Continuity of Functions

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Introduction

A general function from $\mathbb{R}$ to $\mathbb{R}$ can be very convoluted indeed, which means that we will not be able to make many meaningful statements about general functions. To develop a useful theory, we must instead restrict the class of functions we consider. Intuitively, we require that the functions be sufficiently 'nice', and see what properties we can deduce from such restrictions.

The study of continuous functions is a case in point - by requiring a function to be continuous, we obtain enough information to deduce powerful theorems, such as the Intermediate Value Theorem. However, the definition of continuity is flexible enough that there are a wide, and interesting, variety of continuous functions. Indeed, many functions that come up in real-world problems are continuous, which makes the definition pleasing from both an aesthetic and practical point of view.

Definitions of Continuity

The book provides the following definition, based on sequences:

<u>Definition:</u>
A function f is *continuous at* x_0 in its domain if for every sequence (x_n) with x_n in the domain of f for every n and $\lim x_n = x_0$, we have $\lim f(x_n) = f(x_0)$. We say that f is *continuous* if it is continuous at every point in its domain.

What does this say? It says that any time a sequence converges in the domain, the image of the sequence in the range also converges. In other words, we could either take the limit first, and then apply the function, or apply the function first, and then take the limits. Informally, f is continuous if

$$\lim f(x_n) = f(\lim x_n)$$

This is a powerful definition because we have spent a lot of time studying sequences and limits, so we can use what we know to deduce results about continuity. In particular, we can use all the limit rules to avoid tedious calculations.

However, there is a $\epsilon - \delta$ definition, similar to the definition of a limit, which goes as follows:

<u>Definition:</u>
A function f is *continuous at* x_0 in its domain if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever x is in the domain of f and $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$. Again, we say f is *continuous* if it is continuous at every point in its domain.

How should we understand this definition? Let us examine the role played by the parameters ϵ and δ. ϵ measures how far $f(x)$ is from $f(x_0)$ in the range of the function - it is there to control to error you make when you estimate $f(x)$ by $f(x_0)$. δ measures how close x is to x_0 in the domain. The definition holds for every positive ϵ - so ϵ can be taken to be arbitrary. Given x_0 and ϵ, we then have to show there is one δ which works; in particular, δ can depend on x_0 and ϵ. What we are asking is: how close do I have to be to x_0 before my function value $f(x)$ is close to $f(x_0)$? Compare this to the definition of a limit, where we ask: how far along my sequence do I have to go before my terms are close to the limit? You should see that δ plays a comparable role to that of N in the limit definition.

Theorem 17.2 on page 116 proves that the two definitions are equivalent, so it does not matter which definition you use. It is important to understand these definitions, because they tell you what can do with continuous functions. A good test is to see if you can understand the proof of Theorem 17.2. If you can understand the main ideas of the proof, well enough that you could write it out on your own or explain it in your own words to a friend[1], then you should be comfortable with the ideas in this chapter.

Proving Continuity

The definition of continuity gives you a fair amount of information about a function, but this is all a waste of time unless you can show the function you are interested in is continuous. Fortunately for us, a lot of natural functions are continuous, and it is not too difficult to illustrate this is the case. There are three main techniques:

To show a function is continuous, we can do one of three things:

(i) Show it satisfies the sequence definition of continuity

(ii) Show it satisfies the $\epsilon - \delta$ definition of continuity

(iii) Decompose it into simpler functions we already know to be continuous

The most direct approach is (i): to show a function is continuous, we show it satisfies the definition of continuity. In other words, if we have a convergent sequence in the domain, then the image of the sequence converges (to the right limit). Note that this has to hold for *every* convergent sequence - you cannot show it works for just one. This is often a nice and clean approach for simple functions, as we can use the limit rules.

For example, suppose we have the function

$$f(x) = \frac{x^2 - 2x + 2}{x^4 + 1}$$

Note that the domain of this function is $\mathbb{R}$: the numerator and denominator are polynomials, hence defined for all real numbers. Moreover, as $x^4 \geq 0$, the denominator is

[1]Or, for that matter, any sentient being - if you can teach your dog continuity, you're doing well.

always at least 1, so we never divide by 0.

