

Systems of Linear Equations  
Math 130 Linear Algebra  
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You already know how to solve systems of linear equations, and we've looked at the ancient Chinese method of elimination to solve them. We'll analyze that algorithm and introduce some terminology associated with it.

Although we'll have a lot more use for matrices later, for the time being we'll use them as a shorthand notation when describing systems of linear equations.

**Standard notation and terminology.** When we've got a system of linear equations, we'll usually let  $m$  denote the number of equations, and  $n$  the number of variables, and say the system is a  $m \times n$  system of equations. For instance, here's an example  $3 \times 2$  system of equations:

$$\begin{cases} 5x + 2y = 12 \\ 3x - y = 5 \\ x + 3y = 5 \end{cases}$$

Here,  $m = 3$  since there are 3 equations, and  $n = 2$  since the two variables are  $x$  and  $y$ . (Typically, when a system has more equations than unknowns, then there are no solutions. Does this system have any solutions?)

When there are many variables, we usually subscript them, but when there are few variables we usually use different letters for the variables. For instance, if  $n = 6$ , we might use the 6 variables  $x_1, x_2, x_3, x_4, x_5$ , and  $x_6$ , but if  $n = 3$ , we might use the three variables  $x, y$ , and  $z$ .

A general  $m \times n$  system looks like this:

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

where the  $n$  variables are  $x_1, x_2, \dots, x_n$ . The  $m$  constants that appear on the right side of the  $m$  equations are denoted  $b_1, b_2, \dots, b_m$ . Since each of the  $m$  equations has  $n$  coefficients for the  $n$  variables, there are  $mn$  coefficients in all, and these are denoted with doubly subscripts, so, for instance  $a_{35}$  is the coefficient of the variable  $x_5$  in the third equation.

A *solution* to the system is an  $n$ -tuple of values for the  $n$  variables that make all the equations simultaneously true. For example, in the  $3 \times 2$  system above, a solution is  $(x, y) = (2, 1)$ . In fact, it's the only solution for that system.

**Matrices.** There are two different matrices we'll use to analyze systems of linear equations. Since the system is determined by its coefficients and its constants, and those can naturally be displayed in a matrix.

Consider the  $3 \times 2$  system of equations above. Its coefficients can be displayed in the  $3 \times 2$  matrix

$$\begin{bmatrix} 5 & 2 \\ 3 & -1 \\ 1 & 3 \end{bmatrix}$$

called the *coefficient matrix* for the system. If you want to include the constants in a matrix, too, you can add another column to get the  $3 \times 3$  matrix

$$\begin{bmatrix} 5 & 2 & 12 \\ 3 & -1 & 5 \\ 1 & 3 & 5 \end{bmatrix}$$

called the *augmented matrix* for the system. Frequently, a vertical line is added to separate the coefficients from the constants, as in

$$\left[ \begin{array}{cc|c} 5 & 2 & 12 \\ 3 & -1 & 5 \\ 1 & 3 & 5 \end{array} \right]$$

We'll only draw the vertical line to emphasize the meanings of the different parts of the matrix. We'll usually omit it.

**Matrices in MATLAB.** We've seen how easy it is to enter into MATLAB the coefficient matrix **A** and constant column vector **b** of a system of linear equations, and then compute the solution. Let's see what MATLAB does for the  $3 \times 2$  system of equations above.

```
>> A = [5 2; 3 -1; 1 3]
A =
     5     2
     3    -1
     1     3
```

```
>> b = [12; 5; 5]
b =
    12
     5
     5
```

```
>> A\b
ans =
    2.0000
    1.0000
```

Let's try MATLAB on this system

$$\begin{cases} 5x + 2y + z = 12 \\ 3x - y - z = 5 \end{cases}$$

```
>> A=[5 2 1;3 -1 -1]
A =
     5     2     1
     3    -1    -1
```

```
>> b=[12;5]
b =
    12
     5
```

```
>> A\b
ans =
    2.0000
    1.0000
     0
```

Thus,  $(x, y, z) = (2, 1, 0)$  is a solution to the system of equations. But it's not the only solution. Another is  $(x, y, z) = (3, -7, 11)$ , and there are infinitely more solutions to this indeterminate system.

On the other hand, the system

$$\begin{cases} 3x + 2y = 4 \\ 6x + 4y = 5 \end{cases}$$

is inconstant, that is, it has no solutions. MATLAB gives

```
>> A=[3 2;6 4]
A =
     3     2
     6     4
```

```
>> b=[4;5]
b =
     4
     5
```

```
>> A\b
Warning: Matrix is singular.
```

```
ans =
   -Inf
    Inf
```

We'll see later how MATLAB can be used to help you find all the solutions to a system.

