

Function and Concept¹

[This lecture was given to the *Jenaische Gesellschaft für Medicin und Naturwissenschaft* on 9 January 1891, and subsequently published by Frege as a separate work (Jena: Hermann Pohle, 1891). Besides providing Frege's fullest account of his notion of a function, it also marks the first appearance of his distinction between *Sinn* and *Bedeutung*.]

Preface

I publish this lecture separately in the hope of finding readers who are unfamiliar with the Proceedings of the Jena Society for Medicine and Science. It is my intention, in the near future, as I have indicated elsewhere, to explain how I express the fundamental definitions of arithmetic in my *Begriffsschrift*, and how I construct proofs from these solely by means of my symbols. For this purpose it will be useful to be able to refer [*berufen*] to this lecture so as not to be drawn then into discussions which many might condemn as not directly relevant, but which others might welcome. As befitting the occasion, my lecture was not addressed only to mathematicians; and I sought to express myself in as accessible a way as the time available and the subject allowed. May it then arouse interest in the matter in wider learned circles, particularly amongst logicians.

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Rather a long time ago^A I had the honour of addressing this Society about the symbolic system that I entitled *Begriffsschrift*. Today I should

^A On 10 January 1879 and 27 January 1882. [The reference here is to *APCN* and *ACN*, respectively.]

¹ Translated by Peter Geach (*TPW*, pp. 21–41/*CP*, pp. 137–56; Preface translated by Michael Beaney from *KS*, p. 125). Page numbers in the margin are from the original publication. The translated text here is from the third edition of *TPW*, with minor revisions made in accordance with the policy adopted in the present volume – in particular, 'Bedeutung' (and cognates such as 'bedeutungslos') being left untranslated, and 'bedeuten' being rendered as 'stand for' as in the second edition (but with the German always in square brackets following it), unless otherwise indicated. For discussion of this policy, and the problems involved in translating 'Bedeutung' and its cognates, see the Introduction, §4 above.

like to throw light upon the subject from another side, and tell you about some supplementations and new conceptions, whose necessity has occurred to me since then. There can here be no question of setting forth my *Begriffsschrift* in its entirety, but only of elucidating some fundamental ideas.

My starting-point is what is called a function in mathematics. The original *Bedeutung* of this word was not so wide as that which it has since obtained; it will be well to begin by dealing with this first usage, and only then consider the later extensions. I shall for the moment be speaking only of functions of a single argument. The first place where a scientific expression appears with a clear-cut *Bedeutung* is where it is required for the statement of a law. This case arose as regards functions upon the discovery of higher Analysis. Here for the first time it was a matter of setting forth laws holding for functions in general. So we must go back to the time when higher Analysis was discovered, if we want to know how the word 'function' was originally understood. The answer that we are likely to get to this question is: 'A function of x was taken to be a mathematical expression containing x , a formula containing the letter x .'

Thus, e.g., the expression

$$2x^3 + x$$

would be a function of x , and

$$2 \cdot 2^3 + 2$$

would be a function of 2. This answer cannot satisfy us, for here no distinction is made between form and content, sign and thing signified [*Bezeichnetes*]; a mistake, admittedly, that is very often met with in mathematical works, even those of celebrated authors. I have already pointed out on a previous occasion^B the defects of the current formal theories in arithmetic. We there have talk about signs that neither have nor are meant to have any content, but nevertheless properties are ascribed to them which are unintelligible except as belonging to the content of a sign. So also here; a mere expression, the form for a content, cannot be the heart of the matter; only the content itself can be that. Now what is the content, the *Bedeutung* of ' $2 \cdot 2^3 + 2$ '? The same as of '18' or '3.6'. What is expressed in the equation ' $2 \cdot 2^3 + 2 = 18$ ' is that the right-hand complex of signs has the same *Bedeutung* as the left-hand one. I must here combat the view that, e.g., $2 + 5$ and $3 + 4$ are equal but not the same. This view is grounded in the same confusion of form and content, sign and thing signified. It is as though one wanted to regard the sweet-smelling violet as differing from *Viola odorata* because the names

^B *Die Grundlagen der Arithmetik* (1884), §§92ff. [cf. pp. 124–5 above]; 'On Formal Theories of Arithmetic' (1885) [*FTA*].

