

1.3 INTRODUCTION TO FUNCTIONS

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

Definition 1.6. A relation in which each x -coordinate is matched with only one y -coordinate is said to describe y as a **function** of x .

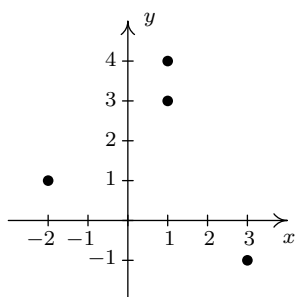
Example 1.3.1. Which of the following relations describe y as a function of x ?

1. $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
2. $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$

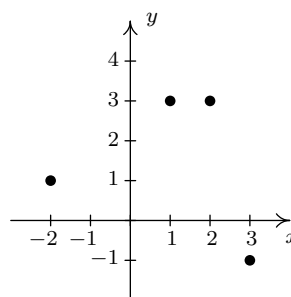
Solution. A quick scan of the points in R_1 reveals that the x -coordinate 1 is matched with two *different* y -coordinates: namely 3 and 4. Hence in R_1 , y is not a function of x . On the other hand, every x -coordinate in R_2 occurs only once which means each x -coordinate has only one corresponding y -coordinate. So, R_2 does represent y as a function of x . \square

Note that in the previous example, the relation R_2 contained two different points with the same y -coordinates, namely $(1, 3)$ and $(2, 3)$. Remember, in order to say y is a function of x , we just need to ensure the same x -coordinate isn't used in more than one point.¹

To see what the function concept means geometrically, we graph R_1 and R_2 in the plane.



The graph of R_1



The graph of R_2

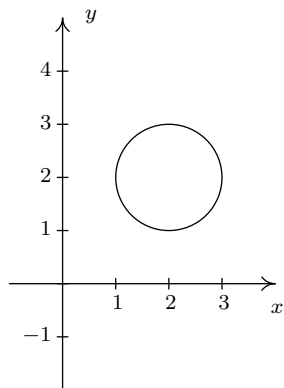
The fact that the x -coordinate 1 is matched with two different y -coordinates in R_1 presents itself graphically as the points $(1, 3)$ and $(1, 4)$ lying on the same vertical line, $x = 1$. If we turn our attention to the graph of R_2 , we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

Theorem 1.1. The Vertical Line Test: A set of points in the plane represents y as a function of x if and only if no two points lie on the same vertical line.

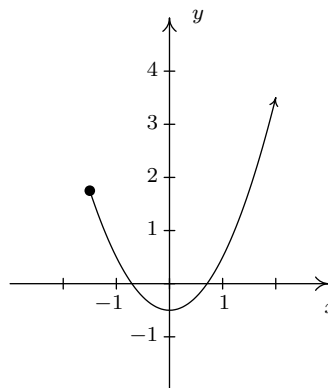
¹We will have occasion later in the text to concern ourselves with the concept of x being a function of y . In this case, R_1 represents x as a function of y ; R_2 does not.

It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of ‘function’ as well as the concept of ‘graph’.

Example 1.3.2. Use the Vertical Line Test to determine which of the following relations describes y as a function of x .



The graph of R

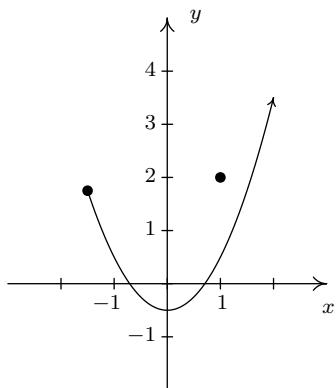


The graph of S

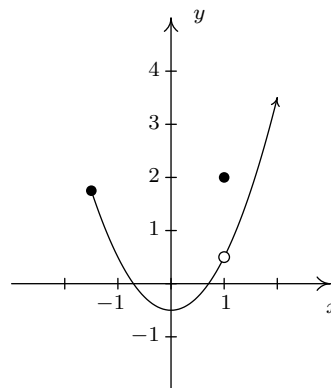
Solution. Looking at the graph of R , we can easily imagine a vertical line crossing the graph more than once. Hence, R does not represent y as a function of x . However, in the graph of S , every vertical line crosses the graph at most once, so S does represent y as a function of x . \square

In the previous test, we say that the graph of the relation R **fails** the Vertical Line Test, whereas the graph of S **passes** the Vertical Line Test. Note that in the graph of R there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is one vertical line that fits the bill, as the next example illustrates.

Example 1.3.3. Use the Vertical Line Test to determine which of the following relations describes y as a function of x .

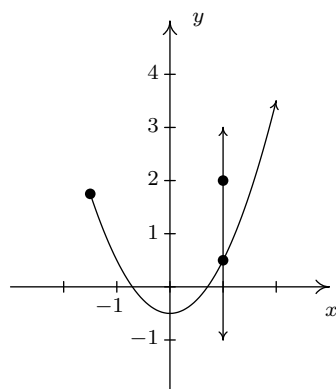


The graph of S_1

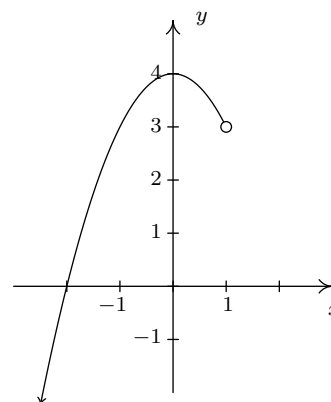


The graph of S_2

Solution. Both S_1 and S_2 are slight modifications to the relation S in the previous example whose graph we determined passed the Vertical Line Test. In both S_1 and S_2 , it is the addition of the point $(1, 2)$ which threatens to cause trouble. In S_1 , there is a point on the curve with x -coordinate 1 just below $(1, 2)$, which means that both $(1, 2)$ and this point on the curve lie on the vertical line $x = 1$. (See the picture below and the left.) Hence, the graph of S_1 fails the Vertical Line Test, so y is not a function of x here. However, in S_2 notice that the point with x -coordinate 1 on the curve has been omitted, leaving an ‘open circle’ there. Hence, the vertical line $x = 1$ crosses the graph of S_2 only at the point $(1, 2)$. Indeed, any vertical line will cross the graph at most once, so we have that the graph of S_2 passes the Vertical Line Test. Thus it describes y as a function of x . \square



S_1 and the line $x = 1$



The graph of G for Ex. 1.3.4

Suppose a relation F describes y as a function of x . The sets of x - and y -coordinates are given special names which we define below.

Definition 1.7. Suppose F is a relation which describes y as a function of x .

- The set of the x -coordinates of the points in F is called the **domain** of F .
- The set of the y -coordinates of the points in F is called the **range** of F .

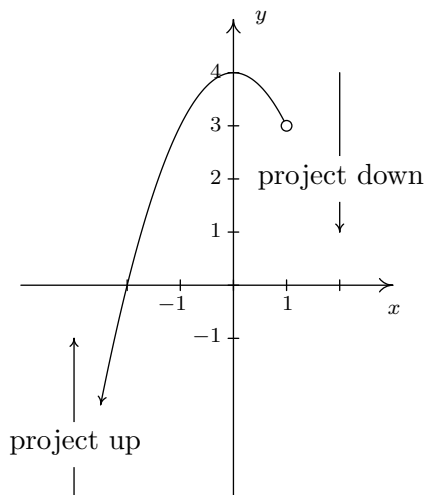
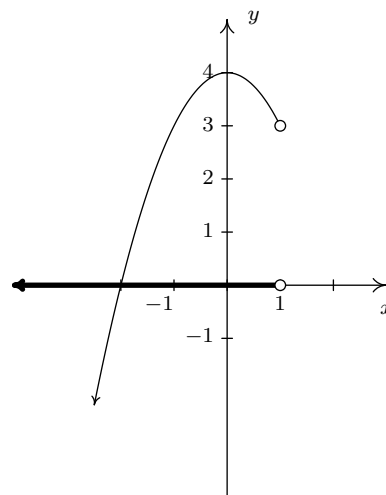
We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

Example 1.3.4. Find the domain and range of the function $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$ and of the function G whose graph is given above on the right.

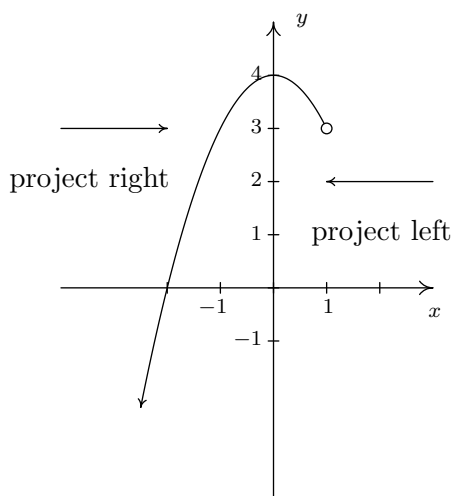
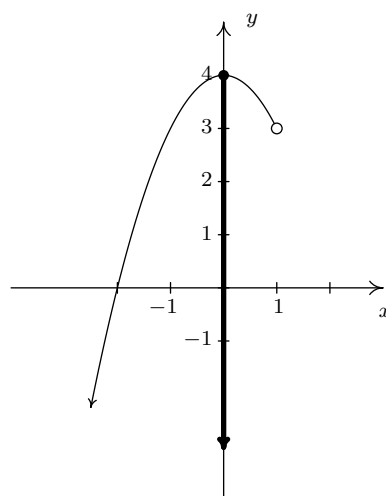
Solution. The domain of F is the set of the x -coordinates of the points in F , namely $\{-3, 0, 4, 5\}$ and the range of F is the set of the y -coordinates, namely $\{1, 2\}$.

To determine the domain and range of G , we need to determine which x and y values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the x -axis and determining the portion of the x -axis that gets covered. This is called **projecting** the curve to the x -axis. Before we start projecting, we need to pay attention to two

subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ isn't on the graph, but all points on the curve leading up to that point are.

The graph of G The graph of G

We see from the figure that if we project the graph of G to the x -axis, we get all real numbers less than 1. Using interval notation, we write the domain of G as $(-\infty, 1)$. To determine the range of G , we project the curve to the y -axis as follows:

The graph of G The graph of G

Note that even though there is an open circle at $(1, 3)$, we still include the y value of 3 in our range, since the point $(-1, 3)$ is on the graph of G . We see that the range of G is all real numbers less than or equal to 4, or, in interval notation, $(-\infty, 4]$. \square

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section 1.2 may or may not describe y as a function of x . The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section 1.4.)

Example 1.3.5. Determine which equations represent y as a function of x .

1. $x^3 + y^2 = 1$

2. $x^2 + y^3 = 1$

3. $x^2y = 1 - 3y$

Solution. For each of these equations, we solve for y and determine whether each choice of x will determine only one corresponding value of y .

