

## ▪ System of Linear Equations in Two Variables.

Definition. Terminology.

Linear system of two equations in two variables can be written in form

$$ax + by = h$$

$$cx + dy = k$$

where  $a, b, c, d, h,$  and  $k$  are real constants.

A pair of numbers  $x = x_0,$  and  $y = y_0$

[also written as an ordered pair  $(x_0, y_0)$ ]

is a *solution* of this system

if each equation is satisfied by the pair.

The set of all such ordered pairs

is called the *solution set* for the system.

To *solve* the system is to find its solution set.

Examples.

a)  $x - 5y = 0$

$10x + 4y = -1$

b)  $3u + 17v = 21$

$12u - 7v = -3$

- **Methods of Solving Linear Systems.**

- **Graphing.**

Each linear equation in two variables defined a straight line. To solve a system of two linear equations in two variables, we graph both equations in the same coordinate system. The coordinates of any points that graphs have in common are solutions to the system, since they satisfy both equations.

**Problem #1.** Solve by graphing and check.

a)  $2x - 4y = 10$

$x + 5y = -9$

b)  $x - 2y = 5$

$2x - 4y = 10$

c)  $x - 2y = -10$

$x - 2y = -6$

- **System of Linear Equations: Basic terms.**

A system of linear equations is *consistent* if it has one or more solutions and *inconsistent* if no solutions exist.

A consistent system is said to be *independent* if it has exactly one solution (often referred to as the *unique solution*).

A system is *dependant* if it has more than one solution.

Two systems of equations are *equivalent* if they have the same solution set.

- Possible Solutions to a Linear System.

### The Linear System

$$ax + by = h$$

$$cx + dy = k$$

Must have

Exactly one solution

*Consistent and independent*

No solution

*Inconsistent*

Infinitely many solutions

*Consistent and dependant*

- Substitution method.

This algebraic method provides exact solutions to a system of two linear equations in two variables provided that solutions exist.

- Description of procedure.

1. We choose one of two equations and solve for one variable.
2. Expression for one variable in terms of other we substitute into second equation and solve the resulting linear equation in one variable.
3. To find second variable we *substitute* the result 2) back into the result of 1).

**Problem #2.** Solve by substitution.

$$2x - 4y = 10$$

$$x + 5y = -9$$

## ▪ Elimination by Edition.

The methods of graphing and substitution both work well for the systems involving two variables.

***Elimination by addition*** is the more important method of solution because it may be generalized for larger systems and is the basis for computer-based solution methods.

## ▪ Equivalent Systems.

Operations that Produce Equivalent Systems.

A system of linear equations is transformed into an equivalent system if

- a) Two equations are interchanged.
- b) An equation is multiplied by a nonzero constant.
- c) A constant multiple of one equation is added to another equation.

**Problem #3.** Solve the following systems using elimination by addition.

$$a) \quad 3x - 2y = 8$$

$$2x + 5y = -1$$

$$b) \quad 5x - 2y = 12$$

$$2x + 3y = 1$$

▪ Applications.

Typical applied problems that can be solved using a linear system are Supply and Demand problem, Cost-Revenue functions and Break-even analysis problems (##55-58, pp. 190-191 PART II), Delivery charges problems, Nutrition problems, Resources allocation problems.

**Problem #4.** Nutrition (#64 p. 195 Part II).

(Problem is similar to the Example #6, pp. 189-190 Part II)

A fruit grower can use two types of fertilizer in the orange Grove, brand A and brand B. Each bag of brand A contains 8 pounds of nitrogen and 4 pounds of phosphoric acid.

Each bag of brand B contains 7 pounds of nitrogen and 6 pounds of phosphoric acid. Tests indicate that the grove needs 720 pounds of nitrogen and 500 pounds of phosphoric acid. How many bags of each brand should be used to provide the required amount of nitrogen and phosphoric acid?

**Problem #5.** Supply and Demand (# 54, p. 194, PART II).

(Problem is similar to the Example #7, pp. 190-191 PART II).

At \$1.40 per bushel, the daily supply for oats is 850 bushels, and the daily demand is 580 bushels. When the price falls to \$1.20 per bushel, the daily supply decreases to 350 bushels,

And the daily demand increases to 980 bushels.

Assume that the supply and demand equations are linear.

- a) Find the supply equation.
- b) Find the demand equation.
- c) Find the equilibrium price and quantity.
- d) Graph the two equations in the same coordinate system and identify the supply curve, demand curve and equilibrium point.

- With each linear system  $2 \times 2$

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

we will associate two rectangular tables called matrices:

The matrix of coefficients of the system,  $A$ , and

augmented matrix,  $\tilde{A}$ .

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \tilde{A} = \left( \begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right)$$

For “elimination by addition” we can work with the augmented matrix  $\tilde{A}$ , instead of working with the equations.

Actions we were using to transform/simplify system,

- a) Two equations are interchanged.
- b) An equation is multiplied by a nonzero constant.
- c) A constant multiple of one equation is added to another equation.

mean we are doing some transformations with rows of the augmented matrix. Such transformations are called ***row operations***.

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following ***row operations***:

- a) Two row are interchanged ( $R_i \leftrightarrow R_j$ ).
- b) An row is multiplied by a nonzero constant ( $kR_i \rightarrow R_i$ )
- c) A constant multiple of one row is added to another row ( $kR_j + R_i \rightarrow R_i$ ).

