3.2 (SUMMARY) — GENERAL SOLUTIONS OF LINEAR EQUATIONS

Consider the n-th order linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \tag{1}$$

where $p_0, \ldots, p_{n-1}$ and f are continuous functions.

Example. $y''' + x^4y'' - 3y' + y = \sin(x)$.

Theorem 2. *(Existence and uniqueness)* For each choice of numbers $a, b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}$, there exists exactly one solution of (1) that satisfies the initial conditions

$$y(a) = b_0,\ y'(a) = b_1,\ \ldots,\ y^{(n-1)}(a) = b_{n-1}.$$

Proof: omitted.

Remark. To find this solution, we first have to treat the **homogeneous** ($f = 0$) equation

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0} \tag{2}$$

Theorem 1. *(Superposition)* If $y_1, \ldots, y_n$ are solutions of the homogeneous DE (2) and $c_1, \ldots, c_n \in \mathbb{R}$ then the linear combination

$$y = c_1y_1 + \cdots + c_ny_n \tag{3}$$

is also a solution of the homogeneous DE. *Proof:* exercise.

Note. Superposition is valid for (2), not for the nonhomogeneous DE (1).

Example. If y_1 and y_2 are solutions of (2), then so is $5y_1 - 3y_2$.

Theorem 4. *(General solution)* Every solution of the homogeneous DE (2) can be written as a linear combination (3) of the solutions $y_1, \ldots, y_n$, provided that $y_1, \ldots, y_n$ are **linearly independent**. *Proof:* omitted.

Thus we can call $y = c_1y_1 + \cdots + c_ny_n$ the general solution of the homogeneous DE.

Linear independence means that

no one of $y_1, \ldots, y_n$ is equal to a linear combination of the others.

[When $n = 2$, this new definition of linear independence for y_1, y_2 is the same as the old one, which says that neither y_1 nor y_2 is a multiple of the other.]

Conclusion. We should first try to find n linearly independent solutions of the homogeneous equation (2). Then since every solution can be written as a linear combination of $y_1, \ldots, y_n$, we can find the desired solution by applying the initial conditions to $y = c_1y_1 + \cdots + c_ny_n$ and solving for the coefficients $c_1, \ldots, c_n$.

Question. For constant coefficient homogeneous equations, we find solutions by $\boxed{y = e^{rx}}$. Then the characteristic polynomial has degree n, so it has n roots (counting repeated roots and complex roots). So our guess $y = e^{rx}$ will yield n solutions $y_1, \ldots, y_n$ of the DE (provided we deal properly with repeated roots; see Sec. 3.1 and 3.3).

But are these solutions linearly independent...???

Problems (examinable)

1. Suppose $r_1 < r_2 < r_3$. Show that the functions $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}, y_3 = e^{r_3 x}$ are linearly independent.

2. Suppose $r_1 < r_2 < \ldots < r_n$. Show that the functions $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}, \ldots, y_n = e^{r_n x}$ are linearly independent.

3. Let $r \in \mathbb{R}$. Show that $y_1 = e^{rx}, y_2 = xe^{rx}, y_3 = x^2 e^{rx}$ are linearly independent.

4. Show that the functions $y_1 = \sin x, y_2 = \sin(2x)$ are linearly independent.

Remark. By arguments like above, one can show that the n solutions of a constant coefficient homogeneous linear equation that we find by trying $y = e^{rx}$ will always be linearly independent.

Wronskians — don't use them!

The textbook develops a method for checking linear independence of $y_1, \ldots, y_n$ by using the Wronskian. This clever method relies on matrix algebra and the Existence and Uniqueness Theorem (or on other deep arguments).

We will **not** use Wronskians in this course. Instead we emphasize the *meaning* of linear independence (that no one function can be written as a linear combination of the others, like in the Problems above).