MATH 116, LECTURE 8: Heaviside Function

1 Heaviside function

Many applications require us to consider different functions over different domains. This is frequently the case when a force is turned "on" or "off" at specific times. For example, an applied current in an electrical circuit, or a mechanical force like a motor.

We have already dealt with functions which are defined in this manner. They are called **piecewise-defined functions** and are given by the form

$$f(x) = \begin{cases} f_1(x), & \text{for } x_0 \leq x < x_1 \\ f_2(x), & \text{for } x_1 \leq x < x_2 \\ \cdots & \cdots \\ f_n(x), & \text{for } x_{n-1} \leq x \leq x_n. \end{cases}$$

In this set up, we can think of $f_i(x)$ being turned "on" in the interval $x_{i-1} \leq x < x_i$ and "off" everywhere else.

There is another approach to handling functions defined in this manner. It is by making use of the **Heaviside function** $H(x)$ defined according to

$$H(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x \geq 0. \end{cases}$$

The heaviside function is turned "off" (i.e. takes the value zero) for $x < 0$ and is turned "on" (i.e. takes the value one) for $x \geq 0$.

Important modifications to the heaviside function are the following:

1. The **shifted heaviside function** $H(x-a)$ is defined according to

$$H(x-a) = \begin{cases} 0, & \text{for } x < a \\ 1, & \text{for } x \geq a. \end{cases}$$

   It is turned "off" for values $x < a$ and "on" for values $x \geq a$.

2. We can restrict the domain of the heaviside function to a finite interval $[a, b]$ with

$$H(x-a) - H(x-b) = \begin{cases} 0, & \text{for } x < a \\ 1, & \text{for } a \leq x < b \\ 0, & \text{for } x \geq b. \end{cases}$$

This is only turned "on" in the interval $[a, b)$.

The advantage of the heaviside function is that we can use it to write piecewise functions in a single line. There are also many operations which we are able to perform on it which cannot be performed on piecewise functions otherwise defined (although most of those will have to wait for upper year courses).

Example 1: Write the following piecewise defined function using heaviside functions:

$$f(x) = \begin{cases} x^3, & \text{for } x < 0 \\ -(x-1)^2 + 1, & \text{for } 0 \leq x < 2 \\ 0, & \text{for } x \geq 2. \end{cases}$$

Solution: We need to x^3 to be "on" in the interval $x < 0$, $-(x-1)^2+1$ to be on in the interval $0 \leq x < 2$ and everything to "off" for $x \geq 2$.

We can turn x^3 "on" for $x < 0$ and "off" for $x \geq 0$ with

$$x^3 - x^3 H(x).$$

We can turn $-(x-1)^2 + 1$ "on" at $x = 0$ and "off" at $x = 2$ with

$$(-(x-1)^2 + 1)[H(x) - H(x-2)].$$

It follows that

$$\begin{aligned} f(x) &= x^3 - x^3 H(x) + (-(x-1)^2 + 1)[H(x) - H(x-2)] \\ &= x^3 + (-(x-1)^2 + 1 - x^3)H(x) - (-(x-1)^2 + 1)H(x-2). \end{aligned}$$

Either way of writing this is fine.

Example 2: Find the piecewise definition of the following

$$f(x) = \sin(x)H(x) + \sin(x)H(x-\pi) - 2\sin(x)H(x-2\pi).$$

Solution: When $x < 0$, nothing is "on". For $0 \leq x < \pi$, $H(x)$ is "on" but the other functions are "off". In $\pi \leq x < 2\pi$, $H(x)$ and $H(x-\pi)$ are both "on", and then for $x \geq 2\pi$ all the heaviside functions are "on". This gives

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ \sin(x), & \text{for } 0 \leq x < \pi \\ 2\sin(x), & \text{for } \pi \leq x < 2\pi \\ 0, & \text{for } x \geq 2\pi. \end{cases}$$

Example 3: Graph the function $H(x^2 - 3x - 10)$.

Solution: From the definition of $H(x)$ we have

$$H(x^2 - 3x - 10) = \begin{cases} 0, & \text{for } x^2 - 3x - 10 < 0 \\ 1, & \text{for } x^2 - 3x - 10 \geq 0. \end{cases}$$

We need to determine where $x^2 - 3x + 10$ changes signs. We have

$$x^2 - 3x - 10 = (x - 5)(x + 2).$$

This equals zero when $x = -2$ and $x = 5$. The overall sign is given by

	$(-\infty, -2)$	$(-2, 5)$	$(5, \infty)$
$(x + 2)$	$-$	$+$	$+$
$(x - 5)$	$-$	$-$	$+$
$x^2 - 3x + 10$	$+$	$-$	$+$

It follows that

$$H(x^2 - 3x - 10) = \begin{cases} 0, & \text{for } x < -2 \\ 1, & \text{for } -2 \leq x \leq 5 \\ 0, & \text{for } x > 5. \end{cases}$$

(See lecture notes for the graph.)