Linear Algebra

1.1 Systems of Linear Equations

A *linear equation* in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{1}$$

where b and the coefficients a_i $(1 \le i \le n)$ are constants.

A *solution* to the linear equation (1) is an ordered set $(s_1, s_2, \ldots, s_n)$ of numbers with the property that $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$ holds. The *solution set* of (1) is the set containing all such solutions. To *solve* (1) is to find the solution set.

A *system* of linear equations, also called a *linear system*, is a collection of $m > 1$ linear equations in the n variables $x_1, x_2, \ldots, x_n$ that we want to solve simultaneously, i.e., we look for a set $\{x_1, x_2, \ldots, x_n\}$ that simultaneously solves each equation in the set. For example, the system

$$\left\{ \begin{array}{rcrcr} x & + & 2y & = & 0 \\ 2x & + & y & = & 3 \end{array} \right. \tag{2}$$

has the solution $x = 2,\ y = -1$.

1.1.1 Equivalent Systems

We say that two systems of linear equations are *equivalent* if they have identical solution sets. For example, the system

$$\left\{ \begin{array}{rcrcr} x & + & 2y & = & 0 \\ -4x & - & 2y & = & -6 \end{array} \right.$$

is equivalent to (2).

In this course, we almost always represent systems of equations by means of their associated matrices. A *matrix* is a 2-dimensional array of numbers. Given a system of equations, there are two matrices of interest. One is the *coefficient matrix*, which contains the coefficients from the left-hand side of the system. If we label this matrix A, then a_{ij}, the entry in row i, column j, is simply the coefficient on the ith variable in the jth equation. The *augmented matrix* for the system includes the right-hand side as an additional column. For example, the coefficient and augmented matrices for (2) are

$$A = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right] \text{ and } [A|\mathbf{b}] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 3 \end{array} \right].$$

1.1.2 Elementary Row Operations

The elementary row operations, applicable to both systems of equations and to matrices, are as follows:

1. interchange two rows.
2. multiply a row by a nonzero real constant. (This is called *scaling*.)
3. add a multiple of one row to another.

The author refers to these as Row Operations 1, 2, and 3, respectively.

Exercises in 1.1: 1b, 2b, 5b, 6d, 9, 10.

1.2 Row Echelon Form

An $m \times n$ matrix A is said to be in *row echelon form* if

1. the first nonzero in any row is a 1,
2. for any $1 \leq k < n$, the number of leading zeros in row $k+1$ is greater than the number of leading zeros in row k (unless row k is a row of zeros), and
3. nonzero rows precede zero rows.

Example: Here is a 4×4 matrix A, in row-echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that condition (1) is not universally required, i.e., the leading nonzero in any row need not be a 1. It is useful to keep in mind, though, that Leon expects only 1s in the leading positions.

1.2.1 Gaussian Elimination

The easiest method for putting a matrix into row-echelon form is to use these elementary row operations iteratively in a procedure called Gaussian Elimination. Once a matrix is in row-echelon form, the associated equation, if consistent, can be solved by back-substitution.

1.2.2 Overdetermined and underdetermined Systems

An overdetermined system is one with more equations than unknowns, while an underdetermined system has more unknowns than equations. Overdetermined systems are often inconsistent, while the consistency of an underdetermined system depends upon its right-hand side. If consistent, an underdetermined system has infinitely many solutions. In particular, if the right-hand side consists only of zeros (i.e., if the system is homogeneous), there will be infinitely many solutions. See Leon, pp. 15–18 for a discussion.

Exercises in 1.2: 2, 4, 5e, 8.

1.3 Matrix Algebra

You should be comfortable with both the vocabulary and the operations of elementary matrix algebra. A **vector** is either an $n \times 1$ or a $1 \times n$ matrix. The first is sometimes called a *column vector* (think of the columns in a matrix), while the second is almost always called a *row vector* to emphasize that it has (and is!) one row, not one column. We will be doing most of our work in $\mathbb{R}^n$, also called *Euclidean n-space*, where vectors will typically be represented as columns.

The basics of doing algebra with matrices consist of addition, subtraction, and multiplication of matrices. You should have mastered these skills before registering for MA3042. To make sure that you remember how it all works, try these exercises from section 1.3: 1–4, 11–14.