sound different. Difference of sign cannot by itself be a sufficient ground for difference of the thing signified. The only reason why in our case the matter is less obvious is that the *Bedeutung* of the numeral 17 is not anything perceptible to the senses. There is at present a very widespread tendency not to recognize as an object anything that cannot be perceived by means of the senses; this leads here to numerals' being taken to be numbers, the proper objects of our discussion;^c and then, I admit, 7 and $2 + 5$ would indeed be different. But such a conception is untenable, for we cannot speak of any arithmetical properties of numbers whatsoever without going back to the *Bedeutung* of the signs. For example, the property belonging to 1, of being the result of multiplying itself by itself, would be a mere myth; for no microscopical or chemical investigation, however far it was carried, could ever detect this property in the possession of the innocent character that we call a figure one. Perhaps there is talk of a definition; but no definition is creative in the sense of being able to endow a thing with properties that it has not already got – apart from the one property of expressing and signifying something in virtue of the definition.^d The characters we call numerals have, on the other hand, physical and chemical properties depending on the writing material. One could imagine the introduction some day of quite new numerals, just as, e.g., the Arabic numerals superseded the Roman. Nobody is seriously going to suppose that in this way we should get quite new numbers, quite new arithmetical objects, with properties still to be investigated. Thus we must distinguish between numerals and their *Bedeutungen*; and if so, we shall have to recognize that the expressions '2', '1 + 1', '3 - 1', '6:3' all have the same *Bedeutung*, for it is quite inconceivable where the difference between them could lie. Perhaps you say: 1 + 1 is a sum, but 6:3 is a quotient. But what is 6:3? The number that when multiplied by 3 gives the result 6. We say 'the number', not 'a number'; by using the definite article, we indicate that there is only a single number. Now we have:

$$(1 + 1) + (1 + 1) + (1 + 1) = 6,$$

and thus (1 + 1) is the very number that was designated as (6:3). The different expressions correspond to different conceptions and aspects, but nevertheless always to the same thing. Otherwise the equation $x^2 = 4$

^c Cf. the essays: H. von Helmholtz, 'Zählen und Messen erkenntnistheoretisch betrachtet' ['Numbering and Measuring from an Epistemological Viewpoint', in *Epistemological Writings* (Dordrecht and Boston, 1977), p. 72], and Leopold Kronecker, 'Über den Zahlbegriff' ['On the Concept of Number'], in *Philosophische Aufsätze: Eduard Zeller zu seinem fünfzigjährigen Doctorjubiläum gewidmet* (Leipzig, 1887).

^d In definition it is always a matter of associating with a sign a sense or a *Bedeutung*. Where sense and *Bedeutung* are missing, we cannot properly speak either of a sign or of a definition.

would not just have the roots 2 and -2, but also the root $(1 + 1)$ and countless others, all of them different, even if they resembled one another in a certain respect. By recognizing only two real roots, we are rejecting the view that the sign of equality does not stand for [*bedeute*] complete coincidence but only partial agreement. If we adhere to this truth, we see that the expressions:

$$\begin{aligned} &'2.1^3 + 1', \\ &'2.2^3 + 2', \\ &'2.4^3 + 4', \end{aligned}$$

stand for [*bedeuten*] numbers, viz. 3, 18, 132. So if a function were really the *Bedeutung* of a mathematical expression, it would be just be a number; and nothing new would have been gained for arithmetic [by speaking of functions]. Admittedly, people who use the word 'function' ordinarily have in mind expressions in which a number is just indicated indefinitely by the letter x , e.g.

$$'2.x^3 + x';$$

but that makes no difference; for this expression likewise just indefinitely indicates a *number*, and it makes no essential difference whether I write it down or just write down ' x '.

All the same, it is precisely by the notation that uses ' x ' to indicate [a number] indefinitely that we are led to the right conception. People call x the argument, and recognize the same function again in

$$\begin{aligned} &'2.1^3 + 1', \\ &'2.4^3 + 4', \\ &'2.5^3 + 5', \end{aligned}$$

only with different arguments, viz. 1, 4, and 5. From this we may discern that it is the common element of these expressions that contains the essential peculiarity of a function; i.e. what is present in

$$'2.x^3 + x'$$

over and above the letter ' x '. We could write this somewhat as follows:

$$'2.()^3 + ()'.$$

I am concerned to show that the argument does not belong with a function, but goes together with the function to make up a complete whole; for a function by itself must be called incomplete, in need of supplementation, or unsaturated [*ungesättigt*]. And in this respect functions differ fundamentally from numbers. Since such is the essence of functions, we can explain why, on the one hand, we recognize the same function in ' $2.1^3 + 1$ ' and ' $2.2^3 + 2$ ', even though these expressions stand for [*bedeuten*] different numbers, whereas, on the other hand, we

do not find one and the same function in ' $2 \cdot 1^3 + 1$ ' and ' $4 - 1$ ' in spite of their equal numerical values. Moreover, we now see how people are easily led to regard the form of an expression as what is essential to a function. We recognize the function in the expression by imagining the latter as split up, and the possibility of thus splitting it up is suggested by its structure.

The two parts into which a mathematical expression is thus split up, the sign of the argument and the expression of the function, are dissimilar; for the argument is a number, a whole complete in itself, as the function is not. (We may compare this with the division of a line by a point. One is inclined in that case to count the dividing-point along with both segments; but if we want to make a clean division, i.e. so as not to count anything twice over or leave anything out, then we may only count the dividing-point along with one segment. This segment thus becomes fully complete in itself, and may be compared to the argument; whereas the other is lacking in something – viz. the dividing-point, which one may call its endpoint, does not belong to it. Only by completing it with this endpoint, or with a line that has two endpoints, do we get from it something entire.) For instance, if I say 'the function $2 \cdot x^3 + x$ ', x must not be considered as | belonging to the function; this letter only serves to indicate the kind of supplementation that is needed; it enables one to recognize the places where the sign for the argument must go in.