**Problem #6.** Solve the following systems using augmented matrix methods.

a)

$$3x + 4y = 1$$

$$x - 2y = -3$$

b)

$$3x - 6y = -9$$

$$x - 2y = 1$$

c)

$$3x - 6y = -9$$

$$x - 2y = -3$$

- Possible Final Matrix forms for a Linear System of two equations in two variables

Form 1: Unique solution (system is consistent and independent).

$$\left( \begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array} \right)$$

Form 2: Infinitely many solutions (system is consistent and dependent).

$$\left( \begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array} \right)$$

Form 3: No solution (system is inconsistent).

$$\left( \begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array} \right)$$

- More about terminology.

***Particular solution*** to the system.

Solution of a dependent system in two variable contains some parameter (common notation is  $k$ ).

Replacing  $k$  with a real number produces a ***particular solution*** to the system.

**Problem #7.** Find particular solutions to the system c) in the Problem #6 that correspond with  $k = -2$ ,  $k = 11$ .

- Matrices associated with a System of Linear Equations (general case).
- Linear System with arbitrary number of equations ( $m$ ) and arbitrary number of variables ( $n$ ).

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = k_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = k_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = k_m$$

With this system are associated the matrix of coefficients  $A$ , the column matrix of constant terms  $K$ , and the augmented matrix  $\tilde{A}$ .

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad K = \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix}$$

$$\tilde{A} = \left( \begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & k_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & k_m \end{array} \right)$$

- For larger systems it is not feasible to explicitly list *all possible simplified form*. Instead, we state a general definition of a simplified form called a ***reduced matrix*** that can be applied to the systems and matrices of any size.
  
- Definition of Reduced Form.

A matrix is said to be in reduced row echelon form, or, more simple, in reduced form, if

1. Each row consisting entirely of zeros is below any row having at least one non-zero element.
  
2. The leftmost nonzero element in each row is 1.
  
3. All other elements in the column containing the leftmost 1 of a given row are zeros.
  
4. The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

**Problem #8.** Check if the following matrices are in reduced form. Convince yourself that the conditions in the definition are met.

$$\text{a) } \left( \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right) \quad \text{b) } \left( \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right) \quad \text{c) } \left( \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

$$\text{d) } \left( \begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right) \quad \text{e) } \left( \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right) \quad \text{f) } \left( \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

**Problem #9.** The following matrices are not in reduced form. Indicate which condition in the definition is violated for each matrix.

$$\text{a) } \left( \begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 3 \end{array} \right) \quad \text{b) } \left( \begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right) \quad \text{c) } \left( \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 7 \end{array} \right)$$

$$\text{d) } \left( \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

**Problem #10.**

State the row operation(s) required to transform the matrices in Problem #10 into a reduced form and find the reduced form.

**Problem #11.**

What is the number of solutions for linear systems corresponding to the following augmented matrices?

$$a) \left( \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad b) \left( \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad c) \left( \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array} \right)$$

**Gauss-Jordan Elimination.****▪ Main idea.**

Given a linear system, write its augmented matrix and transform this matrix into a reduced form (using row operations).

System corresponding to a reduced augmented matrix (reduced system) is easy to solve. Reduced system is equivalent to the given linear system.

**Problem #12.** Solve a system using Gauss-Jordan Elimination.

$$4x + 4y - 4z = 24$$

$$2x - y + z = -9$$

$$x - 2y + 3z = 1$$

- Description of the procedure of Gauss-Jordan Elimination.

**Step 1.** Choose the *leftmost nonzero column* and use appropriate row operation(s) to get 1 on the top.

**Step 2.** Use multiples of the row containing 1 on the top from step 1 to get zeros in all remaining places in the column containing this 1.

**Step 3.** Repeat step 1 with *submatrix* formed by deleting (mentally) the row used in step 1 and all rows above this row.

**Step 4.** Repeat step 2 with the *entire* matrix.

Continue this process until the entire matrix is in reduced form.

If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop process because we have a contradiction:  $0 = n$ ,  $n \neq 0$ . We can conclude That the system has no solution (inconsistent).

- Uniqueness of reduced form.  
Each matrix has *unique* reduced form.  
Sequence of row operations transforming a matrix into a reduced form is *not unique*.

**Problem #13.** Solve the following system using Gauss-Jordan elimination (problem is similar to the Example #3, p. 210, PART II).

$$3x - 6y + 3z = 11$$

$$2x + y - z = 2$$

$$5x - 5y + 2z = 6$$

- Systems with infinitely many solutions.  
Leftmost variables and remaining variables.

**Problem #14.** Solve the following system using Gauss-Jordan elimination (problem is similar to the Example #4, p. 211).

$$x + 2y - z = 0$$

$$3x - y + z = 6$$

$$-2x - 4y + 2z = 0$$

- Generalization.

If the number of leftmost 1's in the reduced augmented matrix is *less than the number of variables* in the system and there are no contradictions, then the system is dependent and has *infinitely many solutions*.

- Applications of System of Linear Equations.

Common applications are problems about production schedules, business leases, taxable income, nutrition.

**Problem #15.** Business leases.

(The problem is similar to the Example #6, p. 211, PART II).

A chemical manufacturer wants to lease a fleet of 24 railroad tank cars with a combined carrying capacity of 520,000 gallons.

Tank cars with three different carrying capacity are available: 8,000 gallons, 16,000 gallons, and 24,000 gallons.

How many of each type of tank car should be used?

**Problem #16.** Nutrition. (# 74 p. 219, PART II).

A dietitian in a hospital is to arrange a special diet composed of three basic foods. The diet is to include exactly 400 units of calcium, 160 units of iron, and 240 units of vitamin A.

The number of units per ounce of each special ingredient for each of the foods is indicated in the table.

	Units per Ounce		
	Food A	Food B	Food C
Calcium	30	10	20
Iron	10	10	20
Vitamin A	10	30	20

How many ounces of each food must be used to meet the diet requirements?