1.4 Elementary Matrices

If M is a nonsingular $m \times m$ matrix and A an $m \times n$ matrix, then the linear equations $A\mathbf{x} = \mathbf{b}$ and $MA\mathbf{x} = M\mathbf{b}$ have the same solutions, i.e., the underlying linear systems are equivalent. Our goal is to find an equivalent system that is easier to solve (a *triangular* system), and the notion of multiplying both sides of the equation by a nonsingular matrix can be useful, both theoretically and practically. An *elementary matrix* is obtained from the identity matrix by performing a single row operation. The author uses them to describe Gaussian Elimination as a sequence of matrix multiplications, as well as to simplify the discussion of the triangular factorization $A = LU$. We'll see elementary matrices once or twice during the quarter, so know what they are and what they do (pp.61–64).

The full discussion of the LU factorization of a matrix A is beyond the scope of these notes, but can be found in the text on pages 67–68. In practice, we can find the LU factorization by simply adding a book-keeping step to Gaussian elimination. Given an $n \times n$ matrix A, in some situations it is advantageous to factor A as a product $A = LU$, where L is $n \times n$, lower triangular, with 1s on the main diagonal, and U is $n \times n$ and upper triangular. Solving $A\mathbf{x} = \mathbf{b}$ then reduces to the consecutive solution of two triangular systems. For an example, consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & 10 & 20 \end{bmatrix}.$$

Applying Gaussian Elimination to A, we find the upper triangular matrix

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix}.$$

So what is this matrix L? It's remarkably simple. In performing Gaussian elimination, we add 1 times row 1 to row 2, so the multiplier is 1. Placing a minus sign in front of that 1, we record (-1) in row 2, column 1 of L. (Note that the multiplier occupies the same position in L as the element just eliminated from A.) In eliminating the entry in row 3, column 1, we add -2 times row 1 to row 3; placing a minus sign in front of -2, we record 2 in row 3, column 1. That's it for column 1. Let's let A' represent the state of our matrix after completing the elimination step in column 1. We have

$$A' = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 6 & 18 \end{bmatrix}.$$

To eliminate the 6 in the $3,2$-position, we add -3 times row 2 to row 3. Placing a minus sign in front of this multiplier, we record a 3 in the $3,2$-position of L. The final state of what started out as A is U, shown above. The final state of L is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

Note that L is lower triangular with 1s on the main diagonal and with the negatives of the multipliers occupying the lower triangle. Note: You might find it useful to think in terms of subtraction rather than addition when applying Row Operation 3, since then you don't have to multiply the multipliers by -1 when constructing L. (The reason for the sign change in the multipliers has to do with the inverses of elementary matrices, so if this is too mysterious you can find the whole story in the text.)

The remarkable thing about this simple process is that

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & 10 & 20 \end{bmatrix} = A.$$

Exercises 1.4: 1–3, 6, 8, 10b,e, 18.

1.5 Partitioned Matrices

A matrix A can be partitioned by using horizontal lines to partition the set of rows and vertical lines to partition the set of columns. It's hard to motivate, but it isn't hard to describe. Suppose A is $m \times n$ and B is $n \times p$. Suppose we partition A as $A = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]$ and B as $B = \left[\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right]$. Then

$$AB = \left[\begin{array}{c|c} C_1D_1 + C_2D_3 & C_1D_2 + C_2D_4 \\ \hline C_3D_1 + C_4D_3 & C_3D_2 + C_4D_4 \end{array}\right],$$

provided the partitioning of A and B is done in such a way that the required subproducts are defined.

The author uses some notation from MATLAB in this section. While MA3042 doesn't require that you know any MATLAB, it makes better sense to borrow MATLAB notation than to invent something new. The borrowed notation used is used to represent rows in a matrix: Let's suppose that A is $m \times n$ and B is $n \times p$. The rows of A are $\mathbf{a}(1,:), \mathbf{a}(2,:), \ldots, \mathbf{a}(m,:)$ (this is MATLAB's notation), and the columns of B are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ (this is standard). It can verified without too much work that the jth column of AB is $A\mathbf{b}_j$, and the ith row of AB is $\mathbf{a}(i,:)B$.

Of special interest are inner products and outer-product expansions. The *inner product* of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is the 1×1 matrix (a scalar) $\mathbf{x}^T\mathbf{y}$. The inner product of $\mathbf{x}$ and $\mathbf{y}$ is the $n \times n$ matrix $\mathbf{x}\mathbf{y}^T$. For example, if $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then their inner product is $\mathbf{x}^T\mathbf{y} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 + 8 = 11$, and their outer product is

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}.$$

An outer product expansion is the description of a matrix product AB obtained by partitioning A as a "row" of columns and B as a"column" of rows. See page 78 in the text.