We give the name 'the value of a function for an argument' to the result of completing the function with the argument. Thus, e.g., 3 is the value of the function $2 \cdot x^3 + x$ for the argument 1, since we have: $2 \cdot 1^3 + 1 = 3$.

There are functions, such as $2 + x - x$ or $2 + 0 \cdot x$, whose value is always the same, whatever the argument; we have $2 = 2 + x - x$ and $2 = 2 + 0 \cdot x$. Now if we counted the argument as belonging with the function, we should hold that the number 2 is this function. But this is wrong. Even though here the value of the function is always 2, the function itself must nevertheless be distinguished from 2; for the expression for a function must always show one or more places that are intended to be filled up with the sign of the argument.

The method of analytic geometry supplies us with a means of intuitively representing the values of a function for different arguments. If we regard the argument as the numerical value of an abscissa, and the corresponding value of the function as the numerical value of the ordinate of a point, we obtain a set of points that presents itself to intuition (in ordinary cases) as a curve. Any point on the curve corresponds to an argument together with the associated value of the function. |

Thus, e.g.,

$$y = x^2 - 4x$$

yields a parabola; here 'y' indicates the value of the function and the numerical value of the ordinate, and 'x' similarly indicates the argument and the numerical value of the abscissa. If we compare with this the function

$$x(x - 4),$$

we find that they have always the same value for the same argument. We have generally:

$$x^2 - 4x = x(x - 4),$$

whatever number we take for x . Thus the curve we get from

$$y = x^2 - 4x$$

is the same as the one that arises out of

$$y = x(x - 4).$$

I express this as follows: the function $x(x - 4)$ has the same value-range² as the function $x^2 - 4x$.

If we write

$$x^2 - 4x = x(x - 4),$$

we have not put one function equal to the other, but only the values of one equal to those of the other. And if we so understand this equation that it is to hold whatever argument may be substituted for x , then we have thus expressed that an equality holds generally. But we can also say: 'the value-range of the function $x(x - 4)$ is equal to that of | the function $x^2 - 4x$ ', and here we have an equality between value-ranges. The possibility of regarding the equality holding generally between values of functions as a [particular] equality, viz. an equality between value-ranges is, I think, indemonstrable; it must be taken to be a fundamental law of logic.^{E3}

¹ In many phrases of ordinary mathematical terminology, the word 'function' certainly corresponds to what I have here called the value-range of a function. But function, in the sense of the word employed here, is the logically prior [notion].

² Frege's term 'Wertverlauf' is here translated as 'value-range'. Alternative translations are 'course-of-values' (Furth, in *BLA*) and 'graph' (Geach, in the third edition of *TPW*). Despite Frege's initial explanation of the term in a geometrical context, 'graph' is inappropriate, since the notion of a function has been generalized, and Frege was insistent that our logical and arithmetical knowledge outstrips our powers of geometrical intuition. But both alternative renderings do have the virtue of indicating that Frege has in mind a set of pairings of arguments with values, and not just the range of values themselves. So although 'value-range' is perhaps the simplest and most literal translation, and seems to have become the most widely adopted, it must be remembered, as Frege makes clear here, that it refers to a set of pairings.

³ This is the first formulation in Frege's work of Axiom V of the *Grundgesetze*, the Axiom that Frege admitted he had never been utterly convinced was a law of logic and that he

We may further introduce a brief notation for the value-range of a function. To this end I replace the sign of the argument in the expression for the function by a Greek vowel, enclose the whole in brackets, and prefix to it the same Greek letter with a smooth breathing. Accordingly, e.g.,

$$\acute{\epsilon}(\epsilon^2 - 4\epsilon)$$

is the value-range of the function $x^2 - 4x$ and

$$\acute{\alpha}(\alpha(\alpha - 4))$$

is the value-range of the function $x(x - 4)$, so that in

$$' \acute{\epsilon}(\epsilon^2 - 4\epsilon) = \acute{\alpha}(\alpha(\alpha - 4)) '$$

we have the expression for: the first value-range is the same as the second. A different choice of Greek letters is made on purpose, in order to indicate that there is nothing that obliges us to take the same one. |

$$'x^2 - 4x = x(x - 4)',$$

understood as above, expresses the same sense, but in a different way.⁴ It presents the sense as an equality holding generally; whereas the newly-introduced expression is simply an equation, whose right side, as well its left, has a *Bedeutung* that is complete in itself. In

$$'x^2 - 4x = x(x - 4)'$$

the left side considered in isolation indicates a number only indefinitely, and the same is true of the right side. If we just had ' $x^2 - 4x$ ' we could write instead ' $y^2 - 4y$ ' without altering the sense; for ' y ' like ' x ' indicates a number only indefinitely. But if we combine the two sides to form an equation, we must choose the same letter for both sides, and we thus express something that is not contained in the left side by itself, nor in the right side, nor in the 'equals' sign; viz. generality. Admittedly, what we express is the generality of an equality; but primarily it is a generality.

