Interpreting Summation Notation

Plan

Learning Goals

Students will be able to:

- Understand how the parts of a limit of a Riemann sum expression relate to the parts of a definite integral expression
- Convert the limit of a Riemann sum to an integral expression

These activities lay the foundation for deep understanding of the meaning of a definite integral, both in terms of a signed area and as a limit of Riemann sums. Since the integral is one of the two major operations of calculus, understanding its meaning will be fundamental to understanding the course concepts and being able to transfer knowledge and skills to other subjects and topics that students will encounter.

This lesson helps to build skills 2.C and 4.C by having students identify errors in existing Riemann sums and write definite integral expressions that represent a given Riemann sum. Students will also see definite integrals connected to the concept of area under a curve, and will practice skill 1.E by calculating those areas.

Student misunderstandings

- The student may struggle to look for Δx when relating the Δx in sigma notation to the *dx* in the definite integral.
- The student may find Δx incorrectly (e.g., $\frac{1}{n}$ instead of $\frac{(b-a)}{n}$).
- The student may struggle to represent x_k as $x_0 + k\Delta x$. Students typically leave off the x_0 and only account for $k\Delta x$
- The student may struggle to find the limits of integration based on sigma notation (i.e., he or she may understand that sigma notation is shorthand for a sum, but may not know how to get the limits for the definite integral from the sigma notation).
- The student may struggle to identify the Δx in the sigma notation, especially when there are other fractions in the expression.
- The student may not know where to begin (e.g., may not know how to find limits of integration, may not know what function serves as the integrand, etc.).
- The student may not know how to write x_i .

Materials

The following supplies are needed:

- student activity sheets (1 per student)

Mind Your P's and Two's

Recall the definition of the definite integral as a limit of a Riemann sum:

$$\int_a^b f(x)dx = \lim_{x\to\infty}\sum_{i=1}^{n} f(a+\Delta x\cdot i)\Delta x \text{ , where } \Delta x = \frac{b-a}{n} \text{ .}$$

In the chart below, there is an error in the expression for the limit of the Riemann sum: one of the numbers needs to be replaced with a 2. Circle the error and replace with a 2, explaining why the 2 should be there.

Definite Integral	Limit of Riemann sum	Explanation
$\int_4^6 (3x^2+5)dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(4+\frac{2i}{n}\right)^2+5\right]\frac{6}{n}$	Answer: The width of each interval, Δx, is given by $\frac{b-a}{n}$, so in this case, $\frac{6-4}{n}=\frac{2}{n}$.
$\int_2^7 (4x-1)dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[4\left(4+\frac{5i}{n}\right)^2-1\right]\frac{5}{n}$	Answer: The value of a is 2 and the height of the rectangle is given by $f(a+\Delta x\cdot i)$.
$\int_3^4 (6x^2+3x)dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[6\left(3+\frac{1i}{n}\right)^3+3\left(3+\frac{1i}{n}\right)\right]\frac{1}{n}$	Answer: The exponent in the integrand is a 2, not a 3.
$\int_5^9 (8-2x)dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[8-4\left(5+\frac{4i}{n}\right)\right]\frac{4}{n}$	Answer: The coefficient in the integrand is a 2, not a 4.
$\int_{-1}^1 3x^3dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(-1+\frac{1i}{n}\right)^3\right]\frac{2}{n}$	Answer: The value of Δx is $\frac{2}{n}$ and the height of the rectangle is given by $f(a+\Delta x\cdot i)$, so that 1 in the numerator should be replaced with a 2.

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Teach

Engage

Make sure students understand that identifying delta x is often the best way to begin a problem.

Remind them that $\Delta x = \frac{b-a}{n}$

Teach students to write out the integrand portion of the summation leaving empty parentheses for every variable, and then substituting $(a+\Delta x\cdot i)$ into each set of empty parentheses. For example:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}[f(a+\Delta x\cdot i)]\Delta x$$

$$\int_{-1}^1 3x^3dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(\boxed{-1+\frac{2}{n}\cdot i}\right)\right]\frac{2}{n}$$

$$\int_{-1}^1 3x^3dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[3(\qquad)^3\right]\frac{2}{n}$$

$$\int_{-1}^1 3x^3dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(-1+\frac{2}{n}\cdot i\right)^3\right]\frac{2}{n}$$

Do several examples of this and then have different students come to the board to do a few examples with their peers providing constructive criticism.

AP CALCULUS STUDENT HANDOUT

Apply Your Understanding of Summation Notation

In the chart below, the limit of a Riemann sum has been provided for you. Write the corresponding definite integral.

