow 0$ or equivalently $n \rightarrow \infty$:

$$\int_1^3 5x \, dx = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n 5x_i \Delta x = \lim_{n \rightarrow \infty} 20 + \frac{10}{n} = 20.$$

We computed this to show in principle that Riemann sums can be evaluated directly, but this is far from the easiest way to compute an integral. Geometrically, the integral equals the trapezoid area below the graph $y = 5x$ and above the interval $[1, 3]$ on the x -axis. Since (trapezoid area) = (width) \times (average height), we get that the integral is $A = (3-1)\left(\frac{5(1)+5(3)}{2}\right) = 20$.

Physically, if $v(t) = 5t$ is a velocity, then the integral $\int_1^3 v(t) \, dt$ is the distance traveled from $t = 1$ to $t = 3$. Since the position $s(t)$ is an antiderivative, we must have $s(t) = \frac{5}{2}t^2 + C$, so the distance traveled is $s(3) - s(1) = \frac{5}{2}(3^2) - \frac{5}{2}(1^2) = 20$.