Exercises in §1.5: 3, 4, 6a, 8.

2.1 The Determinant

The determinant is a function that takes a square matrix as input and produces a real number as its output. The most fundamental application of the determinant is to tell us whether a matrix A is singular, but this is more important from a theoretical point of view than from a practical point of view. (This is a confusing remark. If your application is to learn linear algebra, most of us view the determinant as highly practical. If your application is some real-world problem, the determinant is probably of little use, but it has by then helped you learn enough linear algebra to solve your problem!) The determinant of a 1×1 matrix $A = [a]$ is defined to be $\det(A) = a$.

The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $\det(A) = ad - bc$.

We pause for a change of notation. A competing, and equally popular, notation for the determinant uses the pipes that we use for absolute value, so $|A|$ represents the determinant of A. Now back to business. It turns out that A is invertible if and only if $|A| \neq 0$. We could continue in this fashion, ending up with a characterization of the determinant as a sum of signed diagonal products from A, but there is an equivalent (and easier) way that is recursive in nature. Given an $n \times n$ matrix A, let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A. The *i, j-minor* of A, also called the minor of a_{ij}, is $|M_{ij}|$, the determinant of M_{ij}. The associated *cofactor* A_{ij} is defined by $A_{ij} = (-1)^{i+j}|M_{ij}|$, a signed minor. Here is how we use it: given the $n \times n$ matrix A, we find A using the *cofactor expansion* $|A| = a_{11}A_{11} + a12A_{12} + \cdots + a_{1n}A_{1n}$, equivalent to $|A| = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^n a_{1n}A_{1n}$. See pp.90–95.

The preceding definition used row 1 of A, but it turns out that the determinant can be computed in this fashion using any row, or for that matter any column, of A. Some basic properties of the determinant are mentioned at the end of section 2.1, and are listed here. Let A be any square matrix. Then

1. $|A| = |A^T|$.

2. if A is triangular, then $|A| = \Pi_{i=1}^{n} a_{ii}$, the product of the diagonal entries of A.

3. if A has a line (a row or a column) of zeros, then $|A| = 0$.

4. if A has two identical rows or two identical columns, then $|A| = 0$.

Exercises Section 2.1: 3, 6, 9.

2.1.1 Properties of the Determinant

As forecast by the 1×1 case and 2×2 cases mentioned above, we have a theorem that tells us that a square matrix A is singular if and only if the determinant of A is

0. Perhaps the more remarkable and useful property of the determinant is given in the following theorem, which is proven on page 103 in the text.

THEOREM: **If A and B are $n \times n$ matrices, then**

$$|AB| = |A||B|.$$

Exercises Section 2.2: 1, 4, 7.

3.2 Extra problems for Chapters 1 and 2

1. Which of the following equations is linear?
   (a) $x - 2xy = 6$
   (b) $11x - 26y + z - \cos s = e^t$
   (c) $8x + 6y - 47z = 5$
   (d) $5x^2 - 3 = 0$

2. Find the Row Echelon form of the matrix corresponding to the following linear system, and use this form and back substitution to find the solution of the system.
$$\begin{cases} x_1 - x_2 + x_3 = 6 \\ 4x_1 + 2x_2 - x_3 = 0 \\ 5x_1 + x_2 + x_3 = 12. \end{cases}$$

3. Solve each of the following systems, if possible, by substitution.
   (a) $\begin{cases} 3x_2 = 6 \\ 3x_1 + 3x_2 = 1 \end{cases}$

   (b) $\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + 8x_2 + x_3 = 12 \\ -x_1 + 2x_2 - 5x_3 = 2 \end{cases}$

   (c) $\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + 8x_2 + x_3 = 12 \\ -x_1 + 2x_2 - 3x_3 = 2 \end{cases}$

4. Solve each of the solvable systems from (3), by applying Gaussian elimination.

5. Perform Gaussian elimination on the augmented matrix for each of the following systems. If any solution involves free variables, introduce parameters as needed. Describe the general solution to each system.