Now suppose we have a convergent sequence (x_n) with $\lim x_n = x_0$. Then, using the limit rules, we have

$$\begin{aligned}
\lim f(x_n) &= \lim \frac{x_n^2 - 2x_n + 2}{x_n^4 + 1} \\
&= \frac{\lim(x_n^2 - 2x_n + 2)}{\lim(x_n^4 + 1)} \\
&= \frac{\lim x_n^2 - \lim 2x_n + \lim 2}{\lim x_n^4 + \lim 1} \\
&= \frac{(\lim x_n)^2 - 2(\lim x_n) + 2}{(\lim x_n)^4 + 1} \\
&= \frac{x_0^2 - 2x_0 + 2}{x_0^4 + 1} = f(x_0)
\end{aligned}$$

Hence f satisfies the definition of continuity, and is thus continuous.

Since Theorem 17.2 tells us that the two definitions of continuity are equivalent, we could also try to show that the $\epsilon - \delta$ definition holds, which is technique (ii). This will usually involve a bit of calculation, and so will not be as clean as (i). Thus it should be used in two situations: when we cannot compute the limit, or when we need some information about how continuous[2]the function is.

For example, consider the function

$$f(x) = 4x^2 - 3x + 2$$

Suppose we wish to show this is continuous using the $\epsilon - \delta$ definition. Then we need to control the error $|f(x) - f(x_0)|$ in terms of $|x - x_0|$. We have

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|(4x^2 - 3x + 2) - (4x_0^2 - 3x_0 + 2)\right| \\
&= \left|4x^2 - 4x_0^2 - 3x + 3x_0 + 2 - 2\right| \\
&= |4(x - x_0)(x + x_0) - 3(x - x_0)| \\
&= |(x - x_0)\,(4(x + x_0) - 3)| \\
&= |x - x_0|\,|4(x + x_0) - 3| \\
&\leq |x - x_0|\,(4|x + x_0| + 3)
\end{aligned}$$

where in the last inequality we use the triangle inequality. The first factor is in the form we want, as it is in terms of $x - x_0$. However, the second term also depends on x, so we must rewrite it in terms of $x - x_0$. We can then use the triange inequality again to obtain an upper bound.

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |x - x_0|\,(4|x + x_0| + 3) \\
&= |x - x_0|\,(4|x - x_0 + 2x_0| + 3) \\
&\leq |x - x_0|\,(4|x - x_0| + 8|x_0| + 3)
\end{aligned}$$

[2]This does not make much sense at the moment, but when we study uniform continuity, we will see that knowing how δ depends on ϵ and x_0 gives us more information about the function f.

Thus we have bounded the error $|f(x) - f(x_0)|$ in terms of $|x - x_0|$. We now need to show that given any $\epsilon > 0$, we can choose $\delta > 0$ small enough so that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. Indeed, suppose $|x - x_0| < 1$. Then we have

$$|f(x) - f(x_0)| \leq |x - x_0| \left(4|x - x_0| + 8|x_0| + 3\right) \leq |x - x_0| \left(8|x_0| + 7\right)$$

If we further have $|x - x_0| < \frac{\epsilon}{8|x_0|+7}$, then

$$|f(x) - f(x_0)| \leq |x - x_0| \left(8|x_0| + 7\right) < \epsilon$$

as desired. Hence we may take $\delta = \min\left\{1, \frac{\epsilon}{8|x_0|+7}\right\}$ in the $\epsilon - \delta$ definition, showing that f is continuous. Note that in this case δ depends on both ϵ and x_0, but this is permitted.

The final method, of decomposing a function into simple continuous functions, is the simplest, but requires that you have a set of basic continuous functions to start with - somewhat akin to using limit rules to find limits. Theorems 17.4 and 17.5 show that, where defined, the sum, product, quotient and composition of continuous functions is continuous. So, for example, if we know that both $g(x) = x$ and the constant function $h(x) = k$ (for $k \in \mathbb{R}$) are continuous[3], then we can show that

$$f(x) = \frac{x^2 - 2x + 2}{x^4 + 1}$$

is continuous, since it is the quotient of $f_1(x) = x^2 - 2x + 2$ and $f_2(x) = x^4 + 1$. Now x^2 and $2x$ are the products of continuous functions, hence continuous, and 2 is a continuous function, so $f_1(x) = x^2 - 2x + 2$ is continuous. As x^2 is continuous, x^4 is continuous (either since it is x^2 multiplied by itself, or x^2 composed with itself), and so $f_2(x) = x^4 + 1$ is continuous. As $f_2(x) \neq 0$ for all x, the quotient $f(x) = \frac{f_1(x)}{f_2(x)}$ is continuous everywhere.

