5.3 Soving Systems of Linear Equations

The solution set of a system of linear equations can be;

(i) Empty

(ii) A one point set, i.e. a unique solution

(iii) Infinite

An example where the solution set is empty.

EXAMPLE. Consider the system

$$\begin{aligned} x + 2y &= 4 \\ x + 2y &= 8 \end{aligned}$$

The augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 2 & 8 \end{array}\right) \overset{R_2=-R_1+R2}{\longrightarrow} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 4 \end{array}\right)$$

We get the absurd relation $0 = 4$. This implies that the system has no solutions. Geometrically the two equations are two parallel lines and hence there is no point of inersection.

An example where the solution set is infinite.

EXAMPLE. Consider the linear system of equations

$$\begin{aligned} x_1 + 2x_2 \qquad\quad &= 4 \\ x_2 - \quad x_3 &= 0 \\ x_1 \qquad + 2x_3 &= 4. \end{aligned}$$

The associated augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right)$$

We now reduce the matrix to row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right) \xrightarrow{R_3=-R_1+R3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3=2R_2+R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have a bottom row of zeros. The second row gives $x_2 = x_3$ and the first row gives $x_1 + 2x_2 = 4$. We can express x_1 in terms of x_3 to get $x_1 = 4 - 2x_3$. Thus we can write the solution set for the system in the following manner:

$$\begin{aligned} S &= \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} | x_2 = x_3 \text{ and } x_1 = 4 - 2x_3 \} \\ &= \{ \begin{pmatrix} 4 & -2x_3 \\ & x_3 \\ & x_3 \end{pmatrix} \} \\ &= \{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} x_3 | x_3 \in \mathbb{R} \} \end{aligned}$$

Example. Consider the linear system

$$\begin{aligned} x + y + \ z - \ w &= 1 \\ y - \ z + \ w &= -1 \\ 3x \qquad + 6z - 6w &= 6 \\ - y + \ z - \ w &= 1 \end{aligned}$$

The associated augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 3 & 0 & 6 & -6 & 6 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right)$$

Gauss' Method:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 3 & 0 & 6 & -6 & 6 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right) \overset{R_3=-3R_1+R3}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -3 & 3 & -3 & 3 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right)$$

$$\overset{R_3=3R_2+R_3, R_4=R_2+R_4}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x + y + z - w &= 1 \\ y - z + w &= -1 \end{aligned}$$

From the second equation we get, $y = -1 + z - w$ and substituting this in equation one we get, $x + (-1 + z - w) + z - w = 1$. Solving for x we get $x = 2 - 2z + 2w$. The soution set:

$$\begin{aligned} S &= \{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} | y = -1 + z - w \text{ and } x = 2 - 2z + 2w\} \\ &= \{\begin{pmatrix} 2 & -2z & +2w \\ -1 & +z & -w \\ & z & \\ & & w \end{pmatrix}\} \\ &= \{\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} w | z, w \in \mathbb{R}\} \end{aligned}$$

6 Linear Independence and Gram Schmidt

6.1 Linear Combination of Vectors

A linear combination of vectors in $\mathbb{R}^2$ is a sum of the form

$$\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

e.g.

$$2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-4) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A linear combination of vectors in $\mathbb{R}^3$ is a sum of the form

$$\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

e.g.

$$2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + (-4) \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

Example. Write $\begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

Solution: We need to find scalars like x, y, z such that

$$\begin{aligned} x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix} \\ \begin{pmatrix} x & +2y & +3z \\ 2x & -y & +z \\ 3x & & -z \end{pmatrix} &= \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix} \end{aligned}$$

We get a system of three linear equations:

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - \ y + \ z &= 8 \\ 3x \qquad - \ z &= 3 \end{aligned}$$

We can use the methods of the last chapter to find the solution set of this system of linear equations. The associated augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right)$$

We now reduce the matrix to row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \overset{R_2=-2R_1+R2, R_3=-3R_1+R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right)$$

$$\overset{R_2=\frac{-1}{5}R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right)$$

$$\overset{R_3=6R_2+R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right)$$

$$\overset{R_3=\frac{-1}{4}R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Therefore we get the relations:

$$\begin{aligned} z &= 3 \\ y &= 2 - z = 2 - 3 = -1 \\ x &= 9 - 2y - 3z = 9 - 2(-1) - 3(3) = 2 \end{aligned}$$

Therefore we write,

$$(2)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + (3)\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$$

6.2 Linear Independence of Vectors

Definition. *A set of vectors S is linearly independent if there is no non-trivial linear combination of vectors that sums to zero.*

Remark. If a set is not linearly independent, it is linearly dependent..