	Definite Integral	Limit of Riemann Sum
1.	Answer: $\int_0^6 \sqrt{2x+1}\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{2\left(\frac{6}{n}\right)+1}\right]\frac{6}{n}$
2.	Answer: $\int_{-2}^{3} x^2-3\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(-2+\frac{5i}{n}\right)^2-3\right]\frac{5}{n}$
3.	Answer: $\int_1^6 3x-4\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(1+\frac{5i}{n}\right)-4\right]\frac{5}{n}$
4.	Answer: $\int_{-2}^{4} x^3\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(-2+\frac{6i}{n}\right)^3\right]\frac{6}{n}$
5.	Answer: $\int_{-2}^{0} \sqrt{x^2+1}\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{\left(-2+\frac{2i}{n}\right)^2+1}\right]\frac{2}{n}$
6.	Answer: $\int_2^6 5x+7\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[5\left(2+\frac{4i}{n}\right)+7\right]\frac{4}{n}$
7.	Answer: $\int_0^4 6x^2-2\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[6\left(\frac{4i}{n}\right)^2-2\right]\frac{4}{n}$
8.	Answer: $\int_1^3 4x^3-1\,dx$	$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{2i}{n}\right)^3-1\right]\frac{2}{n}$

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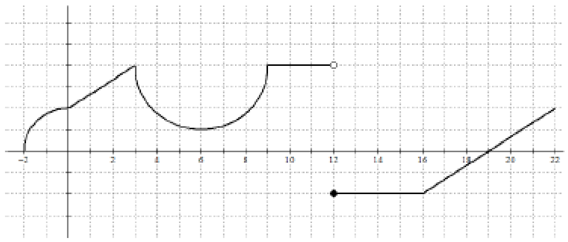
Guided Practice

Read through the directions together, then have students work with a partner to complete the table. Go over any questions at the end of the exercise.

In checking students' work, notice if they are making consistent errors in notation, such as leaving off the *dx* of the integral expression. Circling errors, but not correcting them, and asking students to determine the mistake, will help them develop their ability to proofread their own work.

AP CALCULUS TEACHING STRATEGIES

Translating Notation and Finding Definite Integral Values



The graph above consists of a quarter circle, a half circle and four line segments. For each of the expressions below, fill in the missing definite integrals. Then determine the value of each definite integral using geometric formulas (without using a calculator).

Limit of Riemann Sum	Definite Integral	Value of Definite Integral
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(-2)\left(\frac{4}{n}\right)$	Answer: $\int_{12}^{16}(-2)\,dx$	Answer: -8
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(4-\sqrt{9-\left(\left(3+\frac{6i}{n}\right)-6\right)^2}\right)\left(\frac{6}{n}\right)$	Answer: $\int_{3}^{9}4-\sqrt{9-(x-6)^2}\,dx$	Answer: $24-\frac{9\pi}{2}$
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{2}{3}\left(\frac{3i}{n}\right)+2\right)\left(\frac{3}{n}\right)$	Answer: $\int_{0}^{3}\left(\frac{2}{3}x+2\right)dx$	Answer: 9
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\sqrt{4-\left(-2+\frac{2i}{n}\right)^2}\right)\left(\frac{2}{n}\right)$	Answer: $\int_{-2}^{0}\sqrt{4-x^2}\,dx$	Answer: π
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(4)\left(\frac{3}{n}\right)$	Answer: $\int_{9}^{12}4\,dx$	Answer: 12
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{2}{3}\left(\left(16+\frac{6i}{n}\right)-19\right)\right)\left(\frac{6}{n}\right)$	Answer: $\int_{16}^{22}\frac{2}{3}(x-19)\,dx$	Answer: 0

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Independent Practice

Have students work individually to complete the table. If time does not permit for them to complete the assignment in class, this could be given as homework.

If students are struggling with this activity, you can provide them with part of the answer, such as the limits of integration or the integrand, and have them fill in other parts. Point out to them how specific parts of the limit expression match up with the definite integral expression such as the lower limit of integration or the width of the interval.

If students are struggling with determining the value of the definite integral, encourage them to show what geometric formula they are using.

Assess

Direct students to complete the Topic Questions.

Additional Teacher Notes

This skill is best developed through repetition and consistency. Every time students encounter a definite integral, have them translate to the associated limit expression and vice versa. These problems work well for warm-up problems at the beginning of class, or as transitional problems when moving from one activity to another, or whenever there are a few minutes left at the end of class. Make sure that they have used correct notation. For example, only writing *lim* without including $n\to\infty$ would be considered incorrect. Details are very important.

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Student handouts with answers:

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Recall the definition of the definite integral as a limit of a Riemann sum:

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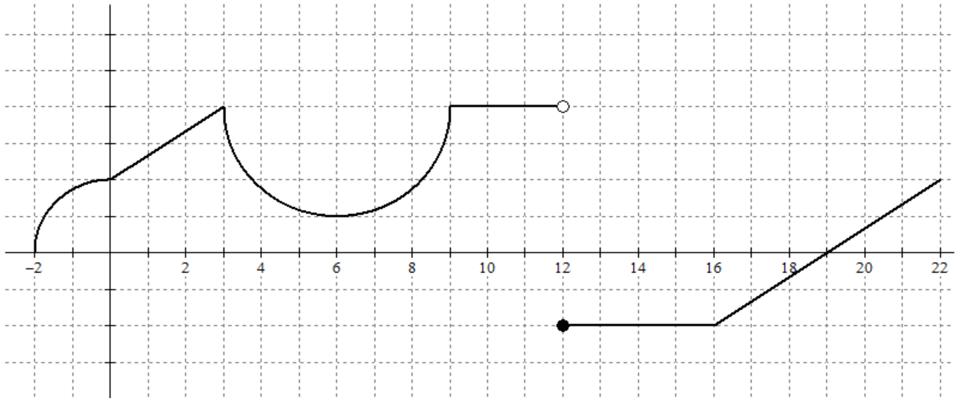

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