Graphing Functions using Transformations

The most common **parent functions** include:

- Linear function $f(x) = x$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \frac{1}{x}$
- Root function $f(x) = \sqrt{x}$
- Sine function $f(x) = \sin(x)$
- Cosine function $f(x) = \cos(x)$
- Tangent function $f(x) = \tan(x)$

Using **transformations**, many other functions can be obtained from these parents functions.

The following general form outlines the possible transformations:

$$\boxed{f(x) = a\, f[\, b(x - h)] + k}$$

a > 1 → Vertical stretch by a factor of **a**.

0 < |a| < 1 → Vertical compression by a factor of **a**.

a is –ve → Vertical reflection (reflection in the x-axis).

b > 1 → Horizontal compression by a factor of $\frac{1}{b}$.

0 < |b| < 1 → Vertical stretch by a factor of $\frac{1}{b}$.

b is –ve → Horizontal reflection (reflection in the y-axis).

h > 1 (i.e. +ve) → horizontal translation **h** units to the right.

h < 1 (i.e. –ve) → horizontal translation **h** units to the left.

Note: Pay special attention to the "–"sign inside the brackets!

k > 1 (i.e. +ve) → vertical translation **k** units up.

k < 1 (i.e. –ve) → vertical translation **k** units down.

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Example 1:

What transformations have been applied to the parent function, $f(x) = \sqrt{x}$ to obtain $g(x) = -3\sqrt{2(x+8)} - 19$?

Solution:

$a = -3$, Indicates a vertical stretch by a factor of 3 and a reflection in the x-axis.

$b = 2$, Indicates a horizontal compression by a factor of $\frac{1}{2}$.

$h = -8$, Indicates a translation 8 units to the left.

$k = -19$, Indicates a translation 19 units down.

Example 2:

Write an equation for $f(x) = \frac{1}{x}$ after the following transformations are applied: vertical stretch by a factor of 4, horizontal stretch by a factor of 2, reflection in the y-axis, translation 3 units up and 2 units right.

Solution:

Vertical stretch by a factor of 4 means that $a = 4$

Horizontal stretch by a factor of 2 and reflection in the y-axis means that $b = -\frac{1}{2}$

Translation 3 units up means that $k = 3$

Translation 2 units right means that $h = 2$

Plugging these values into the general form $f(x) = a\,f[\,b(x-h)] + k$ where $f(x) = \frac{1}{x}$, we get $f(x) = 4[\frac{1}{-\frac{1}{2}(x-2)}] + 3$. This can be simplified to $f(x) = \frac{4}{-\frac{1}{2}(x-2)} + 3$.

The **mapping rule** is useful when graphing functions with transformations.

Any point **(x, y)** of a parent function becomes **$(\frac{x}{b} + h, ay + k)$** after the transformations have been applied.

$$(x, y) \longrightarrow \left(\frac{x}{b} + h, ay + k\right)$$

(Notice that the "horizontal transformations" b and h affect only the x values, while the "vertical transformations" a and k affect only the y values.)

Note: When using the mapping rule to graph functions using transformations you should be able to graph the parent function and list the "main" points.

Example 3:

Use transformations to graph the following functions:

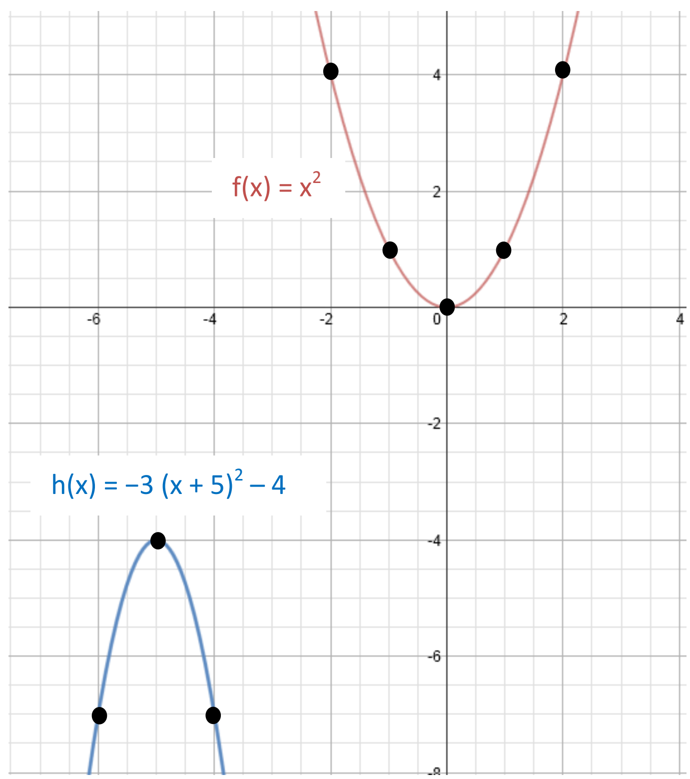
a) $h(x) = -3\,(x + 5)^2 - 4$
b) $g(x) = 2\cos(-x + 90°) + 8$