Example. 1. Consider the linear combination which sums to zero,

$$\begin{aligned} \alpha\begin{pmatrix}1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}\alpha\\0\end{pmatrix}+\begin{pmatrix}0\\\beta\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}\alpha\\\beta\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \alpha &= 0\\ \beta &= 0 \end{aligned}$$

is a unique solution. Therefore the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent.

2. Consider the linear combination which sums to zero

$$\begin{aligned} x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}+z\begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}x\\0\end{pmatrix}+\begin{pmatrix}0\\y\end{pmatrix}++\begin{pmatrix}z\\z\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}x+z\\y+z\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ x+z &= 0; \quad x=-z\\ y+z &= 0; \quad y=-z \end{aligned}$$

So for any value of z, say $z=1$, we can choose $x=y=-z=-1$ and get

$$(-1)\begin{pmatrix}1\\0\end{pmatrix}+(-1)\begin{pmatrix}0\\1\end{pmatrix}+(1)\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

a non-trivial linear combination which sums to zero.
So the vectors $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ are **not** linearly independent.

3. Determine whether the set of vectors $\{[1,1,0],[1,0,1],[2,1,1]\}$ is linearly independent.

   SOLUTION: Take a linear combination of the vectors which sums to zero.

$$\begin{array}{rcl} x\begin{pmatrix}1\\1\\2\end{pmatrix}+y\begin{pmatrix}1\\0\\1\end{pmatrix}+z\begin{pmatrix}0\\1\\1\end{pmatrix} & = & \begin{pmatrix}0\\0\\0\end{pmatrix} \\ \begin{pmatrix}x & +y & +2z\\ x & & +z\\ & y & +z\end{pmatrix} & = & \begin{pmatrix}0\\0\\0\end{pmatrix}\end{array}$$

   We get a system of three linear equations:

$$\begin{aligned} x+y+2z&=0\\ x\quad\;+\;\;z&=0\\ y+\;\;z&=0 \end{aligned}$$

   We can use the methods of the last chapter to find the solution set of this system of linear equations. If $x=y=z=0$ is a unique solution, then the vectors are linearly independent, other wise we can find a dependency. The associated augmented matrix:

$$\left(\begin{array}{ccc|c}1&1&2&0\\1&0&1&0\\0&1&1&0\end{array}\right)$$

   We now reduce the matrix to row echelon form:

$$\begin{array}{ccc}\left(\begin{array}{ccc|c}1&1&2&0\\1&0&1&0\\0&1&1&0\end{array}\right) & \overset{R_2=-R_1+R2}{\longrightarrow} & \left(\begin{array}{ccc|c}1&1&2&0\\0&-1&-1&0\\0&1&1&0\end{array}\right)\\ & \overset{R_3=R_2+R_3}{\longrightarrow} & \left(\begin{array}{ccc|c}1&1&2&0\\0&1&1&0\\0&0&0&0\end{array}\right)\end{array}$$

   We have a bottom row of zeros and hence cannot have a unique solution. The second row gives $y+z=0$ and the first row gives $x+y+2z=0$. So

$$\begin{aligned} y&=-z\\ x&=-y-2z=-(-z)-2z=-z \end{aligned}$$

If we choose $z = 1$, then $x = -1$ and $y = -1$we get a non-trivial linear combination which sums to zero:

$$(-1)\begin{pmatrix}1\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\0\\1\end{pmatrix} + (1)\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Therefore the set of vectors is **not** linearly independent.

THEOREM *Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a set of n, n-dimensional vectors, then the set S is linearly independent if and only if the determinant of the matrix having the n vectors as columns is non-zero.*

EXAMPLE. 1. We can apply this result to the example above. Let $\{[1,1,0],[1,0,1],[2,1,1]\}$ be three, 3-dimensional vectors.

$$\begin{vmatrix}1 & 1 & 2\\1 & 0 & 1\\0 & 1 & 1\end{vmatrix} = 1(0-1) - 1((1-0) + 2(1-0) = -1 - 1 + 2 = 0.$$

Since the determinant is zero, the set of vectors are not linearly independent.

2. Consider the set $\{[1,-1,1],[1,0,1],[1,1,2]\}$.

$$\begin{vmatrix}1 & 1 & 1\\-1 & 0 & 1\\1 & 1 & 2\end{vmatrix} = 1(0-1) - 1((-1)(2) - (1)(1)) + 1((-1)(1) - 0) = 1 \neq 0.$$

Since the determinant is non-zero, the set of vectors are linearly independent.