(a) $$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 1 \end{array}\right.$$

(b) $$\left\{\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 5 \\ x_1 & & & & & & & + & x_5 & = & 3 \\ x_1 & - & x_2 & & & & & & & = & 3 \end{array}\right.$$

(c) $$\left\{\begin{array}{rcrcrcr} -x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 4x_1 & - & 6x_2 & - & x_3 & = & 7 \\ 3x_1 & + & 4x_2 & + & 2x_3 & = & 15 \end{array}\right.$$

6. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Compute the following:

(a) $\mathbf{x} + \mathbf{y}$, $\mathbf{y} + \mathbf{z}$, and $\mathbf{x} + \mathbf{z}$.

(b) $\mathbf{x} - \mathbf{y}$ and $\mathbf{y} - \mathbf{x}$.

(c) $2\mathbf{x} - 3\mathbf{y} + \mathbf{z}$.

(d) $\mathbf{x} \cdot \mathbf{y}$, $\mathbf{x} \cdot \mathbf{z}$, and $\mathbf{y} \cdot \mathbf{z}$

7. Repeat (6), this time using $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

8. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$. Perform the following:

(a) Compute the sum, difference, and product of each pair of matrices.

(b) Compute $2A - 3B + 4C$.

(c) Find the transpose of each matrix.

(d) For each pair, verify that the transpose of the product is equal to the product of the transposes, but taken in reverse order. (e.g., that $(AB)^T = B^T A^T$)

9. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$. Repeat the operations from the preceding exercise for A and B.

10. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$. Find each of the following, or explain why the indicated operation is not defined.
    (a) $A + B$
    (b) $A + B^T$
    (c) AB
    (d) $A^T B$

11. Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}$.

    Find A^{-1} and B^{-1}, if they exist.

12. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 4 \\ 2 & 3 & 6 \end{bmatrix}$. Find A^{-1} if it exists.

13. Let $A = \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}$, and $\mathbf{b} = (0, -3)^T$. Find the LU factorization of A, and use this factorization to solve $A\mathbf{x} = \mathbf{b}$.

14. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, and $\mathbf{b} = (1, 3, 9)^T$. Find the LU factorization of A, and use this factorization to solve $A\mathbf{x} = \mathbf{b}$.

15. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ -7 & 8 & 9 \end{bmatrix}$.

    Find $\det(A)$, $\det(B)$, and $\det(C)$, using a cofactor expansion.

3.3 Solutions to Exercises

1. Only (c) is linear.

2. Find the Row Echelon form of the matrix corresponding to the following linear system, and use this form and back substitution to find the solution of the system.
$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 6 \\ 4x_1 & + & 2x_2 & - & x_3 & = & 0 \\ 5x_1 & + & x_2 & + & x_3 & = & 12. \end{array}\right.$$

Solution:

Linear System
$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 6 \\ 4x_1 & + & 2x_2 & - & x_3 & = & 0 \\ 5x_1 & + & x_2 & + & x_3 & = & 12. \end{array}\right.$$

Augmented Matrix
$$\left[\begin{array}{rrrr} 1 & -1 & 1 & 6 \\ 4 & 2 & -1 & 0 \\ 5 & 1 & 1 & 12 \end{array}\right].$$

Add (-4) times first equation to second
$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 6 \\ & & 6x_2 & - & 5x_3 & = & -24 \\ 5x_1 & + & x_2 & + & x_3 & = & 12. \end{array}\right.$$

Add (-4) times first row to second.
$$\left[\begin{array}{rrrr} 1 & -1 & 1 & 6 \\ 0 & 6 & -5 & -24 \\ 5 & 1 & 1 & 12 \end{array}\right].$$

Add (-5) times first equation to third
$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 6 \\ & & 6x_2 & - & 5x_3 & = & -24 \\ & & 6x_2 & - & 4x_3 & = & -18. \end{array}\right.$$

Add (-5) times first row to third.
$$\left[\begin{array}{rrrr} 1 & -1 & 1 & 6 \\ 0 & 6 & -5 & -24 \\ 0 & 6 & -4 & -18 \end{array}\right].$$

Add (-1) times second equation to third.
$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 6 \\ & & 6x_2 & - & 5x_3 & = & -24 \\ & & & & x_3 & = & 6. \end{array}\right.$$

Add (-1) times second row to third.
$$\left[\begin{array}{rrrr} 1 & -1 & 1 & 6 \\ 0 & 6 & -5 & -24 \\ 0 & 0 & 1 & 6 \end{array}\right].$$

The solution can now be calculated by back-substitution: The third equation gives $x_3 = 6$. Substitution into the second gives $x_2 = 1$. Substitution of both into the first gives $x_1 = 1$.