1.

$$\begin{aligned} x^3 + y^2 &= 1 \\ y^2 &= 1 - x^3 \\ \sqrt{y^2} &= \sqrt{1 - x^3} && \text{extract square roots} \\ y &= \pm\sqrt{1 - x^3} \end{aligned}$$

If we substitute $x = 0$ into our equation for y , we get $y = \pm\sqrt{1 - 0^3} = \pm 1$, so that $(0, 1)$ and $(0, -1)$ are on the graph of this equation. Hence, this equation does not represent y as a function of x .

2.

$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

For every choice of x , the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** value of y . Hence, this equation describes y as a function of x .

3.

$$\begin{aligned} x^2y &= 1 - 3y \\ x^2y + 3y &= 1 \\ y(x^2 + 3) &= 1 && \text{factor} \\ y &= \frac{1}{x^2 + 3} \end{aligned}$$

For each choice of x , there is only one value for y , so this equation describes y as a function of x . \square

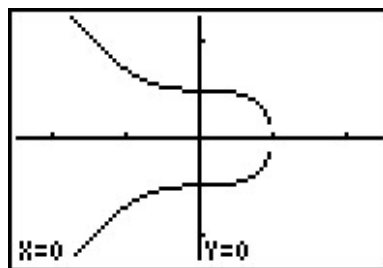
We could try to use our graphing calculator to verify our responses to the previous example, but we immediately run into trouble. The calculator's "Y=" menu requires that the equation be of the form ' $y = \text{some expression of } x$ '. If we wanted to verify that the first equation in Example 1.3.5

does not represent y as a function of x , we would need to enter two separate expressions into the calculator: one for the positive square root and one for the negative square root we found when solving the equation for y . As predicted, the resulting graph shown below clearly fails the Vertical Line Test, so the equation does not represent y as a function of x .

```

Plot1 Plot2 Plot3
Y1=√(1-X^3)
Y2=-√(1-X^3)
Y3=
Y4=
Y5=
Y6=
Y7=

```



Thus in order to use the calculator to show that $x^3 + y^2 = 1$ does not represent y as a function of x we needed to know *analytically* that y was not a function of x so that we could use the calculator properly. There are more advanced graphing utilities out there which can do implicit function plots, but you need to know even more Algebra to make them work properly. Do you get the point we're trying to make here? We believe it is in your best interest to learn the analytic way of doing things so that you are always smarter than your calculator.

1.3.1 EXERCISES

In Exercises 1 - 12, determine whether or not the relation represents y as a function of x . Find the domain and range of those relations which are functions.

1. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$

2. $\{(-3, 0), (1, 6), (2, -3), (4, 2), (-5, 6), (4, -9), (6, 2)\}$

3. $\{(-3, 0), (-7, 6), (5, 5), (6, 4), (4, 9), (3, 0)\}$

4. $\{(1, 2), (4, 4), (9, 6), (16, 8), (25, 10), (36, 12), \dots\}$

5. $\{(x, y) \mid x \text{ is an odd integer, and } y \text{ is an even integer}\}$

6. $\{(x, 1) \mid x \text{ is an irrational number}\}$

7. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \dots\}$

8. $\{\dots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \dots\}$

9. $\{(-2, y) \mid -3 < y < 4\}$

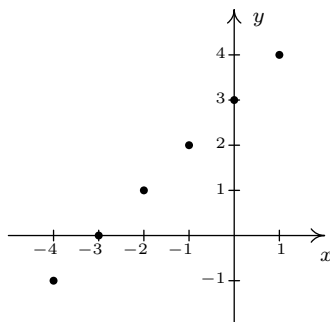
10. $\{(x, 3) \mid -2 \leq x < 4\}$

11. $\{(x, x^2) \mid x \text{ is a real number}\}$

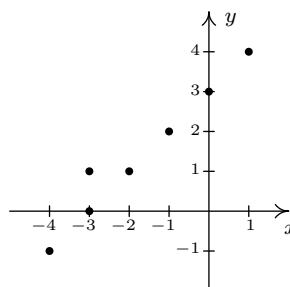
12. $\{(x^2, x) \mid x \text{ is a real number}\}$

In Exercises 13 - 32, determine whether or not the relation represents y as a function of x . Find the domain and range of those relations which are functions.

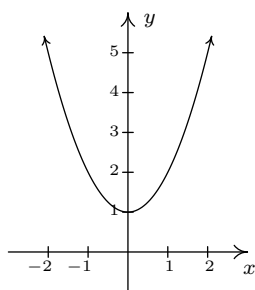
13.



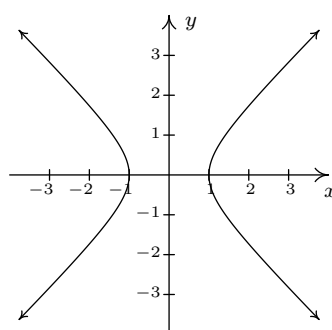
14.



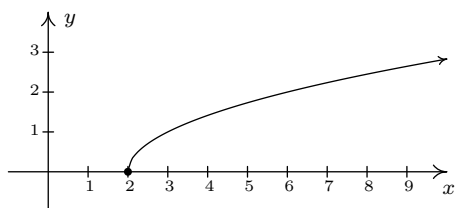
15.



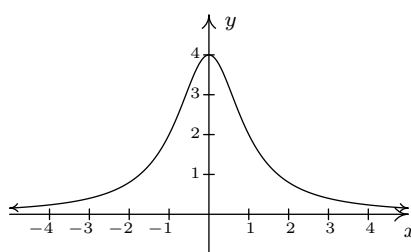
16.



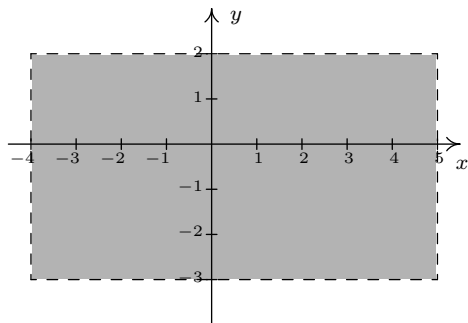
17.



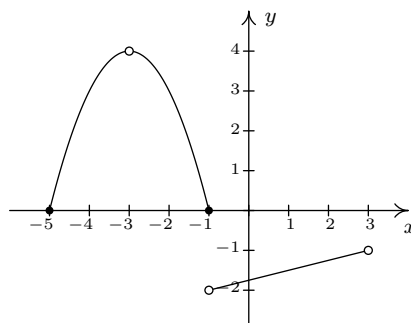
18.



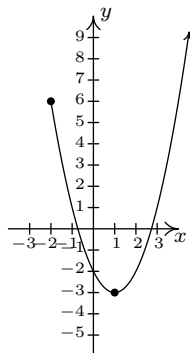
19.



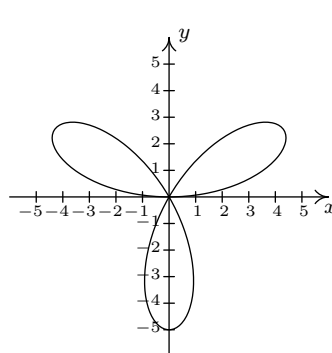
20.



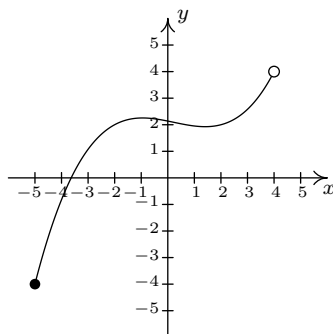
21.



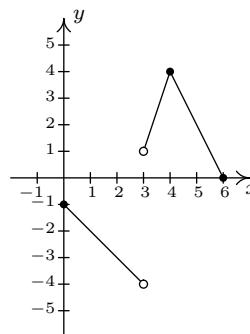
22.



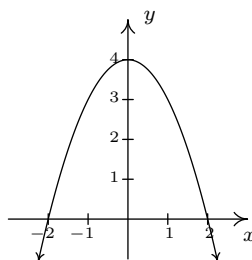
23.



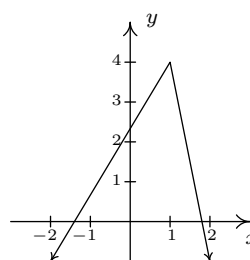
24.



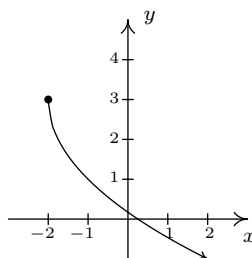
25.



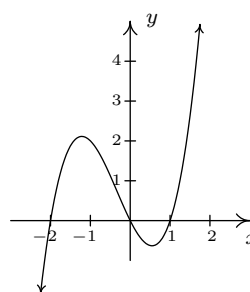
26.



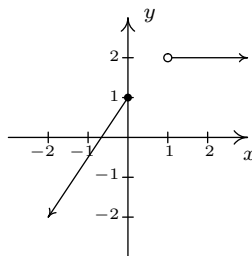
27.



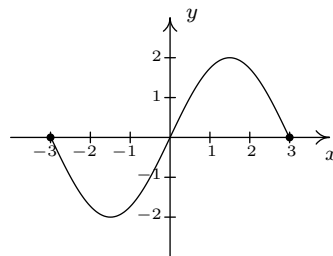
28.



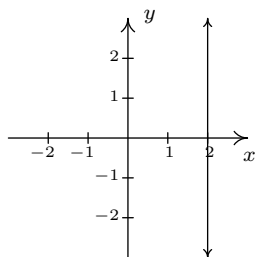
29.



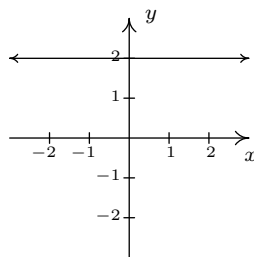
30.



31.



32.



In Exercises 33 - 47, determine whether or not the equation represents y as a function of x .

33. $y = x^3 - x$

34. $y = \sqrt{x - 2}$

35. $x^3y = -4$

36. $x^2 - y^2 = 1$

37. $y = \frac{x}{x^2 - 9}$

38. $x = -6$

39. $x = y^2 + 4$

40. $y = x^2 + 4$

41. $x^2 + y^2 = 4$

42. $y = \sqrt{4 - x^2}$

43. $x^2 - y^2 = 4$

44. $x^3 + y^3 = 4$

45. $2x + 3y = 4$

46. $2xy = 4$

47. $x^2 = y^2$

48. Explain why the population P of Sasquatch in a given area is a function of time t . What would be the range of this function?

49. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?