Just as we indicate a number indefinitely by a letter, in order to express generality, we also need letters to indicate a function indefinitely. To this end people ordinarily use the letters f and F , thus: ' $f(x)$ ', ' $F(x)$ ', where ' x ' replaces the argument. Here the need of a function for sup-

himself held responsible for the contradiction that Russell discovered in his system. See pp. 195, 253-4, 279ff. below. For discussion of Axiom V, see the Introduction, pp. 7-8, 18-20 above.

⁴ This remark has been the subject of much controversy. What Frege is saying here is that the statement that the two functions $x^2 - 4x$ and $x(x - 4)$ have the same value for each argument 'expresses the same sense, but in a different way' as the statement that the value-range of the one function is equal to that of the other. Since what we have here is an instance of what becomes Axiom V of the *Grundgesetze*, the implication is that Axiom V too embodies sameness of *sense* and not just sameness of *Bedeutung*. Cf. fn. 26 on p. 213 below.

plementation is expressed by the fact that the letter f or F | carries along with it a pair of brackets; the space between these is meant to receive the sign for the argument. Thus

$$\acute{\epsilon}f(\epsilon)$$

indicates the value-range of a function that is left undetermined.

Now how has the *Bedeutung* of the word 'function' been extended by the progress of science? We can distinguish two directions in which this has happened.

In the first place, the field of mathematical operations that serve for constructing functions has been extended. Besides addition, multiplication, exponentiation, and their converses, the various means of transition to the limit have been introduced - to be sure, people have not always been clearly aware that they were thus adopting something essentially new. People have gone further still, and have actually been obliged to resort to ordinary language, because the symbolic language of Analysis failed; e.g. when they were speaking of a function whose value is 1 for rational and 0 for irrational arguments.

Secondly, the field of possible arguments and values for functions has been extended by the admission of complex numbers. In conjunction with this, the sense of the expressions 'sum', 'product', etc., had to be defined more widely.

In both directions I go still further. I begin by adding to the signs +, -, etc., which serve for constructing a functional expression, | also signs such as =, >, <, so that I can speak, e.g., of the function $x^2 = 1$, where x takes the place of the argument as before. The first question that arises here is what the values of this function are for different arguments. Now if we replace x successively by -1, 0, 1, 2, we get:

$$\begin{aligned} (-1)^2 &= 1, \\ 0^2 &= 1, \\ 1^2 &= 1, \\ 2^2 &= 1. \end{aligned}$$

Of these equations the first and third are true, the others false. I now say: 'the value of our function is a truth-value', and distinguish between the truth-values of what is true and what is false. I call the first, for short, the True; and the second, the False. Consequently, e.g., what ' $2^2 = 4$ ' stands for [*bedeutet*] is the True just as, say, ' 2^2 ' stands for [*bedeutet*] 4. And ' $2^2 = 1$ ' stands for [*bedeutet*] the False. Accordingly,

$$'2^2 = 4', \quad '2 > 1', \quad '2^4 = 4^2',$$

all stand for the same thing [*bedeuten dasselbe*], viz. the True, so that in

$$(2^2 = 4) = (2 > 1)$$

we have a correct equation.

The objection here suggests itself that ' $2^2 = 4$ ' and ' $2 > 1$ ' nevertheless tell us quite different things, express quite different thoughts; but likewise ' $2^4 = 4^{2^2}$ ' and ' $4.4 = 4^{2^2}$ ' express different thoughts; and yet we can replace ' 2^4 ' by ' 4.4 ', since both signs have the same *Bedeutung*. Consequently, ' $2^4 = 4^{2^2}$ ' and ' $4.4 = 4^{2^2}$ ' likewise have the same *Bedeutung*. We see | from this that from identity [*Gleichheit*] of *Bedeutung* there does not follow identity of the thought [expressed]. If we say 'The Evening Star is a planet with a shorter period of revolution than the Earth', the thought we express is other than in the sentence 'The Morning Star is a planet with a shorter period of revolution than the Earth'; for somebody who does not know that the Morning Star is the Evening Star might regard one as true and the other as false. And yet the *Bedeutung* of both sentences must be the same; for it is just a matter of interchange of the words 'Evening Star' and 'Morning Star', which have the same *Bedeutung*, i.e. are proper names of the same heavenly body. We must distinguish between sense and *Bedeutung*. ' 2^4 ' and ' 4.4 ' certainly have the same *Bedeutung*, i.e. are proper names of the same number; but they have not the same sense; consequently, ' $2^4 = 4^{2^2}$ ' and ' $4.4 = 4^{2^2}$ ' have the same *Bedeutung*, but not the same sense (i.e., in this case: they do not contain the same thought).^F