MATLAB also allows you to do create the identity matrices of various sizes, do addition and subtraction on matrices of the same shape, multiply a scalar and a matrix, take a transpose of a matrix, as well as many other operations, most of which we'll use throughout the course.

```
>> I = eye(3)
I =
     1     0     0
     0     1     0
     0     0     1
```

```
>> A=[4 5; 2 3; 0 1]
```

```
A =
```

```
    4    5
    2    3
    0    1
```

```
>> B=[1 1; 3 0; 4 4]
```

```
B =
```

```
    1    1
    3    0
    4    4
```

```
>> A+B
```

```
ans =
```

```
    5    6
    5    3
    4    5
```

```
>> 2*A-B
```

```
ans =
```

```
    7    9
    1    6
   -4   -2
```

```
>> B'
```

```
ans =
```

```
    1    3    4
    1    0    4
```

### Elementary row operations, and reduced row echelon form for a matrix.

When you want to solve a system of linear equations  $A\mathbf{x} = \mathbf{b}$ , form the augmented matrix by appending the column  $\mathbf{b}$  to the right of the coefficient matrix  $A$ . Then you can solve the system of equations by operating on the rows of the augmented matrix rather than on the actual equations in the system. The three *row operations* are those operations on a matrix that don't change the solution set of the corresponding system of linear equations. The simplest ones are called elementary row operations, and the elementary ones are of three types.

1. Exchange two rows.

2. Multiply or divide a row by a nonzero constant.

3. Add or subtract a multiple of one row from another.

You can use these three operations to convert the augmented matrix to a particularly simple form that allows you to read off the solutions. The idea is to eliminate the variables from the equations so that each variable only occurs once. If there's a unique solution, that works perfectly. If the system is indeterminate and has infinitely many solutions, it doesn't work perfectly, but we'll see how that works.

You can systematize this elimination process to form an algorithm. Work from the leftmost column right one column at a time to simplify the augmented matrix (and, therefore, the system it represents). The column you're working on is called the *pivot column*. Look down the column for the first nonzero entry that has all 0s to its left. Call that entry the *pivot element*, or more simply the *pivot*. There are three things to do corresponding to the three elementary row operations.

1. Exchange the pivot row with the highest row that has 0s in the pivot column and all columns to the left. (Do nothing if there is no such row.)
2. Divide the pivot row by the pivot so that the value of the pivot becomes 1.
3. Subtract multiples of the pivot row from the other rows to clear out the pivot column (except the pivot itself).

When you're all done with this algorithm, the matrix will be in something called reduced row echelon form. The word echelon comes from a particular stairstep formation of troops. For us, a matrix is in *reduced row echelon form* if

1. the rows of all zeros (if any) appear at the bottom of the matrix
2. the first nonzero entry of a nonzero row is 1

3. that leading 1 appears to the right of leading 1s in higher rows
4. all the other entries in a column that has a leading 1 are 0

If the first three conditions hold, the matrix is said to be in *row echelon form*.

The reduction algorithm to convert an augmented matrix to reduced row echelon form goes by the name Gauss-Jordan reduction. The partial algorithm that stops with a row echelon form goes by the name Gaussian elimination. As we've seen, this reduction was known to the ancient Chinese.

Two matrices are *row equivalent* if their corresponding systems of linear equations have the same solutions. It's easy to show that means you can perform a sequence of elementary row operations on one to eventually get the other.

**Solutions of systems of homogeneous linear equations.** A polynomial is said to be *homogeneous* if all its terms have the same degree. For example  $4x^3 + 5x^2y - 8xyz$  is a homogeneous cubic polynomial, whereas  $4x^3 + 5xy$  is a cubic polynomial which is not homogeneous because it has a quadratic term. The word homogeneous comes from the Greek and means of the same kind.

Likewise, an equation is *homogeneous* if its terms have the same degree.  $4x^3 = 5y^2z$  is homogeneous, but  $4x^3 = 5xy$  is not. A linear equation is one whose terms are all degree 1 or less, and it's homogeneous if all its terms are degree 1. Here's a typical system of homogeneous equations

$$\begin{cases} 4x + 2y - z = 0 \\ x - y - 4z = 0 \\ 2x + 3z = 0 \end{cases}$$

The set of solutions  $V$  to any system of homogeneous linear equations is a vector space. That's because  $\mathbf{0} = (0, 0, 0)$  is always a solution, if  $\mathbf{v}$  is a solution, then any scalar multiple of  $\mathbf{v}$  is a solution, and if both  $\mathbf{v}$  and  $\mathbf{w}$  are solutions, then so is  $\mathbf{v} + \mathbf{w}$ .

**Row reduction in MATLAB.** You can actually perform individual elementary row operations in MATLAB, either by explicitly manipulating the rows of a matrix or by using the routine `reduce`.

If you're only interested in the resulting reduced row echelon form of a matrix, then you can use the command `rref`. For example, here it's used for the system of linear equations

$$\begin{cases} 2v + w - 3x + 4y + 5z = 7 \\ v - 2w + 2x + 3y = 4 \\ 4v + 5x + 2y + z = -4 \end{cases}$$

```
>> A=[2 1 -3 4 5 7;1 -2 2 3 0 4;4 0 5 2 1 -4]
A =
```

```
     2     1    -3     4     5     7
     1    -2     2     3     0     4
     4     0     5     2     1    -4
```

```
>> rref(A)
ans =
```

```
     1     0     0     1.5366     1.3171     1.8049
     0     1     0    -1.5610    -0.1951    -3.3415
     0     0     1    -0.8293    -0.8537    -2.2439
```

From that reduced row echelon form, we can determine the general solution. For that  $y$  and  $z$  may be freely chosen, and

$$\begin{aligned} v &= -1.5366y - 1.3171z + 1.8049 \\ w &= 1.5610y + 0.1951z - 3.3415 \\ x &= 0.8293y + 0.8537z - 2.2439 \end{aligned}$$

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