The process given in Example 1.3.5 for determining whether an equation of a relation represents y as a function of x breaks down if we cannot solve the equation for y in terms of x . However, that does not prevent us from proving that an equation fails to represent y as a function of x . What we really need is two points with the same x -coordinate and different y -coordinates which both satisfy the equation so that the graph of the relation would fail the Vertical Line Test 1.1. Discuss with your classmates how you might find such points for the relations given in Exercises 50 - 53.

50. $x^3 + y^3 - 3xy = 0$

51. $x^4 = x^2 + y^2$

52. $y^2 = x^3 + 3x^2$

53. $(x^2 + y^2)^2 = x^3 + y^3$

1.3.2 ANSWERS

1. Function
domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
range = $\{0, 1, 4, 9\}$
2. Not a function
3. Function
domain = $\{-7, -3, 3, 4, 5, 6\}$
range = $\{0, 4, 5, 6, 9\}$
4. Function
domain = $\{1, 4, 9, 16, 25, 36, \dots\}$
 $= \{x \mid x \text{ is a perfect square}\}$
range = $\{2, 4, 6, 8, 10, 12, \dots\}$
 $= \{y \mid y \text{ is a positive even integer}\}$
5. Not a function
6. Function
domain = $\{x \mid x \text{ is irrational}\}$
range = $\{1\}$
7. Function
domain = $\{x \mid x = 2^n \text{ for some whole number } n\}$
range = $\{y \mid y \text{ is any whole number}\}$
8. Function
domain = $\{x \mid x \text{ is any integer}\}$
range = $\{y \mid y = n^2 \text{ for some integer } n\}$
9. Not a function
10. Function
domain = $[-2, 4)$, range = $\{3\}$
11. Function
domain = $(-\infty, \infty)$
range = $[0, \infty)$
12. Not a function
13. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$
14. Not a function
15. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$
16. Not a function
17. Function
domain = $[2, \infty)$
range = $[0, \infty)$
18. Function
domain = $(-\infty, \infty)$
range = $(0, 4]$
19. Not a function
20. Function
domain = $[-5, -3) \cup (-3, 3)$
range = $(-2, -1) \cup [0, 4)$

21. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

23. Function
domain = $[-5, 4)$
range = $[-4, 4)$

25. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

27. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

29. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

31. Not a function

22. Not a function

24. Function
domain = $[0, 3) \cup (3, 6]$
range = $(-4, -1] \cup [0, 4]$

26. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

28. Function
domain = $(-\infty, \infty)$
range = $(-\infty, \infty)$

30. Function
domain = $[-3, 3]$
range = $[-2, 2]$

32. Function
domain = $(-\infty, \infty)$
range = $\{2\}$

33. Function

36. Not a function

39. Not a function

42. Function

45. Function

34. Function

37. Function

40. Function

43. Not a function

46. Function

35. Function

38. Not a function

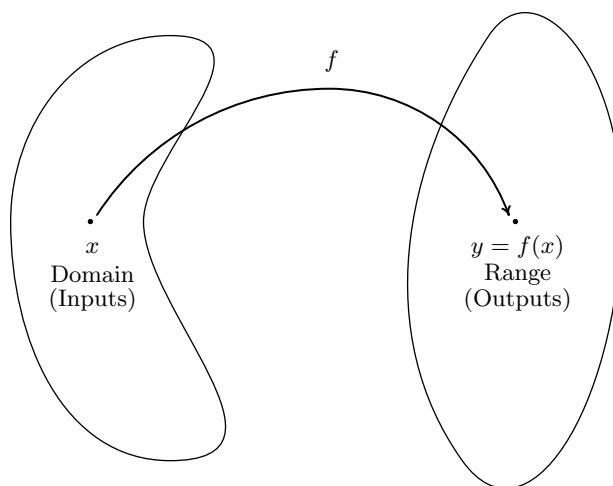
41. Not a function

44. Function

47. Not a function

1.4 FUNCTION NOTATION

In Definition 1.6, we described a function as a special kind of relation — one in which each x -coordinate is matched with only one y -coordinate. In this section, we focus more on the **process** by which the x is matched with the y . If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y . Since the output is completely determined by the input x and the process f , we symbolize the output with **function notation**: ' $f(x)$ ', read ' f of x .' In other words, $f(x)$ is the output which results by applying the process f to the input x . In this case, the parentheses here do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully. This relationship is typically visualized using a diagram similar to the one below.



The value of y is completely dependent on the choice of x . For this reason, x is often called the **independent variable**, or **argument** of f , whereas y is often called the **dependent variable**.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. multiply by 3
2. add 4

If we choose 5 as our input, in step 1 we multiply by 3 to get $(5)(3) = 15$. In step 2, we add 4 to our result from step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying step 1 produces $3x$. Following with step 2 produces $3x + 4$ as our final output. Hence for an input x , we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we replace the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

Example 1.4.1. Suppose a function g is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

Solution. Starting with 5, step 1 gives $5 + 4 = 9$. Continuing with step 2, we get $(3)(9) = 27$. To find a formula for $g(x)$, we start with our input x . Step 1 produces $x + 4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x + 4) = 3x + 12$. Hence, $g(x) = 3x + 12$. We can check our formula by replacing x with 5 to get $g(5) = 3(5) + 12 = 15 + 12 = 27 \checkmark$. \square

Most of the functions we will encounter in College Algebra will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other Math courses.

Example 1.4.2. Let $f(x) = -x^2 + 3x + 4$

1. Find and simplify the following.
 - (a) $f(-1)$, $f(0)$, $f(2)$
 - (b) $f(2x)$, $2f(x)$
 - (c) $f(x + 2)$, $f(x) + 2$, $f(x) + f(2)$
2. Solve $f(x) = 4$.

Solution.

1. (a) To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly, $f(0) = -(0)^2 + 3(0) + 4 = 4$, and $f(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

- (b) To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2(-x^2 + 3x + 4) \\ &= -2x^2 + 6x + 8 \end{aligned}$$

(c) To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -(x^2 + 4x + 4) + (3x + 6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To find $f(x) + 2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x) + 2 &= (-x^2 + 3x + 4) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

From our work above, we see $f(2) = 6$ so that

$$\begin{aligned} f(x) + f(2) &= (-x^2 + 3x + 4) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

2. Since $f(x) = -x^2 + 3x + 4$, the equation $f(x) = 4$ is equivalent to $-x^2 + 3x + 4 = 4$. Solving we get $-x^2 + 3x = 0$, or $x(-x + 3) = 0$. We get $x = 0$ or $x = 3$, and we can verify these answers by checking that $f(0) = 4$ and $f(3) = 4$. \square

A few notes about Example 1.4.2 are in order. First note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the *input* by 2; for $2f(x)$, we are multiplying the *output* by 2. As we see, we get entirely different results. Along these lines, note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three *different* expressions as well. Even though function notation uses parentheses, as does multiplication, there is *no* general ‘distributive property’ of function notation. Finally, note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

Suppose now we wish to find $r(3)$ for $r(x) = \frac{2x}{x^2-9}$. Substitution gives

$$r(3) = \frac{2(3)}{(3)^2 - 9} = \frac{6}{0},$$

which is undefined. (Why is this, again?) The number 3 is not an allowable input to the function r ; in other words, 3 is not in the domain of r . Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3 , the expression $r(x)$ is a real number. Hence, we write our domain in interval notation¹ as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain**² of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 1.4.3. Find the domain³ of the following functions.

$$1. g(x) = \sqrt{4 - 3x}$$

$$2. h(x) = \sqrt[5]{4 - 3x}$$

$$3. f(x) = \frac{2}{1 - \frac{4x}{x - 3}}$$

$$4. F(x) = \frac{\sqrt[4]{2x + 1}}{x^2 - 1}$$

$$5. r(t) = \frac{4}{6 - \sqrt{t + 3}}$$

$$6. I(x) = \frac{3x^2}{x}$$

Solution.

1. The potential disaster for g is if the radicand⁴ is negative. To avoid this, we set $4 - 3x \geq 0$. From this, we get $3x \leq 4$ or $x \leq \frac{4}{3}$. What this shows is that as long as $x \leq \frac{4}{3}$, the expression $4 - 3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $(-\infty, \frac{4}{3}]$.
2. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference – whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h . Hence, the domain is $(-\infty, \infty)$.
3. In the expression for f , there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the ‘small’ denominator, we get $x - 3 = 0$ or $x = 3$. For the ‘large’ denominator

¹See the Exercises for Section 1.1.

²or, ‘implicit domain’

³The word ‘implied’ is, well, implied.

⁴The ‘radicand’ is the expression ‘inside’ the radical.

$$\begin{aligned}
1 - \frac{4x}{x-3} &= 0 \\
1 &= \frac{4x}{x-3} \\
(1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)(\cancel{x-3}) \quad \text{clear denominators} \\
x-3 &= 4x \\
-3 &= 3x \\
-1 &= x
\end{aligned}$$

So we get two real numbers which make denominators 0, namely $x = -1$ and $x = 3$. Our domain is all real numbers except -1 and 3 : $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

4. In finding the domain of F , we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a fourth (even-indexed) root. Our strategy is to determine the restrictions imposed by each part and select the real numbers which satisfy both conditions. To satisfy the fourth root, we require $2x+1 \geq 0$. From this we get $2x \geq -1$ or $x \geq -\frac{1}{2}$. Next, we round up the values of x which could cause trouble in the denominator by setting the denominator equal to 0. We get $x^2 - 1 = 0$, or $x = \pm 1$. Hence, in order for a real number x to be in the domain of F , $x \geq -\frac{1}{2}$ but $x \neq \pm 1$. In interval notation, this set is $[-\frac{1}{2}, 1) \cup (1, \infty)$.
5. Don't be put off by the ' t ' here. It is an independent variable representing a real number, just like x does, and is subject to the same restrictions. As in the previous problem, we have double danger here: we have a square root and a denominator. To satisfy the square root, we need a non-negative radicand so we set $t+3 \geq 0$ to get $t \geq -3$. Setting the denominator equal to zero gives $6 - \sqrt{t+3} = 0$, or $\sqrt{t+3} = 6$. Squaring both sides gives $t+3 = 36$, or $t = 33$. Since we squared both sides in the course of solving this equation, we need to check our answer.⁵ Sure enough, when $t = 33$, $6 - \sqrt{t+3} = 6 - \sqrt{36} = 0$, so $t = 33$ will cause problems in the denominator. At last we can find the domain of r : we need $t \geq -3$, but $t \neq 33$. Our final answer is $[-3, 33) \cup (33, \infty)$.
6. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined.⁶ Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$. \square

It is worth reiterating the importance of finding the domain of a function *before* simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to

⁵Do you remember why? Consider squaring both sides to 'solve' $\sqrt{t+1} = -2$.

⁶More precisely, the fraction $\frac{0}{0}$ is an 'indeterminant form'. Calculus is required tame such beasts.