As you can see, this last method allows you to very quickly assemble a large collection of continuous functions. However, you have to start with some continuous functions, and so it is necessary to be able to use the definitions.

Proving Discontinuity

Knowing that a function is continuous gives us quite a lot of power, so, as we might expect, there is a price to pay - not all functions are continuous. It is important to be able to recognise when a function is discontinuous, and as before, there are three methods we can apply:

(i) Show that the function fails the sequence definition

(ii) Show that the function fails the $\epsilon - \delta$ definition

(iii) Prove by contradiction by relating it to a function known to be discontinuous

[3]This is quite easy to see by using the sequence definition of continuity.

For example, consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 10 \\ 1 & \text{if } x > 10 \end{cases}$$

Suppose we wish to show this is discontinuous at $x_0 = 10$ using the sequence definition. If f were continuous, then whenever we had $x_n \to x_0$, we must have $f(x_n) \to f(x_0)$. Since we are trying to show this is discontinuous, we need this definition to fail, or for the opposite to be true. If you negate the statement, that means that there is *a* sequence x_n such that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$. Note that we need only *one* counterexample to show discontinuity.

In our case, $f(x_0) = f(10) = 0$, so we need to find a sequence such that $x_n \to 10$ but $f(x_n) \not\to f(10) = 0$. Notice that for any x, $f(x)$ is either 0 or 1. Thus to make a sequence not converging to 0, we need $f(x_n) = 1$ infinitely often[4]. But for this function $f(x) = 1$ if and only if $x > 10$. Hence it suffices to find a sequence with $x_n > 10$ for all n that converges to 10. Any such sequence will do - in particular, we can take $x_n = 10 + \frac{1}{n}$. Then we have $x_n \to 10$ as $n \to \infty$, but $f(x_n) = 1$ for all n, as $x_n = 10 + \frac{1}{n} > 10$, and so $f(x_n) \to 1 \neq 0 = f(10)$. Hence the function f is discontinuous at $x_0 = 10$.

Once again, to emphasise the key point, to prove a function is discontinuous at a point, we just need to find *one* bad sequence. In order to do so, we need to study the values the function takes, and pick a sequence accordingly.

To show a function fails the $\epsilon - \delta$ definition is a lot more involved. This is because when we negate the definition, we need to show that there is an $\epsilon > 0$ such that for *every* $\delta > 0$ there is an x with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon$. Since we have to show this holds for *every* positive delta, this essentially amounts to constructing a bad sequence as in the earlier case. For example, consider the function from before. To show it is discontinuous at $x_0 = 10$, we need to pick an ϵ for which every δ fails. Let us choose $\epsilon = 1$. Then suppose there was a $\delta > 0$ that satisfied the $\epsilon - \delta$ condition. Choose $10 < x < 10 + \delta$. Then we have $|x - 10| = x - 10 < \delta$, but as $x > 10$, $f(x) = 1$, so $|f(x) - f(10)| = f(x) = 1 \geq \epsilon$.

It is usually easier, and less confusing, to negate the sequence definition, and so I would recommend not using the $\epsilon - \delta$ definition for showing a function is discontinuous.

Finally, we can try using proof by contradiction. Assume your function is continuous, and then show that you can use it to build a function that you know is discontinuous. Applying Theorems 17.4 and 17.5, this would imply that the other function was continuous, which is a contradiction[5].

Consider the function

$$h(x) = \begin{cases} 0 & \text{if } x \leq 10 \\ x^2 & \text{if } x > 10 \end{cases}$$

[4]A little side remark - by passing to a subsequence where $f(x_{n_k}) = 1$ for all k, we need only consider sequences where $f(x_n) = 1$ for all n, not just infinitely many.

[5]This is quite similar to how we showed some numbers were irrational.