Solutions:

a) The parent function is $f(x) = x^2$

The following transformations have been applied:
$a = -3$ (Vertical stretch by a factor of 3 and reflection in the x-axis)
$h = -5$ (Translation 5 units to the left)
$k = -4$ (Translation 4 units down)

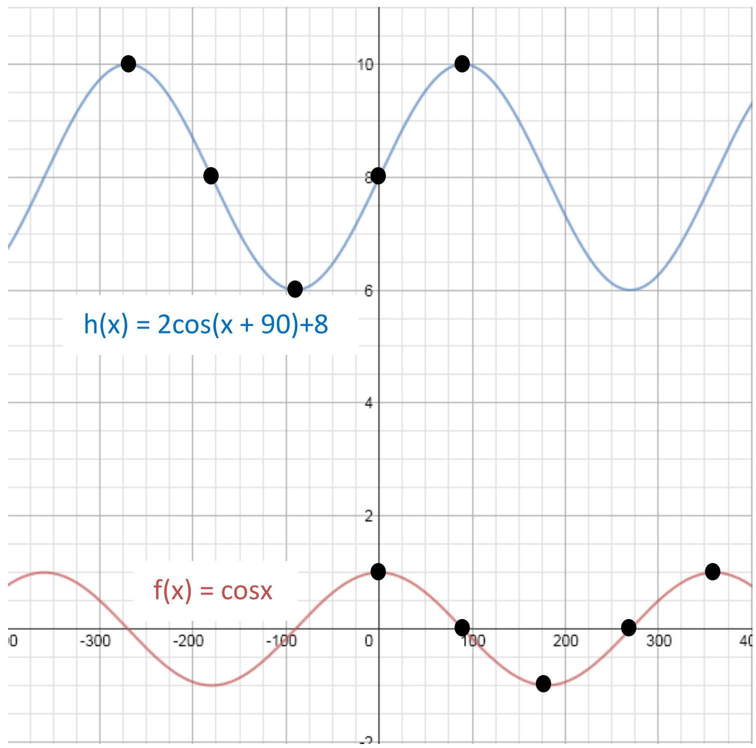
(x, y)	**$(\frac{x}{b} + h,\ ay + k)$**
$(-2, 4)$	$\frac{x}{b} + h = -2-5 = -7$ $ay + k = -3(4)-4 = -16$ $(-7, -16)$
$(-1, 1)$	$\frac{x}{b} + h = -1-5 = -6$ $ay + k = -3(1)-4 = -7$ $(-6, -7)$
$(0, 0)$	$\frac{x}{b} + h = 0-5 = -5$ $ay + k = -3(0)-4 = -4$ $(-5, -4)$
$(1, 1)$	$\frac{x}{b} + h = 1-5 = -4$ $ay + k = -3(1)-4 = -7$ $(-4, -7)$
$(2, 4)$	$\frac{x}{b} + h = 2-5 = -3$ $ay + k = -3(4)-4 = -16$ $(-3, -16)$



b) In this particular example, first factor out the –ve sign inside the brackets.
$g(x) = 2\cos[-(x - 90°)] + 8$
The parent function is $f(x) = \cos(x)$

The following transformations have been applied:
$a = 2$ (Vertical stretch by a factor of 2)
$b = -1$ (Reflection in the y-axis)
$h = 90°$ (Translation 90° to the right)
$k = 8$ (Translation 8 units up)

(x, y)	**$(\frac{x}{b} + h, ay + k)$**
(0°, 1)	$\frac{x}{b} + h = \frac{0}{-1} + 90 = 90°$ $ay + k = 2(1) + 8 = 10$ (90°, 10)
(90°, 0)	$\frac{x}{b} + h = \frac{90}{-1} + 90 = 0°$ $ay + k = 2(0) + 8 = 8$ (0°, 8)
(180°, −1)	$\frac{x}{b} + h = \frac{180}{-1} + 90 = -90°$ $ay + k = 2(-1) + 8 = 6$ (−90°, 6)
(270°, 0)	$\frac{x}{b} + h = \frac{270}{-1} + 90 = -180°$ $ay + k = 2(0) + 8 = 8$ (−180°, 8)
(360°, 1)	$\frac{x}{b} + h = \frac{360}{-1} + 90 = -270°$ $ay + k = 2(1) + 8 = 10$ (−270°, 10)