$3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

1.4.1 MODELING WITH FUNCTIONS

The importance of Mathematics to our society lies in its value to approximate, or **model** real-world phenomenon. Whether it be used to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends of various and sundry real and mythical beasts,⁷ Mathematics is second only to literacy in the importance humanity's development.⁸

It is important to keep in mind that anytime Mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for \$1.50 per pound. Then one pound of grapes costs \$1.50, two pounds of grapes cost \$3.00, and so forth. Suppose we want to develop a formula which relates the cost of buying grapes to the amount of grapes being purchased. Since these two quantities vary from situation to situation, we assign them variables. Let c denote the cost of the grapes and let g denote the amount of grapes purchased. To find the cost c of the grapes, we multiply the amount of grapes g by the price \$1.50 dollars per pound to get

$$c = 1.5g$$

In order for the units to be correct in the formula, g must be measured in *pounds* of grapes in which case the computed value of c is measured in *dollars*. Since we're interested in finding the cost c given an amount g , we think of g as the independent variable and c as the dependent variable. Using the language of function notation, we write

$$c(g) = 1.5g$$

where g is the amount of grapes purchased (in pounds) and $c(g)$ is the cost (in dollars). For example, $c(5)$ represents the cost, in dollars, to purchase 5 pounds of grapes. In this case, $c(5) = 1.5(5) = 7.5$, so it would cost \$7.50. If, on the other hand, we wanted to find the *amount* of grapes we can purchase for \$5, we would need to set $c(g) = 5$ and solve for g . In this case, $c(g) = 1.5g$, so solving $c(g) = 5$ is equivalent to solving $1.5g = 5$. Doing so gives $g = \frac{5}{1.5} = 3.\bar{3}$. This means we can purchase exactly $3.\bar{3}$ pounds of grapes for \$5. Of course, you would be hard-pressed to buy exactly $3.\bar{3}$ pounds of grapes,⁹ and this leads us to our next topic of discussion, the **applied domain**¹⁰ of a function.

Even though, mathematically, $c(g) = 1.5g$ has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of g that don't make any physical sense. For example, $g = -1$ corresponds to 'purchasing' -1 pounds of grapes.¹¹ Also, unless the 'local market' mentioned is the State of California (or some other exporter of grapes), it also doesn't make much sense for $g = 500,000,000$, either. So the reality of the situation limits what g can be, and

⁷See Sections 2.5, 11.1, and 6.5, respectively.

⁸In Carl's humble opinion, of course ...

⁹You could get close... within a certain specified margin of error, perhaps.

¹⁰or, 'explicit domain'

¹¹Maybe this means *returning* a pound of grapes?

these limits determine the applied domain of g . Typically, an applied domain is stated explicitly. In this case, it would be common to see something like $c(g) = 1.5g$, $0 \leq g \leq 100$, meaning the number of pounds of grapes purchased is limited from 0 up to 100. The upper bound here, 100 may represent the inventory of the market, or some other limit as set by local policy or law. Even with this restriction, our model has its limitations. As we saw above, it is virtually impossible to buy exactly $3.\bar{3}$ pounds of grapes so that our cost is exactly \$5. In this case, being sensible shoppers, we would most likely ‘round down’ and purchase 3 pounds of grapes or however close the market scale can read to $3.\bar{3}$ without being over. It is time for a more sophisticated example.

Example 1.4.4. The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if } 0 \leq t \leq 20 \\ 0, & \text{if } t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.
2. Solve $h(t) = 375$ and interpret your answers.

Solution.

1. We first note that the independent variable here is t , chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. Since $t = 10$ satisfies the inequality $0 \leq t \leq 20$, we use the first formula listed, $h(t) = -5t^2 + 100t$, to find $h(10)$. We get $h(10) = -5(10)^2 + 100(10) = 500$. Since t represents the number of seconds since lift-off and $h(t)$ is the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t , so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.
2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces. For $0 \leq t \leq 20$, $h(t) = -5t^2 + 100t$, so for these values of t , we solve $-5t^2 + 100t = 375$. Rearranging terms, we get $5t^2 - 100t + 375 = 0$, and factoring gives $5(t - 5)(t - 15) = 0$. Our answers are $t = 5$ and $t = 15$, and since both of these values of t lie between 0 and 20, we keep both solutions. For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$. In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude *twice* – once 5 seconds after launch, and again 15 seconds after launch.¹² \square

¹²What goes up ...

The type of function in the previous example is called a **piecewise-defined** function, or ‘piecewise’ function for short. Many real-world phenomena (e.g. postal rates,¹³ income tax formulas¹⁴) are modeled by such functions.

By the way, if we wanted to avoid using a piecewise function in Example 1.4.4, we could have used $h(t) = -5t^2 + 100t$ on the explicit domain $0 \leq t \leq 20$ because after 20 seconds, the rocket is on the ground and stops moving. In many cases, though, piecewise functions are your only choice, so it’s best to understand them well.

Mathematical modeling is not a one-section topic. It’s not even a one-*course* topic as is evidenced by undergraduate and graduate courses in mathematical modeling being offered at many universities. Thus our goal in this section cannot possibly be to tell you the whole story. What we can do is get you started. As we study new classes of functions, we will see what phenomena they can be used to model. In that respect, mathematical modeling cannot be a topic in a book, but rather, must be a theme of the book. For now, we have you explore some very basic models in the Exercises because you need to crawl to walk to run. As we learn more about functions, we’ll help you build your own models and get you on your way to applying Mathematics to your world.

¹³See the United States Postal Service website <http://www.usps.com/prices/first-class-mail-prices.htm>

¹⁴See the Internal Revenue Service’s website <http://www.irs.gov/pub/irs-pdf/i1040tt.pdf>

1.4.2 EXERCISES

In Exercises 1 - 10, find an expression for $f(x)$ and state its domain.

1. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply 2; (2) add 3; (3) divide by 4.
2. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.
3. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.
4. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply 2; (2) add 3; (3) take the square root.
5. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply 2; (3) take the square root.
6. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.
7. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.
8. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.
9. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.
10. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

In Exercises 11 - 18, use the given function f to find and simplify the following:

- | | | |
|------------|------------|-------------------------------|
| • $f(3)$ | • $f(-1)$ | • $f\left(\frac{3}{2}\right)$ |
| • $f(4x)$ | • $4f(x)$ | • $f(-x)$ |
| • $f(x-4)$ | • $f(x)-4$ | • $f(x^2)$ |

36. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq -1 \\ \sqrt{1-x^2} & \text{if } -1 < x \leq 1 \\ x & \text{if } x > 1 \end{cases}$ Compute the following function values.

(a) $f(4)$

(b) $f(-3)$

(c) $f(1)$

(d) $f(0)$

(e) $f(-1)$

(f) $f(-0.999)$

In Exercises 37 - 62, find the (implied) domain of the function.

37. $f(x) = x^4 - 13x^3 + 56x^2 - 19$

38. $f(x) = x^2 + 4$

39. $f(x) = \frac{x-2}{x+1}$

40. $f(x) = \frac{3x}{x^2 + x - 2}$

41. $f(x) = \frac{2x}{x^2 + 3}$

42. $f(x) = \frac{2x}{x^2 - 3}$

43. $f(x) = \frac{x+4}{x^2 - 36}$

44. $f(x) = \frac{x-2}{x-2}$

45. $f(x) = \sqrt{3-x}$

46. $f(x) = \sqrt{2x+5}$

47. $f(x) = 9x\sqrt{x+3}$

48. $f(x) = \frac{\sqrt{7-x}}{x^2+1}$

49. $f(x) = \sqrt{6x-2}$

50. $f(x) = \frac{6}{\sqrt{6x-2}}$

51. $f(x) = \sqrt[3]{6x-2}$

52. $f(x) = \frac{6}{4 - \sqrt{6x-2}}$

53. $f(x) = \frac{\sqrt{6x-2}}{x^2-36}$

54. $f(x) = \frac{\sqrt[3]{6x-2}}{x^2+36}$

55. $s(t) = \frac{t}{t-8}$

56. $Q(r) = \frac{\sqrt{r}}{r-8}$

57. $b(\theta) = \frac{\theta}{\sqrt{\theta-8}}$

58. $A(x) = \sqrt{x-7} + \sqrt{9-x}$

59. $\alpha(y) = \sqrt[3]{\frac{y}{y-8}}$

60. $g(v) = \frac{1}{4 - \frac{1}{v^2}}$

61. $T(t) = \frac{\sqrt{t}-8}{5-t}$

62. $u(w) = \frac{w-8}{5-\sqrt{w}}$

63. The area A enclosed by a square, in square inches, is a function of the length of one of its sides x , when measured in inches. This relation is expressed by the formula $A(x) = x^2$ for $x > 0$. Find $A(3)$ and solve $A(x) = 36$. Interpret your answers to each. Why is x restricted to $x > 0$?
64. The area A enclosed by a circle, in square meters, is a function of its radius r , when measured in meters. This relation is expressed by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?
65. The volume V enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides x , when measured in centimeters. This relation is expressed by the formula $V(x) = x^3$ for $x > 0$. Find $V(5)$ and solve $V(x) = 27$. Interpret your answers to each. Why is x restricted to $x > 0$?
66. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r , when measured in feet. This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?
67. The height of an object dropped from the roof of an eight story building is modeled by: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?
68. The temperature T in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.
69. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.
70. Using data from the [Bureau of Transportation Statistics](#), the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use your calculator to find $F(0)$, $F(14)$ and $F(28)$. Round your answers to two decimal places and interpret your answers to each.
71. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \frac{150t}{t+15}$, where t represents the number of years since 1803. Find and interpret $P(0)$ and $P(205)$. Discuss with your classmates what the applied domain and range of P should be.
72. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if } 1 \leq n \leq 25 \\ 13.50n & \text{if } 25 < n \leq 50 \\ 12n & \text{if } n > 50 \end{cases}$$

- (a) Find and interpret $C(20)$.

- (b) How much does it cost to order 50 copies of the book? What about 51 copies?
- (c) Your answer to 72b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

73. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if } 1 \leq n \leq 14 \\ 0 & \text{if } n \geq 15 \end{cases}$$

where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

- (a) What is the cost to ship 10 comic books?
- (b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

74. The cost C (in dollars) to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if } 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if } m > 1000 \end{cases}$$

- (a) How much does it cost to talk 750 minutes per month with this plan?
- (b) How much does it cost to talk 20 hours a month with this plan?
- (c) Explain the terms of the plan verbally.

75. In Section 1.1.1 we defined the set of **integers** as $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.¹⁵ The **greatest integer of x** , denoted by $\lfloor x \rfloor$, is defined to be the largest integer k with $k \leq x$.

- (a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$, and $\lfloor \pi + 6 \rfloor$
- (b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piecewise defined function.