We wish to show this function is discontinuous at $x_0 = 10$. Suppose for contradiction that it was continuous at $x_0 = 10$. Then we notice that $f(x) = \frac{h(x)}{x^2}$, where $f(x)$ is the function from the previous example. As $h(x)$ and x^2 are continuous, and $x_0^2 \neq 0$, Theorem 17.4 implies that $f(x)$ is continuous at $x_0 = 10$. But we have already proven that $f(x)$ is discontinuous at $x_0 = 10$, which gives a contradiction. Hence we must have $h(x)$ also discontinuous at $x_0 = 10$.

Note that it is important to use the function in question to build the known function, not the other way around. It is possible to combine discontinuous functions to make continuous functions.

The Intermediate Value Theorem

Throughout this document I have promised that knowing a function is continuous gives you a lot of information, and in this section we shall see that claim justified. The Intermediate Value Theorem is a powerful theorem that holds for continuous functions, and gives us some insight into what values the function must take.

Theorem:

Let f be a function that is continuous on an interval $[a, b]$. Then, if y is any value strictly between $f(a)$ and $f(b)$, there is some c with $a < c < b$ and $f(c) = y$.

For example, consider the function $f(x) = \frac{x^2-2x+2}{x^4+1}$. By our earlier work, we know that f is continuous on $\mathbb{R}$, and so it is continuous on any subinterval $[a, b]$. Suppose we take $a = 0$; then we have $f(a) = 2$. If we now take $b = 1$, then we have $f(b) = \frac{1}{2}$. Since $\frac{1}{2} < \frac{3}{4} < 2$, the Intermediate Value Theorem tells us that there is some c with $0 < c < 1$ with $f(c) = \frac{3}{4}$.

Notice that the theorem does not tell us what c is, just that such a c exists[6]. This feature of the Intermediate Value Theorem means that it is very useful for showing that solutions to equations exist. Moreover, as we have $a < c < b$, we have some control over where the solution lives. Indeed, if you have ever studied the Bisection Method, you will have seen how you can use the Intermediate Value Theorem to approximate solutions to equations.

To see some more examples of how to use the Intermediate Value Theorem, consider the following:

Suppose we need to show that $5x = e^x$ has a solution in the open interval $(0, 1)$. To use the Intermediate Value Theorem, we want to have a constant on the right-hand side (since the theorem says that a continuous function takes on any *fixed* value). Hence we rearrange to consider $f(x) = 5x - e^x$. This is clearly a continuous function on the interval $[0, 1]$. At the endpoints, we have $f(0) = 0 - 1 = -1 < 0$ and $f(1) = 5 - e > 0$,

[6] Trying to show such a c existed directly would be difficult, as we would have to solve a fourth-degree polynomial - no mean feat!

and so 0 lies between the values at the endpoints. By the Intermediate Value Theorem, there is an c with $0 < c < 1$ and $f(c) = 5c - e^c = 0$, so $5c = e^c$. In other words, we have shown this equation has a solution in the interval $(0, 1)$.

For another example, suppose we know that $\arctan x$ is continuous, and we want to show that $\arctan x = 1 - x$ has a solution. As before, we rearrange so that we have a constant on one side. We can rewrite the equation as $\arctan x + x = 1$. Let $f(x) = \arctan x + x$, which is continuous; we need to find a value c such that $f(c) = 1$. However, we cannot yet apply the Intermediate Value Theorem, as we do not have an interval. In order for the theorem to give us a c with $f(c) = 1$, we need 1 to lie between the function values at the endpoints. To find the right interval requires a bit of trial and error.

Suppose we take one endpoint to be 0. Then $\arctan 0 = 0$, so $f(0) = 0 + 0 = 0$. Hence if we can find a point where $f(x) > 1$, we are done. Consider $x = 1$. Then $\arctan x = \frac{\pi}{4}$, and so $f(1) = \frac{\pi}{4} + 1 > 1$. Hence we can use the interval $[0, 1]$ in the Intermediate Value Theorem to find a c such that $f(c) = 1$, which implies $\arctan c = 1 - c$, as required. Note that we also gain the information that the solution c satisfies $0 < c < 1$.

Conclusion

Continuity is one of the central topics in this course, so it is important to take some time now to really understand what the definitions mean, and how the theorems work. I hope that these notes help[7]; please do let me know if anything requires clarification. Have fun working with continuity!

[7] Or, at the very least, do not hinder.