Practice Questions

1. The graph of $f(x) = x^3$ was reflected in the y-axis, compressed vertically by a factor of $\frac{1}{2}$ and translated 4 units up and 6 units to the left. What is the equation for the transformed function? Sketch the parent and the transformed functions.
2. For each of the following functions i) state the parent function and transformations that have been applied and ii) graph the transformed function using the mapping rule.
   a) $f(x) = -3(x + 8)^2 - 5$
   b) $g(x) = 4\sqrt{-2x + 8} + 6$
   c) $h(x) = 2\sin(-x) - 4$

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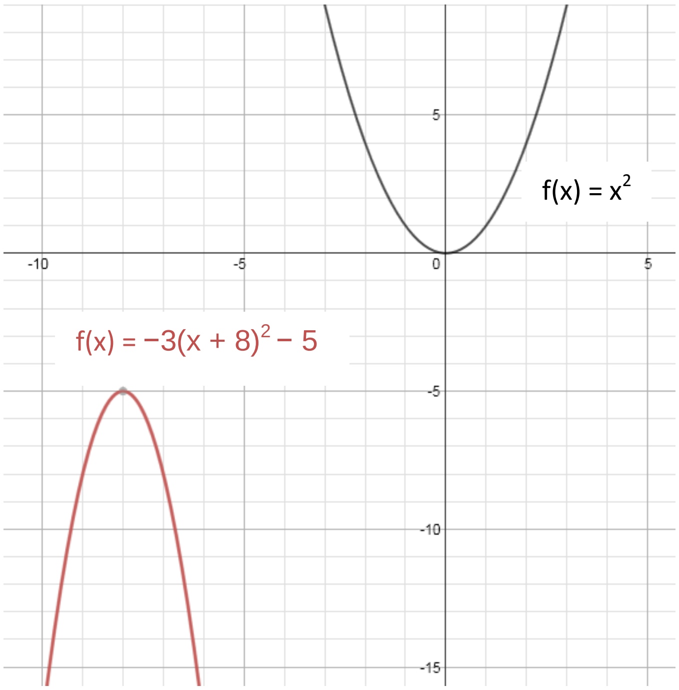
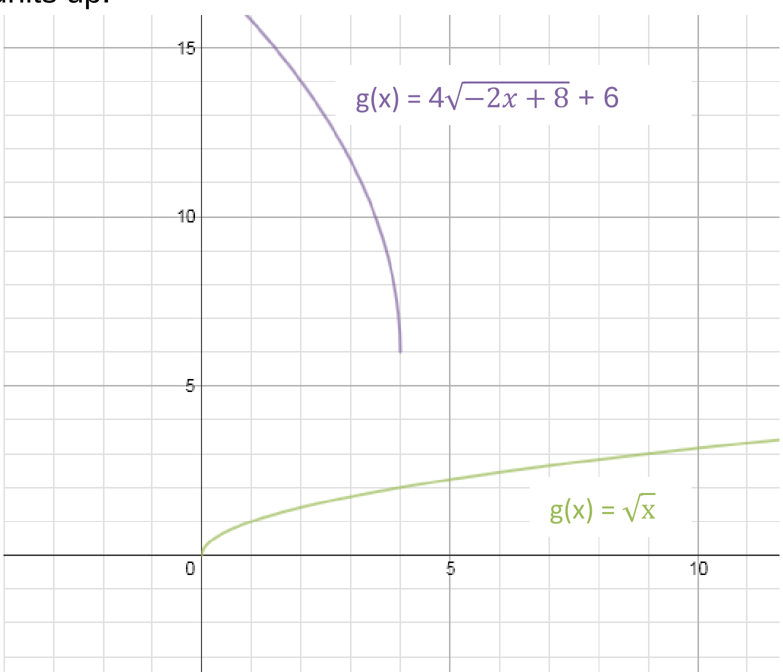
Answers

1. The equation for the transformed function is $f(x) = \frac{1}{2}(-x - 6)^3 + 4$.
2. a) The parent function is $f(x) = x^2$
   The parent function has been reflected in the x-axis, vertically stretched by a factor of 3, translated 8 units to the left and 5 units down.



b) The parent function is $g(x) = \sqrt{x}$
   The parent function has been vertically stretched by a factor of 4, reflected in the y-axis, horizontally compressed by a factor of $\frac{1}{2}$, translated 4 units to the left and 6 units up.



c) The parent function is h(x) = sin(x)
The parent function has been vertically stretched by a factor of 2, reflected in the y-axis and translated 4 units down.