Thus, just as we write:

$$'2^4 = 4.4'$$

we may also write with equal justification

$$'(2^4 = 4^2) = (4.4 = 4^2)'$$

and

$$'(2^2 = 4) = (2 > 1)'. |$$

It might further be asked: What, then, is the point of admitting the signs =, >, <, into the field of those that help to build up a functional expression? Nowadays, it seems, more and more supporters are being won by the view that arithmetic is a further development of logic; that a more rigorous establishment of arithmetical laws reduces them to purely logical laws and to such laws alone. I too am of this opinion, and I base upon it the requirement that the symbolic language of arithmetic must be expanded into a logical symbolism. I shall now have to indicate how this is done in our present case.

We saw that the value of our function $x^2 = 1$ is always one of the two truth-values. Now if for a definite argument, e.g. -1 , the value of the function is the True, we can express this as follows: 'the number -1 has

^F I do not fail to see that this way of putting it may at first seem arbitrary and artificial, and that it would be desirable to establish my view by going further into the matter. Cf. my forthcoming essay 'Über Sinn und Bedeutung' [pp. 151 71 below].

the property that its square is 1'; or, more briefly, ' -1 is a square root of 1'; or ' -1 falls under the concept: square root of 1'. If the value of the function $x^2 = 1$ for an argument, e.g. for 2, is the False, we can express this as follows: '2 is not a square root of 1' or '2 does not fall under the concept: square root of 1'. We thus see how closely that which is called a concept in logic is connected with what we call a function. Indeed, we may say at once: a concept is a function whose value is always a truth-value. Again, the value of the function

$$(x + 1)^2 = 2(x + 1) |$$

is always a truth-value. We get the True as its value, e.g., for the argument -1 , and this can also be expressed thus: -1 is a number less by 1 than a number whose square is equal to its double. This expresses the fact that -1 falls under a concept. Now the functions

$$x^2 = 1 \quad \text{and} \quad (x + 1)^2 = 2(x + 1)$$

always have the same value for the same argument, viz. the True for the arguments -1 and $+1$, and the False for all other arguments. According to our previous conventions we shall also say that these functions have the same value-range, and express this in symbols as follows:

$$\dot{\epsilon}(\epsilon^2 = 1) = \dot{\alpha}((\alpha + 1)^2 = 2(\alpha + 1)).$$

In logic this is called identity [*Gleichheit*] of the extension of concepts. Hence we can designate as an extension the value-range of a function whose value for every argument is a truth-value.

We shall not stop at equations [*Gleichungen*] and inequalities [*Ungleichungen*]. The linguistic form of equations is a statement. A statement contains (or at least purports to contain) a thought as its sense; and this thought is in general true or false; i.e. it has in general a truth-value, which must be regarded as the *Bedeutung* of the sentence, just as, say, the number 4 is the *Bedeutung* of the expression ' $2 + 2$ ' or London the *Bedeutung* of the expression 'the capital of England'. |

Statements in general, just like equations or inequalities or expressions in Analysis, can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or unsaturated. Thus, e.g., we split up the sentence

'Caesar conquered Gaul'

into 'Caesar' and 'conquered Gaul'. The second part is unsaturated – it contains an empty place; only when this place is filled up with a proper name, or with an expression that replaces a proper name, does a complete sense appear. Here too I give the name 'function' to the *Bedeutung* of this unsaturated part. In this case the argument is Caesar.

We see that here we have undertaken to extend [the application of the term] in the other direction, viz. as regards what can occur as an argument. Not merely numbers, but objects in general, are now admissible; and here persons must assuredly be counted as objects. The two truth-values have already been introduced as possible values of a function; we must go further and admit objects without restriction as values of functions. To get an example of this, let us start, e.g., with the expression

‘the capital of the German Empire’.

This obviously takes the place of a proper name, and stands for [*bedeutet*] an object. If we now split it up into the parts |

‘the capital of’

and ‘the German Empire’, where I count the [German] genitive form as going with the first part, then this part is unsaturated, whereas the other is complete in itself. So in accordance with what I said before, I call

‘the capital of x ’

the expression of a function. If we take the German Empire as the argument, we get Berlin as the value of the function.

When we have thus admitted objects without restriction as arguments and values of functions, the question arises what it is that we are here calling an object. I regard a regular definition as impossible, since we have here something too simple to admit of logical analysis. It is only possible to indicate what is meant [*gemeint*]. Here I can only say briefly: an object is anything that is not a function, so that an expression for it does not contain any empty place.

A statement contains no empty place, and therefore we must take its *Bedeutung* as an object. But this *Bedeutung* is a truth-value. Thus the two truth-values are objects.

Earlier on we presented equations between value-ranges, e.g.:

$$\dot{\epsilon}(\epsilon^2 - 4\epsilon) = \dot{\alpha}(\alpha(\alpha - 4)).$$

We can split this up into ‘ $\dot{\epsilon}(\epsilon^2 - 4\epsilon)$ ’ and ‘ $() = \dot{\alpha}(\alpha(\alpha - 4))$ ’.