3. We are to solve each system by substitution, if possible.

(a) $\left\{ \begin{array}{rcrcl} & & 3x_2 & = & 6 \\ 3x_1 & + & 3x_2 & = & 1 \end{array} \right.$

Solution: From the first equation, $x_2 = 2$. Substituting into the second equation, we have $3x_1 + 3(2) = 3x_1 + 6 = 1$, so $3x_1 = -5$ and $x_1 = -5/3$.

(b) $\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 3x_1 & + & 8x_2 & + & x_3 & = & 12 \\ -x_1 & + & 2x_2 & - & 5x_3 & = & 2 \end{array} \right.$

Solution: No solution exists. If one goes through the steps required to discover a solution, in the end one is faced with an impossibility of the form $0x_2 = 4$.

(c) $\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 3x_1 & + & 8x_2 & + & x_3 & = & 12 \\ -x_1 & + & 2x_2 & - & 3x_3 & = & 2 \end{array} \right.$

Solution: Solving the first equation for x_1, we find $x_1 = 2 - 2x_2 - x_3$. Substituting this for x_1 in the second equation, we have $x_2 - x_3 = 3$, or $x_2 = 3 + x_3$, from which we have now have $x_1 = -4 - 3x_3$. We can then eliminate both x_1 and x_2 from the third equation, arriving at $x_3 = -4$. It follows that $x_2 = 3 - 4 = -1$ and $x_1 = -4 - 3(-4) = 8$.

4. We now use Gaussian elimination to solve systems (a) and (c) from the preceding problem.

(a) $\left\{ \begin{array}{rcrcl} & & 3x_2 & = & 6 \\ 3x_1 & + & 3x_2 & = & 1 \end{array} \right.$

Solution: There is really no work to be done here, since x_1 is absent from the first equation.

(c) $\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 3x_1 & + & 8x_2 & + & x_3 & = & 12 \\ -x_1 & + & 2x_2 & - & 3x_3 & = & 2 \end{array} \right.$

Solution: Subtracting three times row one from row two, and adding row one to row three, we obtain a new system:

$$\left\{ \begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ & & 2x_2 & - & 2x_3 & = & 6 \\ & & 4x_2 & - & 2x_3 & = & 4 \end{array} \right.$$

Now subtracting twice row two from row three, we have

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ \qquad 2x_2 - 2x_3 = 6 \\ \qquad\qquad 2x_3 = -8 \end{cases}$$

Backsubstitution gives us $x_3 = -4$, $x_2 = \frac{1}{2}(6-8) = -1$, and $x_1 = 2+2+4 = 8$.

5. We are to perform Gaussian elimination on the augmented matrix for each of the following systems, obtaining the general solution.

(a) $$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ \qquad x_2 + x_3 = 1 \end{cases}$$

(b) $$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 5 \\ x_1 + x_5 = 3 \\ x_1 - x_2 = 3 \end{cases}$$

(c) $$\begin{cases} -x1 + 2x_2 - x_3 = -4 \\ 4x_1 - 6x_2 - x_3 = 7 \\ 3x_1 + 4x_2 + 2x_3 = 15 \end{cases}$$

Solution:

(a) The augmented matrix for system (a) is

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Elimination has no effect, since the matrix is already in row-echelon form. The free variable is x_3, so we let $x_3 = s$ and proceed with back-substitution: $x_2 = 1 - s$, and $x_1 = 1 - s - 2(1 - s) = -1 + s$. The general solution is then

$$(x_1, x_2, x_3) = (-1 + s, 1 - s, s).$$

(b) The augmented matrix for system (b) is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 5 \\ 1 & 0 & 0 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Introducing zeros below the pivot in the first column, we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & -1 & -1 & -1 & 0 & -2 \\ 0 & -2 & -1 & -1 & -1 & -2 \end{bmatrix}.$$