HINT: There are infinitely many pieces!

- (c) Is $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if a or b is an integer? Test some values, make a conjecture, and explain your result.

76. We have through our examples tried to convince you that, in general, $f(a + b) \neq f(a) + f(b)$. It has been our experience that students refuse to believe us so we'll try again with a different approach. With the help of your classmates, find a function f for which the following properties are always true.

- (a) $f(0) = f(-1 + 1) = f(-1) + f(1)$

¹⁵The use of the letter \mathbb{Z} for the integers is ostensibly because the German word *zahlen* means 'to count.'

(b) $f(5) = f(2 + 3) = f(2) + f(3)$

(c) $f(-6) = f(0 - 6) = f(0) - f(6)$

(d) $f(a + b) = f(a) + f(b)$ regardless of what two numbers we give you for a and b .

How many functions did you find that failed to satisfy the conditions above? Did $f(x) = x^2$ work? What about $f(x) = \sqrt{x}$ or $f(x) = 3x + 7$ or $f(x) = \frac{1}{x}$? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**, $f(a + b) \neq f(a) + f(b)$.

1.4.3 ANSWERS

$$1. f(x) = \frac{2x+3}{4}$$

Domain: $(-\infty, \infty)$

$$2. f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$$

Domain: $(-\infty, \infty)$

$$3. f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$$

Domain: $(-\infty, \infty)$

$$4. f(x) = \sqrt{2x+3}$$

Domain: $\left[-\frac{3}{2}, \infty\right)$

$$5. f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$$

Domain: $[-3, \infty)$

$$6. f(x) = 2\sqrt{x+3}$$

Domain: $[-3, \infty)$

$$7. f(x) = \frac{4}{\sqrt{x}-13}$$

Domain: $[0, 169) \cup (169, \infty)$

$$8. f(x) = \frac{4}{\sqrt{x-13}}$$

Domain: $(13, \infty)$

$$9. f(x) = \frac{4}{\sqrt{x}} - 13$$

Domain: $(0, \infty)$

$$10. f(x) = \sqrt{\frac{4}{x}} - 13 = \frac{2}{\sqrt{x}} - 13$$

Domain: $(0, \infty)$

$$11. \text{ For } f(x) = 2x + 1$$

$$\bullet f(3) = 7$$

$$\bullet f(-1) = -1$$

$$\bullet f\left(\frac{3}{2}\right) = 4$$

$$\bullet f(4x) = 8x + 1$$

$$\bullet 4f(x) = 8x + 4$$

$$\bullet f(-x) = -2x + 1$$

$$\bullet f(x-4) = 2x - 7$$

$$\bullet f(x) - 4 = 2x - 3$$

$$\bullet f(x^2) = 2x^2 + 1$$

$$12. \text{ For } f(x) = 3 - 4x$$

$$\bullet f(3) = -9$$

$$\bullet f(-1) = 7$$

$$\bullet f\left(\frac{3}{2}\right) = -3$$

$$\bullet f(4x) = 3 - 16x$$

$$\bullet 4f(x) = 12 - 16x$$

$$\bullet f(-x) = 4x + 3$$

$$\bullet f(x-4) = 19 - 4x$$

$$\bullet f(x) - 4 = -4x - 1$$

$$\bullet f(x^2) = 3 - 4x^2$$

13. For $f(x) = 2 - x^2$

- $f(3) = -7$ • $f(-1) = 1$ • $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$ • $4f(x) = 8 - 4x^2$ • $f(-x) = 2 - x^2$
- $f(x-4) = -x^2 + 8x - 14$ • $f(x) - 4 = -x^2 - 2$ • $f(x^2) = 2 - x^4$

14. For $f(x) = x^2 - 3x + 2$

- $f(3) = 2$ • $f(-1) = 6$ • $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 16x^2 - 12x + 2$ • $4f(x) = 4x^2 - 12x + 8$ • $f(-x) = x^2 + 3x + 2$
- $f(x-4) = x^2 - 11x + 30$ • $f(x) - 4 = x^2 - 3x - 2$ • $f(x^2) = x^4 - 3x^2 + 2$

15. For $f(x) = \frac{x}{x-1}$

- $f(3) = \frac{3}{2}$ • $f(-1) = \frac{1}{2}$ • $f\left(\frac{3}{2}\right) = 3$
- $f(4x) = \frac{4x}{4x-1}$ • $4f(x) = \frac{4x}{x-1}$ • $f(-x) = \frac{x}{x+1}$
- $f(x-4) = \frac{x-4}{x-5}$ • $f(x) - 4 = \frac{x}{x-1} - 4$ • $f(x^2) = \frac{x^2}{x^2-1}$
 $= \frac{4-3x}{x-1}$

16. For $f(x) = \frac{2}{x^3}$

- $f(3) = \frac{2}{27}$ • $f(-1) = -2$ • $f\left(\frac{3}{2}\right) = \frac{16}{27}$
- $f(4x) = \frac{1}{32x^3}$ • $4f(x) = \frac{8}{x^3}$ • $f(-x) = -\frac{2}{x^3}$
- $f(x-4) = \frac{2}{(x-4)^3}$ • $f(x) - 4 = \frac{2}{x^3} - 4$ • $f(x^2) = \frac{2}{x^6}$
 $= \frac{2}{x^3-12x^2+48x-64}$ $= \frac{2-4x^3}{x^3}$

17. For $f(x) = 6$

- $f(3) = 6$ • $f(-1) = 6$ • $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$ • $4f(x) = 24$ • $f(-x) = 6$
- $f(x-4) = 6$ • $f(x) - 4 = 2$ • $f(x^2) = 6$

18. For $f(x) = 0$

- $f(3) = 0$

- $f(-1) = 0$

- $f\left(\frac{3}{2}\right) = 0$

- $f(4x) = 0$

- $4f(x) = 0$

- $f(-x) = 0$

- $f(x - 4) = 0$

- $f(x) - 4 = -4$

- $f(x^2) = 0$

19. For $f(x) = 2x - 5$

- $f(2) = -1$

- $f(-2) = -9$

- $f(2a) = 4a - 5$

- $2f(a) = 4a - 10$

- $f(a + 2) = 2a - 1$

- $f(a) + f(2) = 2a - 6$

- $f\left(\frac{2}{a}\right) = \frac{\frac{4}{a}}{2} - 5$
 $= \frac{4-5a}{a}$

- $\frac{f(a)}{2} = \frac{2a-5}{2}$

- $f(a + h) = 2a + 2h - 5$

20. For $f(x) = 5 - 2x$

- $f(2) = 1$

- $f(-2) = 9$

- $f(2a) = 5 - 4a$

- $2f(a) = 10 - 4a$

- $f(a + 2) = 1 - 2a$

- $f(a) + f(2) = 6 - 2a$

- $f\left(\frac{2}{a}\right) = \frac{5 - \frac{4}{a}}{2}$
 $= \frac{5a-4}{a}$

- $\frac{f(a)}{2} = \frac{5-2a}{2}$

- $f(a + h) = 5 - 2a - 2h$

21. For $f(x) = 2x^2 - 1$

- $f(2) = 7$

- $f(-2) = 7$

- $f(2a) = 8a^2 - 1$

- $2f(a) = 4a^2 - 2$

- $f(a + 2) = 2a^2 + 8a + 7$

- $f(a) + f(2) = 2a^2 + 6$

- $f\left(\frac{2}{a}\right) = \frac{\frac{8}{a^2}}{2} - 1$
 $= \frac{8-a^2}{a^2}$

- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$

- $f(a + h) = 2a^2 + 4ah + 2h^2 - 1$

22. For $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2 + 15a + 16$
- $f(a) + f(2) = 3a^2 + 3a + 14$
- $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2$
 $= \frac{12+6a-2a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
- $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

23. For $f(x) = \sqrt{2x+1}$

- $f(2) = \sqrt{5}$
- $f(-2)$ is not real
- $f(2a) = \sqrt{4a+1}$
- $2f(a) = 2\sqrt{2a+1}$
- $f(a+2) = \sqrt{2a+5}$
- $f(a)+f(2) = \sqrt{2a+1}+\sqrt{5}$
- $f\left(\frac{2}{a}\right) = \sqrt{\frac{4}{a}+1}$
 $= \sqrt{\frac{a+4}{a}}$
- $\frac{f(a)}{2} = \frac{\sqrt{2a+1}}{2}$
- $f(a+h) = \sqrt{2a+2h+1}$

24. For $f(x) = 117$

- $f(2) = 117$
- $f(-2) = 117$
- $f(2a) = 117$
- $2f(a) = 234$
- $f(a+2) = 117$
- $f(a) + f(2) = 234$
- $f\left(\frac{2}{a}\right) = 117$
- $\frac{f(a)}{2} = \frac{117}{2}$
- $f(a+h) = 117$

25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1$
 $= \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

26. For $f(x) = \frac{2}{x}$

$$\bullet f(2) = 1$$

$$\bullet f(-2) = -1$$

$$\bullet f(2a) = \frac{1}{a}$$

$$\bullet 2f(a) = \frac{4}{a}$$

$$\bullet f(a+2) = \frac{2}{a+2}$$

$$\bullet f(a) + f(2) = \frac{2}{a} + 1 = \frac{a+2}{2}$$

$$\bullet f\left(\frac{2}{a}\right) = a$$

$$\bullet \frac{f(a)}{2} = \frac{1}{a}$$

$$\bullet f(a+h) = \frac{2}{a+h}$$

27. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$

28. For $f(x) = 3 - \frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$

29. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$

30. For $f(x) = x^2 - x - 12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$

31. For $f(x) = \sqrt{x+4}$, $f(0) = 2$ and $f(x) = 0$ when $x = -4$

32. For $f(x) = \sqrt{1-2x}$, $f(0) = 1$ and $f(x) = 0$ when $x = \frac{1}{2}$

33. For $f(x) = \frac{3}{4-x}$, $f(0) = \frac{3}{4}$ and $f(x)$ is never equal to 0

34. For $f(x) = \frac{3x^2-12x}{4-x^2}$, $f(0) = 0$ and $f(x) = 0$ when $x = 0$ or $x = 4$

35. (a) $f(-4) = 1$

(b) $f(-3) = 2$

(c) $f(3) = 0$

(d) $f(3.001) = 1.999$

(e) $f(-3.001) = 1.999$

(f) $f(2) = \sqrt{5}$

36. (a) $f(4) = 4$

(b) $f(-3) = 9$

(c) $f(1) = 0$

(d) $f(0) = 1$

(e) $f(-1) = 1$

(f) $f(-0.999) \approx 0.0447$

37. $(-\infty, \infty)$

38. $(-\infty, \infty)$

39. $(-\infty, -1) \cup (-1, \infty)$

40. $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$

41. $(-\infty, \infty)$

42. $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$

43. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$

44. $(-\infty, 2) \cup (2, \infty)$

45. $(-\infty, 3]$

46. $[-\frac{5}{2}, \infty)$

47. $[-3, \infty)$
48. $(-\infty, 7]$
49. $[\frac{1}{3}, \infty)$
50. $(\frac{1}{3}, \infty)$
51. $(-\infty, \infty)$
52. $[\frac{1}{3}, 3) \cup (3, \infty)$
53. $[\frac{1}{3}, 6) \cup (6, \infty)$
54. $(-\infty, \infty)$
55. $(-\infty, 8) \cup (8, \infty)$
56. $[0, 8) \cup (8, \infty)$
57. $(8, \infty)$
58. $[7, 9]$
59. $(-\infty, 8) \cup (8, \infty)$
60. $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$
61. $[0, 5) \cup (5, \infty)$
62. $[0, 25) \cup (25, \infty)$
63. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(x) = 36$ are $x = \pm 6$. Since x is restricted to $x > 0$, we only keep $x = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since x represents a length, $x > 0$.
64. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is 4π square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. Since r is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be 16π square meters, the radius needs to be 4 meters. Since r represents a radius (length), $r > 0$.
65. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(x) = 27$ is $x = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to be 3 centimeters. Since x represents a length, $x > 0$.
66. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to be 2 feet. Since r represents a radius (length), $r > 0$.
67. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. Since we restrict $0 \leq t \leq 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \leq t \leq 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.
68. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