This latter part needs supplementation, since on the left of the ‘equals’ sign it contains an empty | place. The first part, ‘ $\dot{\epsilon}(\epsilon^2 - 4\epsilon)$ ’, is fully complete in itself and thus stands for [*bedeutet*] an object. Value-ranges of functions are objects, whereas functions themselves are not. We gave the name ‘value-range’ also to $\dot{\epsilon}(\epsilon^2 = 1)$, but we could also have termed

it the extension of the concept: square root of 1. Extensions of concepts likewise are objects, although concepts themselves are not.

After thus extending the field of things that may be taken as arguments, we must get more exact specifications as to the *Bedeutungen* of the signs already in use. So long as the only objects dealt with in arithmetic are the integers, the letters a and b in ‘ $a + b$ ’ indicate only integers; the plus sign need be defined only between integers. Every widening of the field to which the objects indicated by a and b belong obliges us to give a new definition of the plus sign. It seems to be demanded by scientific rigour that we ensure that an expression never becomes *bedeutungslos*; we must see to it that we never perform calculations with empty signs in the belief that we are dealing with objects. People have in the past carried out invalid procedures with divergent infinite series. It is thus necessary to lay down rules from which it follows, e.g., what

‘ $\odot + 1$ ’

stands for [*bedeutet*], if ‘ \odot ’ stands for [*bedeutet*] the Sun. What rules we lay down is a matter of comparative | indifference; but it is essential that we should do so – that ‘ $a + b$ ’ should always have a *Bedeutung*, whatever signs for definite objects may be inserted in place of ‘ a ’ and ‘ b ’. This involves the requirement as regards concepts, that, for any argument, they shall have a truth-value as their value; that it shall be determinate, for any object, whether it falls under the concept or not. In other words: as regards concepts we have a requirement of sharp delimitation; if this were not satisfied it would be impossible to set forth logical laws about them. For any argument x for which ‘ $x + 1$ ’ were *bedeutungslos*, the function $x + 1 = 10$ would likewise have no value, and thus no truth-value either, so that the concept:

‘what gives the result 10 when increased by 1’

would have no sharp boundaries. The requirement of the sharp delimitation of concepts thus carries along with it this requirement for functions in general that they must have a value for every argument.

We have so far considered truth-values only as values of functions, not as arguments. By what I have just said, we must get a value of a function when we take a truth-value as the argument; but as regards the signs already in common use, the only point, in most cases, of a rule to this effect is that there should *be* a rule; it does not much matter what is determined upon. But now we must deal with certain functions that are of importance to us precisely when their argument is a truth-value. |

I introduce the following as such a function:

— x .

I lay down the rule that the value of this function shall be the True if the True is taken as argument, and that contrariwise, in all other cases the value of this function is the False – i.e. both when the argument is the False and when it is not a truth-value at all. Accordingly, e.g.

— $1 + 3 = 4$

is the True, whereas both

— $1 + 3 = 5$

and also

— 4

are the False. Thus this function has as its value the argument itself, when that is a truth-value. I used to call this horizontal stroke the content stroke – a name that no longer seems to me appropriate.⁵ I now wish to call it simply the horizontal.

If we write down an equation or inequality, e.g. $5 > 4$, we ordinarily wish at the same time to express a judgement; in our example, we want to assert that 5 is greater than 4. According to the view I am here presenting, '5 > 4' and '1 + 3 = 5' just give us expressions for truth-values, without making any assertion. This separation of the act from the subject matter of judgement seems to be indispensable; for otherwise we could not express a mere supposition – the putting of a case without a simultaneous | judgement as to its arising or not. We thus need a special sign in order to be able to assert something. To this end I make use of a vertical stroke at the left end of the horizontal, so that, e.g., by writing

| — $2 + 3 = 5$

we assert that 2 + 3 equals 5. Thus here we are not just writing down a truth-value, as in

$2 + 3 = 5$,

but also at the same time saying that it is the True.⁶

⁶ The judgement stroke [*Urteilsstrich*] cannot be used to construct a functional expression, for it does not serve, in conjunction with other signs, to designate an object. '| — $2 + 3 = 5$ ' does not designate [*bezeichnet*] anything; it asserts something.

⁵ For Frege's earlier account, see *BS*, §2 (pp. 52–3 above). Given the bifurcation of 'content' into 'Sinn' and 'Bedeutung', the term 'content stroke' is indeed now inappropriate. But even though Frege is concerned here with the level of *Bedeutung* rather than *sense*, 'Bedeutung stroke' would also be inappropriate, since, as Frege has just explained, the expression that results from inserting a name into the argument-place of the functional expression '— x ' only stands for [*bedeutet*] the argument itself when that argument is a truth-value; in all other cases, the *Bedeutung* of the completed expression is the False.