Introducing a zero below the pivot in the second column, we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & -1 & -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{bmatrix},$$

and the elimination is complete. Free variables are x_4 and x_5, so we set $x_4 = s$ and $x_5 = t$, say. It follows that $x_3 = 2-s+t$, $x_2 = 2-s-(2-s+t) = -t$, and $x_1 = 5+t-(2-s+t)-s-t = 3-t$. So the general solution is

$$(x_1, x_2, x_3, x_4, x_5) = (3-t, -t, 2-s+t, s, t).$$

(c) The augmented matrix for system (c) is

$$A = \begin{bmatrix} -1 & 2 & -1 & -4 \\ 4 & -6 & -1 & 7 \\ 3 & 4 & 2 & 15 \end{bmatrix}.$$

Introducing zeros below the pivot in the first column, we have

$$A = \begin{bmatrix} -1 & 2 & -1 & -4 \\ 0 & 2 & -5 & -9 \\ 0 & 10 & -1 & 3 \end{bmatrix}.$$

Introducing a zero below the pivot in the second column, we obtain

$$A = \begin{bmatrix} -1 & 2 & -1 & -4 \\ 0 & 2 & -5 & -9 \\ 0 & 0 & 24 & 48 \end{bmatrix}.$$

There are no free variables, so we continue with backsubstitution, obtaining

$$(x_1, x_2, x_3) = (3, 1/2, 2).$$

6. We are given $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, and must compute the following:

   (a) $\mathbf{x}+\mathbf{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{y}+\mathbf{z} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and $\mathbf{x}+\mathbf{z} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

   (b) $\mathbf{x}-\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{y}-\mathbf{x} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

   (c) $2\mathbf{x}-3\mathbf{y}+\mathbf{z} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$.

   (d) The dot products:

$$\begin{aligned} x \cdot y &= 1 \cdot (-2) + 2 \cdot 1 = -2 + 2 = 0 \\ x \cdot z &= 1 \cdot 2 + 2 \cdot (-2) = 2 - 4 = -2 \\ y \cdot z &= -2 \cdot 2 + 1 \cdot (-2) = -4 - 2 = -6 \end{aligned}$$

   (e)

$$\begin{aligned} x \cdot y &= 1 \cdot (-2) + 2 \cdot 1 + (-1) \cdot 0 = -2 + 2 + 0 = 0 \\ x \cdot z &= 1 \cdot 2 + 2 \cdot (-2) + (-1) \cdot 3 = 2 - 4 - 3 = -5 \\ y \cdot z &= -2 \cdot 2 + 1 \cdot (-2) + 0 \cdot 3 = -4 - 2 + 0 = -6 \end{aligned}$$

7. We repeat (6), this time using $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, obtaining

   (a) $\mathbf{x}+\mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{y}+\mathbf{z} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$, and $\mathbf{x}+\mathbf{z} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

   (b) $\mathbf{x}-\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}-\mathbf{x} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$.

   (c) $2\mathbf{x}-3\mathbf{y}+\mathbf{z} = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix}$

8. We are given

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix},$$

and are to perform the following:

(a) Compute the sum, difference, and product of each pair of matrices.

(b) Compute $2A - 3B + 4C$.

(c) Find the transpose of each matrix.

(d) For each pair, verify that the transpose of the product is equal to the product of the transposes, but taken in reverse order. (e.g., that $(AB)^T = B^T A^T$)

Solution:

(a) i. The sums: $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A + C = \begin{bmatrix} 4 & 4 \\ -2 & 2 \end{bmatrix}$, and $B + C = \begin{bmatrix} 4 & 0 \\ -2 & 2 \end{bmatrix}$.

ii. Some differences: $A - B = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$, $B - C = \begin{bmatrix} -2 & -4 \\ 2 & 0 \end{bmatrix}$

iii. Some products: $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $BC = \begin{bmatrix} 7 & 0 \\ -2 & 1 \end{bmatrix}$, $CB = \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}$.

(b) $2A - 3B + 4C = \begin{bmatrix} 11 & 18 \\ -8 & 3 \end{bmatrix}$.

(c) The transposes:

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad \text{and } C^T = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}.$$

(d) Using B and C, we get

$$B^T C^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix} = (CB)^T.$$

9. We are given $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$, and are to repeat operations (a), (c), and (d) from the preceding exercise, using A and B.

Solution:

(a) $A+B = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 4 & 3 & 2 \end{bmatrix}$, $A-B = \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 0 \\ -2 & -1 & 0 \end{bmatrix}$, $B-A = \begin{bmatrix} 0 & -4 & -6 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$,

$$AB = \begin{bmatrix} 13 & 8 & 0 \\ -3 & -1 & -2 \\ 4 & 1 & -2 \end{bmatrix}, \text{ and } BA = \begin{bmatrix} -1 & -2 & 4 \\ -1 & 0 & -2 \\ 4 & 9 & 11 \end{bmatrix}.$$

(c) $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 4 & -1 & 1 \end{bmatrix}$.