69. $C(0) = 27$, so to make 0 pens, it costs¹⁶ \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.
70. $F(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $F(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $F(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.
71. $P(0) = 0$ which means in 1803 (0 years after 1803), there are no Sasquatch in Portage County. $P(205) = \frac{3075}{22} \approx 139.77$, so in 2008 (205 years after 1803), there were between 139 and 140 Sasquatch in Portage County.
72. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
 (b) $C(50) = 675$, so it costs \$675 for 50 copies of the book. $C(51) = 612$, so it costs \$612 for 51 copies of the book.
 (c) 56 books.
73. (a) $S(10) = 17.5$, so it costs \$17.50 to ship 10 comic books.
 (b) There is free shipping on orders of 15 or more comic books.
74. (a) $C(750) = 25$, so it costs \$25 to talk 750 minutes per month with this plan.
 (b) Since 20 hours = 1200 minutes, we substitute $m = 1200$ and get $C(1200) = 45$. It costs \$45 to talk 20 hours per month with this plan.
 (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.
75. (a) $\lfloor 0.785 \rfloor = 0$, $\lfloor 117 \rfloor = 117$, $\lfloor -2.001 \rfloor = -3$, and $\lfloor \pi + 6 \rfloor = 9$

¹⁶This is called the ‘fixed’ or ‘start-up’ cost. We’ll revisit this concept on page 82.

1.5 FUNCTION ARITHMETIC

In the previous section we used the newly defined function notation to make sense of expressions such as ' $f(x) + 2$ ' and ' $2f(x)$ ' for a given function f . It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g .^a

- The **sum** of f and g , denoted $f + g$, is the function defined by the formula

$$(f + g)(x) = f(x) + g(x)$$

- The **difference** of f and g , denoted $f - g$, is the function defined by the formula

$$(f - g)(x) = f(x) - g(x)$$

- The **product** of f and g , denoted fg , is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g , denoted $\frac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

provided $g(x) \neq 0$.

^aThus x is an element of the intersection of the two domains.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f + g)(x) = f(x) + g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f + g$ as the sum of the real number outputs from f and g .

Example 1.5.1. Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \frac{1}{x}$.

1. Find $(f + g)(-1)$
2. Find $(fg)(2)$
3. Find the domain of $g - f$ then find and simplify a formula for $(g - f)(x)$.

4. Find the domain of $\left(\frac{g}{f}\right)$ then find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

Solution.

1. To find $(f + g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that $(f + g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$.
2. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.
3. One method to find the domain of $g - f$ is to find the domain of g and of f separately, then find the intersection of these two sets. Owing to the denominator in the expression $g(x) = 3 - \frac{1}{x}$, we get that the domain of g is $(-\infty, 0) \cup (0, \infty)$. Since $f(x) = 6x^2 - 2x$ is valid for all real numbers, we have no further restrictions. Thus the domain of $g - f$ matches the domain of g , namely, $(-\infty, 0) \cup (0, \infty)$.

A second method is to analyze the formula for $(g - f)(x)$ *before simplifying* and look for the usual domain issues. In this case,

$$(g - f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - (6x^2 - 2x),$$

so we find, as before, the domain is $(-\infty, 0) \cup (0, \infty)$.

Moving along, we need to simplify a formula for $(g - f)(x)$. In this case, we get common denominators and attempt to reduce the resulting fraction. Doing so, we get

$$\begin{aligned} (g - f)(x) &= g(x) - f(x) \\ &= \left(3 - \frac{1}{x}\right) - (6x^2 - 2x) \\ &= 3 - \frac{1}{x} - 6x^2 + 2x \\ &= \frac{3x}{x} - \frac{1}{x} - \frac{6x^3}{x} + \frac{2x^2}{x} && \text{get common denominators} \\ &= \frac{3x - 1 - 6x^3 - 2x^2}{x} \\ &= \frac{-6x^3 - 2x^2 + 3x - 1}{x} \end{aligned}$$

4. As in the previous example, we have two ways to approach finding the domain of $\frac{g}{f}$. First, we can find the domain of g and f separately, and find the intersection of these two sets. In addition, since $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$, we are introducing a new denominator, namely $f(x)$, so we need to guard against this being 0 as well. Our previous work tells us that the domain of g is $(-\infty, 0) \cup (0, \infty)$ and the domain of f is $(-\infty, \infty)$. Setting $f(x) = 0$ gives $6x^2 - 2x = 0$

or $x = 0, \frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x = 0$ and $x = \frac{1}{3}$, or $(-\infty, 0) \cup (0, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

Alternatively, we may proceed as above and analyze the expression $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$ *before* simplifying. In this case,

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

We see immediately from the ‘little’ denominator that $x \neq 0$. To keep the ‘big’ denominator away from 0, we solve $6x^2 - 2x = 0$ and get $x = 0$ or $x = \frac{1}{3}$. Hence, as before, we find the domain of $\frac{g}{f}$ to be $(-\infty, 0) \cup (0, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

Next, we find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

$$\begin{aligned} \left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\ &= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\ &= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \cdot \frac{x}{x} \quad \text{simplify compound fractions} \\ &= \frac{\left(3 - \frac{1}{x}\right)x}{(6x^2 - 2x)x} \\ &= \frac{3x - 1}{(6x^2 - 2x)x} \\ &= \frac{3x - 1}{2x^2(3x - 1)} \quad \text{factor} \\ &= \frac{\cancel{(3x - 1)}^1}{2x^2\cancel{(3x - 1)}} \quad \text{cancel} \\ &= \frac{1}{2x^2} \end{aligned}$$

□

Please note the importance of finding the domain of a function *before* simplifying its expression. In number 4 in Example 1.5.1 above, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we’d just have the formula $\frac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty, 0) \cup (0, \infty)$, since the other troublesome number, $x = \frac{1}{3}$, was canceled away.¹

¹We’ll see what this means geometrically in Chapter 4.

Next, we turn our attention to the **difference quotient** of a function.

Definition 1.8. Given a function f , the **difference quotient** of f is the expression

$$\frac{f(x+h) - f(x)}{h}$$

We will revisit this concept in Section 2.1, but for now, we use it as a way to practice function notation and function arithmetic. For reasons which will become clear in Calculus, ‘simplifying’ a difference quotient means rewriting it in a form where the ‘ h ’ in the definition of the difference quotient cancels from the denominator. Once that happens, we consider our work to be done.

Example 1.5.2. Find and simplify the difference quotients for the following functions

$$1. f(x) = x^2 - x - 2 \qquad 2. g(x) = \frac{3}{2x+1} \qquad 3. r(x) = \sqrt{x}$$

Solution.

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h(2x + h - 1)}{h} && \text{factor} \\ &= \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} && \text{cancel} \\ &= 2x + h - 1. \end{aligned}$$

2. To find $g(x + h)$, we replace every occurrence of x in the formula $g(x) = \frac{3}{2x+1}$ with the quantity $(x + h)$ to get

$$\begin{aligned} g(x + h) &= \frac{3}{2(x + h) + 1} \\ &= \frac{3}{2x + 2h + 1}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{g(x + h) - g(x)}{h} &= \frac{\frac{3}{2x + 2h + 1} - \frac{3}{2x + 1}}{h} \\ &= \frac{\frac{3}{2x + 2h + 1} - \frac{3}{2x + 1}}{h} \cdot \frac{(2x + 2h + 1)(2x + 1)}{(2x + 2h + 1)(2x + 1)} \\ &= \frac{3(2x + 1) - 3(2x + 2h + 1)}{h(2x + 2h + 1)(2x + 1)} \\ &= \frac{6x + 3 - 6x - 6h - 3}{h(2x + 2h + 1)(2x + 1)} \\ &= \frac{-6h}{h(2x + 2h + 1)(2x + 1)} \\ &= \frac{-6\cancel{h}}{\cancel{h}(2x + 2h + 1)(2x + 1)} \\ &= \frac{-6}{(2x + 2h + 1)(2x + 1)}. \end{aligned}$$

Since we have managed to cancel the original ‘ h ’ from the denominator, we are done.

3. For $r(x) = \sqrt{x}$, we get $r(x + h) = \sqrt{x + h}$ so the difference quotient is

$$\frac{r(x + h) - r(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

In order to cancel the ‘ h ’ from the denominator, we rationalize the *numerator* by multiplying by its conjugate.²

²Rationalizing the *numerator*!? How’s that for a twist!

$$\begin{aligned}
\frac{r(x+h) - r(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} && \text{Multiply by the conjugate.} \\
&= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} && \text{Difference of Squares.} \\
&= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{\overset{1}{\cancel{h}}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}$$

Since we have removed the original ‘ h ’ from the denominator, we are done. \square

As mentioned before, we will revisit difference quotients in Section 2.1 where we will explain them geometrically. For now, we want to move on to some classic applications of function arithmetic from Economics and for that, we need to think like an entrepreneur.³

Suppose you are a manufacturer making a certain product.⁴ Let x be the **production level**, that is, the number of items produced in a given time period. It is customary to let $C(x)$ denote the function which calculates the total **cost** of producing the x items. The quantity $C(0)$, which represents the cost of producing no items, is called the **fixed** cost, and represents the amount of money required to begin production. Associated with the total cost $C(x)$ is cost per item, or **average cost**, denoted $\overline{C}(x)$ and read ‘ C -bar’ of x . To compute $\overline{C}(x)$, we take the total cost $C(x)$ and divide by the number of items produced x to get

$$\overline{C}(x) = \frac{C(x)}{x}$$

On the retail end, we have the **price** p charged per item. To simplify the dialog and computations in this text, we assume that *the number of items sold equals the number of items produced*. From a

³Not really, but “entrepreneur” is the buzzword of the day and we’re trying to be trendy.