The next simplest function, we may say, is the one whose value is the False for just those arguments for which the value of — x is the True, and, conversely, is the True for the arguments for which the value of — x is the False. I symbolize it thus:

— x ,

and here I call the little vertical stroke the negation stroke. I conceive of this as a function with the argument — x :

(— x) = (— (— x)),

where I imagine the two horizontal strokes to be fused together. But we also have:

(— (— x)) = (— x), |

since the value of — x is always a truth-value. I thus regard the parts of the stroke in '— x ' to the right and to the left of the negation stroke as horizontals, in the sense of the word that I defined previously. Accordingly, e.g.:

— $2^2 = 5$

stands for [*bedeutet*] the True, and we may add the judgement stroke:

| — $2^2 = 5$;

and in this we assert that $2^2 = 5$ is not the True, or that 2^2 is not 5. But moreover

— 2

is the True, since — 2 is the False:

| — 2 ;

i.e. 2 is not the True.

My way of presenting generality can best be seen in an example. Suppose what we have to express is that every object is equal to itself. In

$x = x$

we have a function, whose argument is indicated by ' x '. We now have to say that the value of this function is always the True, whatever we take as argument. By the sign

— α — $f(\alpha)$

I now understand the True when the function $f(x)$ always has the True as its value, whatever the argument may be; in all other cases |

— α — $f(\alpha)$

is to stand for [bedeuten] the False. For our function $x = x$ we get the first case. Thus

$$\overline{\alpha} f(\alpha)$$

is the True; and we write this as follows:

$$\alpha = \alpha.$$

The horizontal stroke to the right and to the left of the concavity are to be regarded as horizontals in our sense. Instead of 'α', any other Gothic letter could be chosen; except those which are to serve as letters for a function, like \ddot{f} and \ddot{g} .

This notation affords the possibility of negating generality, as in

$$\neg \alpha \alpha^2 = 1.$$

That is to say,

$$\overline{\alpha} \alpha^2 = 1$$

is the False, since not every argument makes the value of the function $x^2 = 1$ to be the True. (Thus, e.g., we get $2^2 = 1$ for the argument 2, and this is the False.) Now if

$$\overline{\alpha} \alpha^2 = 1$$

is the False, then

$$\neg \alpha \alpha^2 = 1$$

is the True, according to the rule that we laid down previously for the negation stroke. Thus we have

$$\neg \alpha \alpha^2 = 1;$$

i.e. 'not every object is a square root of 1', or 'there are objects that are not square roots of 1'.

Can we also express: there are square roots of 1? Certainly: we need only take, instead of the function $x^2 = 1$, the function

$$\neg x^2 = 1.$$

By fusing together the horizontals in

$$\overline{\alpha} \neg \alpha^2 = 1$$

we get

$$\overline{\alpha} \alpha^2 = 1.$$

This refers to [bedeuten] the False, since not every argument makes the value of the function

$$\neg x^2 = 1$$

to be the True. E.g.:

$$\neg 1^2 = 1$$

is the False, for $1^2 = 1$ is the True. Now since

$$\overline{\alpha} \alpha^2 = 1$$

is thus the False,

$$\neg \alpha \alpha^2 = 1$$

is the True:

$$\neg \alpha \alpha^2 = 1;$$

i.e. 'not every argument makes the value of the function

$$\neg x^2 = 1$$

to be the True', or: 'not every argument makes the value of the function $x^2 = 1$ to be the False', or: 'there is at least one square root of 1'.

At this point there may follow a few examples in symbols and words.

$$\neg \alpha \alpha \geq 0,$$

there is at least one positive number; |

$$\neg \alpha \alpha < 0,$$

there is at least one negative number;

$$\neg \alpha \alpha^3 - 3\alpha^2 + 2\alpha = 0,$$

there is at least one root of the equation

$$x^3 - 3x^2 + 2x = 0.$$

From this we may see how to express existential sentences, which are so important. If we use the functional letter f as an indefinite indication of a concept, then

$$\neg \alpha f(\alpha)$$

gives us the form that includes the last examples (if we abstract from the judgement stroke). The expressions

$$\neg \alpha \alpha^2 = 1, \quad \neg \alpha \alpha \geq 0,$$

$$\neg \alpha \alpha < 0, \quad \neg \alpha \alpha^3 - 3\alpha^2 + 2\alpha = 0,$$

arise from this form in a manner analogous to that in which x^2 gives rise to '1', '2', '3'. Now just as in x^2 we have a function whose argument is indicated by 'x', I also conceive of

$$\neg \alpha f(\alpha)$$

as the expression of a function whose argument is indicated by 'f'. Such a function is obviously a fundamentally different one from those we have dealt with so far; for only a function can occur as its argument. Now just as functions are fundamentally different from objects, so also functions whose arguments are and must be functions are fundamentally different from functions whose arguments are objects and cannot be | anything else. I call the latter first-level, the former second-level, functions. In the same way, I distinguish between first-level and second-level concepts.^H Second-level functions have actually long been used in Analysis; e.g. definite integrals (if we regard the function to be integrated as the argument).