(d) $B^T A^T = \begin{bmatrix} 13 & -3 & 4 \\ 8 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = (AB)^T$.

10. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$. Find each of the following, or explain why the indicated operation is not defined.

(a) $A + B$ is undefined, since the dimensions of the matrices do not agree.

(b) $A + B^T = \begin{bmatrix} 2 & 2 & 7 \\ -2 & 2 & 1 \end{bmatrix}$.

(c) $AB = \begin{bmatrix} 13 & 8 \\ -3 & -1 \end{bmatrix}$.

(d) $A^T B$ is undefined, since the number of columns in A^T is not equal to the number of rows in B.

11. Given $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}$, we must find A^{-1} and B^{-1} if they exist.

Solution: Both are invertible. Following are the steps for finding A^{-1}; the same procedure applied to B will produce B^{-1}.

We start with the matrix

$$[\ A \mid I\ ] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right].$$

After the initial elimination step, we have the matrix

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1/2 & -5/2 & 1 \end{array}\right].$$

Multiplying row two by 2 and subtracting from row one, we have

$$\left[\begin{array}{cc|cc} 2 & 0 & 6 & -2 \\ 0 & 1 & -5 & 2 \end{array}\right].$$

Finally dividing row one by 2, we have

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right], \text{ revealing } A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

12. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 4 \\ 2 & 3 & 6 \end{bmatrix}$. Find A^{-1} if it exists.

Solution: As in the preceding problem, both matrices are invertible and, as before, we construct A^{-1}.

Our starting point is

$$[\ A \mid I\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right].$$

Introducing zeros below the pivot in column one, we have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 4 & -1 & 1 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \end{array}\right].$$

Introducing a zero below the pivot in column two leads to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 4 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right].$$

Subtracting row three from row two to put a zero above the pivot in column three, we have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right].$$

Finally, dividing row three by 2 and row two by 3, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & 1 & -2/3 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/2 \end{array}\right],$$

from which we know that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & -2/3 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}.$$

13. Let $A = \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}$, and $\mathbf{b} = (0, -3)^T$. Find the LU factorization of A, and use this factorization to solve $A\mathbf{x} = \mathbf{b}$.

Solution: The multiplier used in reducing $A = \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}$ to the upper triangular form $U = \begin{bmatrix} 4 & 3 \\ 0 & 3/4 \end{bmatrix}$ is $m_{21} = 7/4$. It follows that $L = \begin{bmatrix} 1 & 0 \\ 7/4 & 1 \end{bmatrix}$. We now use the decomposition to solve the indicated system: solving $L\mathbf{y} = \mathbf{b}$, we have $\mathbf{y} = (0, -3)^T$. Solving $U\mathbf{x} = \mathbf{y}$, we find $\mathbf{x} = (3, -4)^T$.

14. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, and $\mathbf{b} = (1, 3, 9)^T$. Find the LU factorization of A, and use this factorization to solve $A\mathbf{x} = \mathbf{b}$.

Solution: The multipliers used in reducing $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix}$ to the upper-triangular form $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ are $m_{21} = 3$, $m_{31} = 2$, and $m_{32} = -1/2$, so $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}$. Solving $L\mathbf{y} = \mathbf{b}$, we have $\mathbf{y} = (1, 0, 7)^T$. Solving $U\mathbf{x} = \mathbf{y}$, we obtain $\mathbf{x} = (-1, 2, 1)^T$.

15. Given $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ -7 & 8 & 9 \end{bmatrix}$, we must find $\det(A)$, $\det(B)$, and $\det(C)$, using cofactor expansions.

Solution: Expanding along row one to exploit the 0's, we find

$$\det(A) = 2 \det \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = 2(12 - 12) = 0.$$

Either row one or column three would be best for B; using row one, we have

$$\det(B) = \det \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \det \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} = -7 - 7 = -14.$$

There being no 0's in C, there are no shortcuts. Using row one, we find

$$\det(C) = \det \begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & -6 \\ -7 & 9 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ -7 & 8 \end{bmatrix} = 306.$$