⁴Poorly designed resin Sasquatch statues, for example. Feel free to choose your own entrepreneurial fantasy.

retail perspective, it seems natural to think of the number of items sold, x , as a function of the price charged, p . After all, the retailer can easily adjust the price to sell more product. In the language of functions, x would be the *dependent* variable and p would be the *independent* variable or, using function notation, we have a function $x(p)$. While we will adopt this convention later in the text,⁵ we will hold with tradition at this point and consider the price p as a function of the number of items sold, x . That is, we regard x as the independent variable and p as the dependent variable and speak of the **price-demand** function, $p(x)$. Hence, $p(x)$ returns the price charged per item when x items are produced and sold. Our next function to consider is the **revenue** function, $R(x)$. The function $R(x)$ computes the amount of money collected as a result of selling x items. Since $p(x)$ is the price charged per item, we have $R(x) = xp(x)$. Finally, the **profit** function, $P(x)$ calculates how much money is earned after the costs are paid. That is, $P(x) = (R - C)(x) = R(x) - C(x)$. We summarize all of these functions below.

Summary of Common Economic Functions

Suppose x represents the quantity of items produced and sold.

- The price-demand function $p(x)$ calculates the price per item.
- The revenue function $R(x)$ calculates the total money collected by selling x items at a price $p(x)$, $R(x) = xp(x)$.
- The cost function $C(x)$ calculates the cost to produce x items. The value $C(0)$ is called the fixed cost or start-up cost.
- The average cost function $\overline{C}(x) = \frac{C(x)}{x}$ calculates the cost per item when making x items. Here, we necessarily assume $x > 0$.
- The profit function $P(x)$ calculates the money earned after costs are paid when x items are produced and sold, $P(x) = (R - C)(x) = R(x) - C(x)$.

It is high time for an example.

Example 1.5.3. Let x represent the number of dOpi media players ('dOpis'⁶) produced and sold in a typical week. Suppose the cost, in dollars, to produce x dOpis is given by $C(x) = 100x + 2000$, for $x \geq 0$, and the price, in dollars per dOpi, is given by $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

1. Find and interpret $C(0)$.
2. Find and interpret $\overline{C}(10)$.
3. Find and interpret $p(0)$ and $p(20)$.
4. Solve $p(x) = 0$ and interpret the result.
5. Find and simplify expressions for the revenue function $R(x)$ and the profit function $P(x)$.
6. Find and interpret $R(0)$ and $P(0)$.
7. Solve $P(x) = 0$ and interpret the result.

⁵See Example 5.2.4 in Section 5.2.

⁶Pronounced 'dopeys'...

Solution.

1. We substitute $x = 0$ into the formula for $C(x)$ and get $C(0) = 100(0) + 2000 = 2000$. This means to produce 0 dOpis, it costs \$2000. In other words, the fixed (or start-up) costs are \$2000. The reader is encouraged to contemplate what sorts of expenses these might be.
2. Since $\overline{C}(x) = \frac{C(x)}{x}$, $\overline{C}(10) = \frac{C(10)}{10} = \frac{3000}{10} = 300$. This means when 10 dOpis are produced, the cost to manufacture them amounts to \$300 per dOpi.
3. Plugging $x = 0$ into the expression for $p(x)$ gives $p(0) = 450 - 15(0) = 450$. This means no dOpis are sold if the price is \$450 per dOpi. On the other hand, $p(20) = 450 - 15(20) = 150$ which means to sell 20 dOpis in a typical week, the price should be set at \$150 per dOpi.
4. Setting $p(x) = 0$ gives $450 - 15x = 0$. Solving gives $x = 30$. This means in order to sell 30 dOpis in a typical week, the price needs to be set to \$0. What's more, this means that even if dOpis were given away for free, the retailer would only be able to move 30 of them.⁷
5. To find the revenue, we compute $R(x) = xp(x) = x(450 - 15x) = 450x - 15x^2$. Since the formula for $p(x)$ is valid only for $0 \leq x \leq 30$, our formula $R(x)$ is also restricted to $0 \leq x \leq 30$. For the profit, $P(x) = (R - C)(x) = R(x) - C(x)$. Using the given formula for $C(x)$ and the derived formula for $R(x)$, we get $P(x) = (450x - 15x^2) - (100x + 2000) = -15x^2 + 350x - 2000$. As before, the validity of this formula is for $0 \leq x \leq 30$ only.
6. We find $R(0) = 0$ which means if no dOpis are sold, we have no revenue, which makes sense. Turning to profit, $P(0) = -2000$ since $P(x) = R(x) - C(x)$ and $P(0) = R(0) - C(0) = -2000$. This means that if no dOpis are sold, more money (\$2000 to be exact!) was put into producing the dOpis than was recouped in sales. In number 1, we found the fixed costs to be \$2000, so it makes sense that if we sell no dOpis, we are out those start-up costs.
7. Setting $P(x) = 0$ gives $-15x^2 + 350x - 2000 = 0$. Factoring gives $-5(x - 10)(3x - 40) = 0$ so $x = 10$ or $x = \frac{40}{3}$. What do these values mean in the context of the problem? Since $P(x) = R(x) - C(x)$, solving $P(x) = 0$ is the same as solving $R(x) = C(x)$. This means that the solutions to $P(x) = 0$ are the production (and sales) figures for which the sales revenue exactly balances the total production costs. These are the so-called '**break even**' points. The solution $x = 10$ means 10 dOpis should be produced (and sold) during the week to recoup the cost of production. For $x = \frac{40}{3} = 13.\overline{3}$, things are a bit more complicated. Even though $x = 13.\overline{3}$ satisfies $0 \leq x \leq 30$, and hence is in the domain of P , it doesn't make sense in the context of this problem to produce a fractional part of a dOpi.⁸ Evaluating $P(13) = 15$ and $P(14) = -40$, we see that producing and selling 13 dOpis per week makes a (slight) profit, whereas producing just one more puts us back into the red. While breaking even is nice, we ultimately would like to find what production level (and price) will result in the largest profit, and we'll do just that ... in Section 2.3. \square

⁷Imagine that! Giving something away for free and hardly anyone taking advantage of it ...

⁸We've seen this sort of thing before in Section 1.4.1.

27. $f(x) = x - x^2$

28. $f(x) = x^3 + 1$

29. $f(x) = mx + b$ where $m \neq 0$

30. $f(x) = ax^2 + bx + c$ where $a \neq 0$

31. $f(x) = \frac{2}{x}$

32. $f(x) = \frac{3}{1-x}$

33. $f(x) = \frac{1}{x^2}$

34. $f(x) = \frac{2}{x+5}$

35. $f(x) = \frac{1}{4x-3}$

36. $f(x) = \frac{3x}{x+1}$

37. $f(x) = \frac{x}{x-9}$

38. $f(x) = \frac{x^2}{2x+1}$

39. $f(x) = \sqrt{x-9}$

40. $f(x) = \sqrt{2x+1}$

41. $f(x) = \sqrt{-4x+5}$

42. $f(x) = \sqrt{4-x}$

43. $f(x) = \sqrt{ax+b}$, where $a \neq 0$.

44. $f(x) = x\sqrt{x}$

45. $f(x) = \sqrt[3]{x}$. **HINT:** $(a-b)(a^2+ab+b^2) = a^3-b^3$

In Exercises 46 - 50, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- Find and interpret $C(0)$.
 - Find and interpret $\overline{C}(10)$.
 - Find and interpret $p(5)$.
 - Find and simplify $R(x)$.
 - Find and simplify $P(x)$.
 - Solve $P(x) = 0$ and interpret.
46. The cost, in dollars, to produce x “I’d rather be a Sasquatch” T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.
47. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.
48. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior’s Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.
49. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.

50. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

In Exercises 51 - 62, let f be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let g be the function defined

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

. Compute the indicated value if it exists.

51. $(f + g)(-3)$

52. $(f - g)(2)$

53. $(fg)(-1)$

54. $(g + f)(1)$

55. $(g - f)(3)$

56. $(gf)(-3)$

57. $\left(\frac{f}{g}\right)(-2)$

58. $\left(\frac{f}{g}\right)(-1)$

59. $\left(\frac{f}{g}\right)(2)$

60. $\left(\frac{g}{f}\right)(-1)$

61. $\left(\frac{g}{f}\right)(3)$

62. $\left(\frac{g}{f}\right)(-3)$

1.5.2 ANSWERS

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$

$$\begin{array}{lll}
 \bullet (f+g)(2) = 9 & \bullet (f-g)(-1) = -7 & \bullet (g-f)(1) = -1 \\
 \bullet (fg)\left(\frac{1}{2}\right) = \frac{35}{4} & \bullet \left(\frac{f}{g}\right)(0) = \frac{1}{4} & \bullet \left(\frac{g}{f}\right)(-2) = -\frac{6}{5}
 \end{array}$$

2. For $f(x) = x^2$ and $g(x) = -2x + 1$

$$\begin{array}{lll}
 \bullet (f+g)(2) = 1 & \bullet (f-g)(-1) = -2 & \bullet (g-f)(1) = -2 \\
 \bullet (fg)\left(\frac{1}{2}\right) = 0 & \bullet \left(\frac{f}{g}\right)(0) = 0 & \bullet \left(\frac{g}{f}\right)(-2) = \frac{5}{4}
 \end{array}$$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$

$$\begin{array}{lll}
 \bullet (f+g)(2) = 10 & \bullet (f-g)(-1) = -9 & \bullet (g-f)(1) = 11 \\
 \bullet (fg)\left(\frac{1}{2}\right) = -\frac{47}{16} & \bullet \left(\frac{f}{g}\right)(0) = 0 & \bullet \left(\frac{g}{f}\right)(-2) = \frac{4}{3}
 \end{array}$$

4. For $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$

$$\begin{array}{lll}
 \bullet (f+g)(2) = 5 & \bullet (f-g)(-1) = 0 & \bullet (g-f)(1) = -8 \\
 \bullet (fg)\left(\frac{1}{2}\right) = -\frac{17}{16} & \bullet \left(\frac{f}{g}\right)(0) = 0 & \bullet \left(\frac{g}{f}\right)(-2) = \frac{3}{16}
 \end{array}$$