I will now add something about functions with two arguments. We get the expression for a function by splitting up the complex sign for an object into a saturated and an unsaturated part. Thus, we split up this sign for the True,

$$'3 > 2',$$

into '3' and ' $x > 2$ '. We can further split up the 'unsaturated' part ' $x > 2$ ' in the same way, into '2' and

$$'x > y',$$

where 'y' enables us to recognize the empty place previously filled up by '2'. In

$$x > y$$

we have a function with two arguments, one indicated by 'x' and the other by 'y'; and in

$$3 > 2$$

we have the value of this function for the | arguments 3 and 2. We have here a function whose value is always a truth-value. We called such functions of one argument concepts; we call such functions of two arguments relations. Thus we have relations also, e.g., in

$$x^2 + y^2 = 9$$

and in

$$x^2 + y^2 > 9,$$

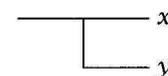
whereas the function

$$x^2 + y^2$$

has numbers as values. We shall therefore not call this a relation.

^H Cf. my *Grundlagen der Arithmetik*. I there used the term 'second-order' instead of 'second-level'. The ontological proof of God's existence suffers from the fallacy of treating existence as a first-level concept. [See *GL*, §53 (pp. 102-3 above).]

At this point I may introduce a function not peculiar to arithmetic. The value of the function



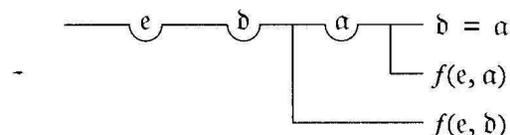
is to be the False if we take the True as the y -argument and at the same time take some object that is not the True as the x -argument; in all other cases the value of this function is to be the True. The lower horizontal stroke, and the two parts that the upper one is split into by the vertical, are to be regarded as horizontals [in our sense]. Consequently, we can always regard as the arguments of our function — x and — y , i.e. truth-values.

Among functions of one argument we distinguished first-level and second-level ones. Here, a greater multiplicity is possible. A function of two arguments may be | of the same level in relation to them, or of different levels; there are equal-levelled and unequal-levelled functions. Those we have dealt with up to now were equal-levelled. An example of an unequal-levelled function is the differential quotient, if we take the arguments to be the function that is to be differentiated and the argument for which it is differentiated; or the definite integral, so long as we take as arguments the function to be integrated and the upper limit. Equal-levelled functions can again be divided into first-level and second-level ones. An example of a second-level one is

$$F(f(1)),$$

where 'F' and 'f' indicate the arguments.

In regard to second-level functions with one argument, we must make a distinction, according as the role of this argument can be played by a function of one or of two arguments; for a function of one argument is essentially so different from one with two arguments that the one function cannot occur as an argument in the same place as the other. Some second-level functions of one argument require that this should be a function with one argument; others, that it should be a function with two arguments; and these two classes are sharply divided.



is an example of a second-level function with | one argument, which requires that this should be a function of two arguments. The letter f here indicates the argument, and the two places, separated by a comma,

within the brackets that follow 'f' bring it to our notice that f represents a function with two arguments.⁶

For functions of two arguments there arises a still greater multiplicity.

If we look back from here over the development of arithmetic, we discern an advance from level to level. At first people did calculations with individual numbers, 1, 3, etc.

$$2 + 3 = 5, \quad 2.3 = 6$$

are theorems of this sort. Then they went on to more general laws that hold good for all numbers. What corresponds to this in symbolism is the transition to algebra. A theorem of this sort is

$$(a + b).c = a.c + b.c.$$

At this stage they had got to the point of dealing with individual functions; but were not yet using the word, in its mathematical sense, and had not yet grasped its *Bedeutung*. The next higher level was the recognition of general laws about functions, accompanied by the coinage of the technical term 'function'. What corresponds to this in symbolism is the introduction of letters like f , F , to indicate functions indefinitely. A theorem of this sort is

$$\frac{df(x).F(x)}{dx} = F(x).\frac{df(x)}{dx} + f(x).\frac{dF(x)}{dx}.$$

Now at this point people had | particular second-level functions, but lacked the conception of what we have called second-level functions. By forming that, we make the next step forward. One might think that this would go on. But probably this last step is already not so rich in consequences as the earlier ones; for instead of second-level functions one can deal, in further advances, with first-level functions – as shall be shown elsewhere.⁷ But this does not banish from the world the difference between first-level and second-level functions; for it is not made arbitrarily, but founded deep in the nature of things.

Again, instead of functions of two arguments we can deal with functions of a single but complex argument; but the distinction between functions of one and of two arguments still holds in all its sharpness.

⁶ The second-level function defined here is that of a *many-one* relation. Cf. *BS*, §31, formula 115 (see p. 77 above); *GG*, I, §23.

⁷ Cf. *GG*, I, §§25, 34–7.