5. For $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$

$$\begin{array}{lll}
 \bullet (f+g)(2) = 3 + \sqrt{5} & \bullet (f-g)(-1) = 3 + \sqrt{2} & \bullet (g-f)(1) = -1 \\
 \bullet (fg)\left(\frac{1}{2}\right) = 0 & \bullet \left(\frac{f}{g}\right)(0) = -\sqrt{3} & \bullet \left(\frac{g}{f}\right)(-2) = -5
 \end{array}$$

6. For $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$

$$\begin{array}{lll}
 \bullet (f+g)(2) = 2 + \sqrt{2} & \bullet (f-g)(-1) = -1 + \sqrt{5} & \bullet (g-f)(1) = 0 \\
 \bullet (fg)\left(\frac{1}{2}\right) = \frac{\sqrt{35}}{2} & \bullet \left(\frac{f}{g}\right)(0) = \sqrt{2} & \bullet \left(\frac{g}{f}\right)(-2) = 0
 \end{array}$$

7. For $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$

- $(f+g)(2) = \frac{21}{5}$
- $(f-g)(-1) = -1$
- $(g-f)(1) = -\frac{5}{3}$
- $(fg)(\frac{1}{2}) = \frac{1}{2}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$

- $(f+g)(2) = 7$
- $(f-g)(-1) = \frac{8}{5}$
- $(g-f)(1) = -4$
- $(fg)(\frac{1}{2}) = -\frac{3}{8}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$

9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$

- $(f+g)(2) = \frac{17}{4}$
- $(f-g)(-1) = 0$
- $(g-f)(1) = 0$
- $(fg)(\frac{1}{2}) = 1$
- $\left(\frac{f}{g}\right)(0)$ is undefined.
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

10. For $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

- $(f+g)(2) = \frac{26}{5}$
- $(f-g)(-1) = \frac{3}{2}$
- $(g-f)(1) = -\frac{3}{2}$
- $(fg)(\frac{1}{2}) = 1$
- $\left(\frac{f}{g}\right)(0) = 1$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$

11. For $f(x) = 2x + 1$ and $g(x) = x - 2$

- $(f+g)(x) = 3x - 1$
Domain: $(-\infty, \infty)$
- $(f-g)(x) = x + 3$
Domain: $(-\infty, \infty)$
- $(fg)(x) = 2x^2 - 3x - 2$
Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
Domain: $(-\infty, 2) \cup (2, \infty)$

12. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

- $(f+g)(x) = -2x$
Domain: $(-\infty, \infty)$
- $(f-g)(x) = 2 - 6x$
Domain: $(-\infty, \infty)$
- $(fg)(x) = -8x^2 + 6x - 1$
Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
Domain: $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

13. For $f(x) = x^2$ and $g(x) = 3x - 1$

- $(f + g)(x) = x^2 + 3x - 1$
Domain: $(-\infty, \infty)$

- $(fg)(x) = 3x^3 - x^2$
Domain: $(-\infty, \infty)$

- $(f - g)(x) = x^2 - 3x + 1$
Domain: $(-\infty, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
Domain: $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$

14. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f + g)(x) = x^2 + 6x$
Domain: $(-\infty, \infty)$

- $(fg)(x) = 7x^3 - 7x^2$
Domain: $(-\infty, \infty)$

- $(f - g)(x) = x^2 - 8x$
Domain: $(-\infty, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
Domain: $(-\infty, 0) \cup (0, \infty)$

15. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f + g)(x) = x^2 + 3x + 2$
Domain: $(-\infty, \infty)$

- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
Domain: $(-\infty, \infty)$

- $(f - g)(x) = x^2 - 3x - 10$
Domain: $(-\infty, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
Domain: $(-\infty, -2) \cup (-2, \infty)$

16. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f + g)(x) = x - 3$
Domain: $(-\infty, \infty)$

- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
Domain: $(-\infty, \infty)$

- $(f - g)(x) = -2x^2 + x + 15$
Domain: $(-\infty, \infty)$

- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

17. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f + g)(x) = \frac{x^2+4}{2x}$
Domain: $(-\infty, 0) \cup (0, \infty)$

- $(fg)(x) = 1$
Domain: $(-\infty, 0) \cup (0, \infty)$

- $(f - g)(x) = \frac{x^2-4}{2x}$
Domain: $(-\infty, 0) \cup (0, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
Domain: $(-\infty, 0) \cup (0, \infty)$

18. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f + g)(x) = \frac{x^2 - 2x + 2}{x - 1}$
Domain: $(-\infty, 1) \cup (1, \infty)$

- $(fg)(x) = 1$
Domain: $(-\infty, 1) \cup (1, \infty)$

- $(f - g)(x) = \frac{x^2 - 2x}{x - 1}$
Domain: $(-\infty, 1) \cup (1, \infty)$

- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
Domain: $(-\infty, 1) \cup (1, \infty)$

19. For $f(x) = x$ and $g(x) = \sqrt{x + 1}$

- $(f + g)(x) = x + \sqrt{x + 1}$
Domain: $[-1, \infty)$

- $(fg)(x) = x\sqrt{x + 1}$
Domain: $[-1, \infty)$

- $(f - g)(x) = x - \sqrt{x + 1}$
Domain: $[-1, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x + 1}}$
Domain: $(-1, \infty)$

20. For $f(x) = \sqrt{x - 5}$ and $g(x) = f(x) = \sqrt{x - 5}$

- $(f + g)(x) = 2\sqrt{x - 5}$
Domain: $[5, \infty)$

- $(fg)(x) = x - 5$
Domain: $[5, \infty)$

- $(f - g)(x) = 0$
Domain: $[5, \infty)$

- $\left(\frac{f}{g}\right)(x) = 1$
Domain: $(5, \infty)$

21. 2

22. -3

23. 0

24. $6x + 3h - 1$

25. $-2x - h + 2$

26. $8x + 4h$

27. $-2x - h + 1$

28. $3x^2 + 3xh + h^2$

29. m

30. $2ax + ah + b$

31. $\frac{-2}{x(x + h)}$

32. $\frac{3}{(1 - x - h)(1 - x)}$

33. $\frac{-(2x + h)}{x^2(x + h)^2}$

34. $\frac{-2}{(x + 5)(x + h + 5)}$

35. $\frac{-4}{(4x - 3)(4x + 4h - 3)}$

36. $\frac{3}{(x + 1)(x + h + 1)}$

$$37. \frac{-9}{(x-9)(x+h-9)}$$

$$38. \frac{2x^2 + 2xh + 2x + h}{(2x+1)(2x+2h+1)}$$

$$39. \frac{1}{\sqrt{x+h-9} + \sqrt{x-9}}$$

$$40. \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$

$$41. \frac{-4}{\sqrt{-4x-4h+5} + \sqrt{-4x+5}}$$

$$42. \frac{-1}{\sqrt{4-x-h} + \sqrt{4-x}}$$

$$43. \frac{a}{\sqrt{ax+ah+b} + \sqrt{ax+b}}$$

$$44. \frac{3x^2 + 3xh + h^2}{(x+h)^{3/2} + x^{3/2}}$$

$$45. \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$$

46. • $C(0) = 26$, so the fixed costs are \$26.
 • $\bar{C}(10) = 4.6$, so when 10 shirts are produced, the cost per shirt is \$4.60.
 • $p(5) = 20$, so to sell 5 shirts, set the price at \$20 per shirt.
 • $R(x) = -2x^2 + 30x$, $0 \leq x \leq 15$
 • $P(x) = -2x^2 + 28x - 26$, $0 \leq x \leq 15$
 • $P(x) = 0$ when $x = 1$ and $x = 13$. These are the 'break even' points, so selling 1 shirt or 13 shirts will guarantee the revenue earned exactly recoups the cost of production.
47. • $C(0) = 100$, so the fixed costs are \$100.
 • $\bar{C}(10) = 20$, so when 10 bottles of tonic are produced, the cost per bottle is \$20.
 • $p(5) = 30$, so to sell 5 bottles of tonic, set the price at \$30 per bottle.
 • $R(x) = -x^2 + 35x$, $0 \leq x \leq 35$
 • $P(x) = -x^2 + 25x - 100$, $0 \leq x \leq 35$
 • $P(x) = 0$ when $x = 5$ and $x = 20$. These are the 'break even' points, so selling 5 bottles of tonic or 20 bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.
48. • $C(0) = 240$, so the fixed costs are 240¢ or \$2.40.
 • $\bar{C}(10) = 42$, so when 10 cups of lemonade are made, the cost per cup is 42¢.
 • $p(5) = 75$, so to sell 5 cups of lemonade, set the price at 75¢ per cup.
 • $R(x) = -3x^2 + 90x$, $0 \leq x \leq 30$
 • $P(x) = -3x^2 + 72x - 240$, $0 \leq x \leq 30$
 • $P(x) = 0$ when $x = 4$ and $x = 20$. These are the 'break even' points, so selling 4 cups of lemonade or 20 cups of lemonade will guarantee the revenue earned exactly recoups the cost of production.

49. • $C(0) = 36$, so the daily fixed costs are \$36.
 • $\overline{C}(10) = 6.6$, so when 10 pies are made, the cost per pie is \$6.60.
 • $p(5) = 9.5$, so to sell 5 pies a day, set the price at \$9.50 per pie.
 • $R(x) = -0.5x^2 + 12x$, $0 \leq x \leq 24$
 • $P(x) = -0.5x^2 + 9x - 36$, $0 \leq x \leq 24$
 • $P(x) = 0$ when $x = 6$ and $x = 12$. These are the ‘break even’ points, so selling 6 pies or 12 pies a day will guarantee the revenue earned exactly recoups the cost of production.
50. • $C(0) = 1000$, so the monthly fixed costs are 1000 *hundred* dollars, or \$100,000.
 • $\overline{C}(10) = 120$, so when 10 scooters are made, the cost per scooter is 120 hundred dollars, or \$12,000.
 • $p(5) = 130$, so to sell 5 scooters a month, set the price at 130 hundred dollars, or \$13,000 per scooter.
 • $R(x) = -2x^2 + 140x$, $0 \leq x \leq 70$
 • $P(x) = -2x^2 + 120x - 1000$, $0 \leq x \leq 70$
 • $P(x) = 0$ when $x = 10$ and $x = 50$. These are the ‘break even’ points, so selling 10 scooters or 50 scooters a month will guarantee the revenue earned exactly recoups the cost of production.
51. $(f + g)(-3) = 2$ 52. $(f - g)(2) = 3$ 53. $(fg)(-1) = 0$
54. $(g + f)(1) = 0$ 55. $(g - f)(3) = 3$ 56. $(gf)(-3) = -8$
57. $\left(\frac{f}{g}\right)(-2)$ does not exist 58. $\left(\frac{f}{g}\right)(-1) = 0$ 59. $\left(\frac{f}{g}\right)(2) = 4$
60. $\left(\frac{g}{f}\right)(-1)$ does not exist 61. $\left(\frac{g}{f}\right)(3) = -2$ 62. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$