The term linear function consists of two parts: linear and function. To understand what these terms mean together, we must first understand what a function is. The term function describes a special type of relationship between the independent and dependent variable. These values for these variables are chosen from two sets called the domain and range of the function. The values for the independent variable are chosen from the domain and the values for the dependent variable are chosen from the range.

> A function is a correspondence between the independent and dependent variable such that each value of the independent variable corresponds to one value of the dependent variable.

The phrase "a function of" is used to tell the user what the independent variable is. For instance, the phrase "y as a function of x" indicates that the independent variable is x and the dependent variable is y.

To determine if an equation describes a function, identify the independent and dependent variable. Now determine if each value of the independent variable selected from the domain of the correspondence matches with no more than one value of the dependent variable. If there is no more than one match, then the correspondence is a function.

Example 2   Determine If An Equation Represents a Function

Does the equation $10P + Q = 500$ describe P as a function of Q?

Solution Since this example specifies P as a function of Q, we know that the independent variable is Q and the dependent variable is P. To make it easier to see how P and Q are linked, solve the equation for the dependent variable P:

$$10P + Q = 500$$

$$10P = -Q + 500 \qquad \text{Subtract } Q \text{ from both sides}$$

$$P = -\tfrac{1}{10}Q + 50 \qquad \text{Divide both sides by 10}$$

In this form, we can see that a value like $Q = 100$ corresponds to one value of P, $P = 40$. In fact, for any value of Q you get only one value P. This means that this equation describes a function.

Example 3   Determine If An Equation Represents a Function

Does the equation $x^2 + y^2 = 4$ describe y as a function of x?

Solution This example specifies y as a function of x so we know that the independent variable is x and the dependent variable is y. Like the previous example, solve for the dependent variable y:

$$x^2 + y^2 = 4$$

$$y^2 = -x^2 + 4 \qquad \text{Subtract } x^2 \text{ from both sides}$$

$$y = \pm\sqrt{-x^2 + 4} \qquad \text{Square root both sides}$$

If we try a value like $x = 10$, the radicand becomes $-10^2 + 4$ or -96. Assuming we are using real numbers, we can't take the square root of a negative number. The input to this equation is not a reasonable input because it is not a part of the domain of this function.

To be a part of the domain of this function, the input to this function needs to make the radicand nonnegative. The value of x must satisfy $-x^2 + 4 \geq 0$. Inspecting this inequality, we can see that $-x^2 + 4 = 0$ at x = -2 or 2. Values between -2 and 2 like $x = 0$ make the expression $-x^2 + 4$ positive. This means the domain of this relationship is all real numbers greater than or equal to -2, but less than or equal to 2.

If we pick values from the domain like x = 0, we get two outputs,

$y = \sqrt{-0^2 + 4} = 2$ or $y = -\sqrt{-0^2 + 4} = -2$. Since there is a number in the domain that corresponds to more than one member of the range, the relationship $x^2 + y^2 = 4$ does not describe a function.

A graph of this equation verifies this conclusion. The graph of this equation is a circle.
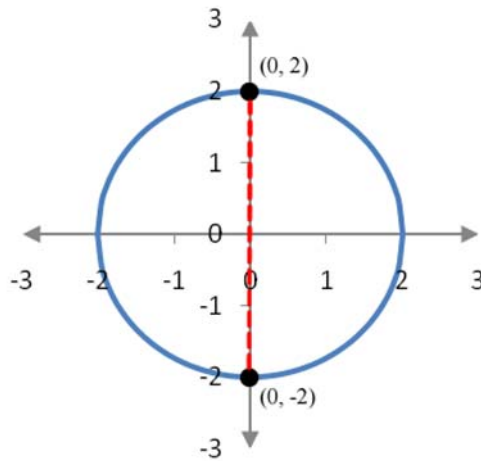
Figure 1 – For this graph, any input between x = -2 and x = 2 leads to two outputs. This means the graph does not correspond to a function.

On this circle, we can see that the line $x = 0$ crosses the graph at the points (0, 2) and (0, -2). This means that the input $x = 0$ corresponds to two different outputs. In fact, any input except for x = 2 or x = -2 corresponds to two outputs since vertical lines cross the graph in two places.

In Example 2, we found that by writing $10P + Q = 500$ in the form $P = -\frac{1}{10}Q + 50$ we could easily check to see if each value of Q led to no more than one value of P. This told us that this relationship was a function since each value of the independent variable led to

no more than one value of the dependent variable. Now let's look at this function a little closer.

You may have thought that the variables P and Q were a bit strange. After all, in most math textbooks you typically work with the variables x and y. In these situations you were working with equations and their corresponding graphs in the x-y plane. You were most concerned with the shapes of these graphs and the equations usually had little basis in an application.

In business and finance, every equation is based on an application. The names of the variables often help you to understand what they represent. For instance, the variables P and Q usually represent the price and quantity of some good. These variables can be related to each other in one of two different ways. A demand function relates the price P of a good to the quantity Q of the good demanded by consumers. The function

$$P = -\tfrac{1}{10}Q + 50$$

is an example of a typical demand function. As Q gets larger, bigger and bigger negative numbers are added to 50 resulting in smaller and smaller values. A graph of this function reflects this characteristic. In Figure 2, the graph is a line that drops as you move from left to right. This means that as the quantity is increased (move left to right), the price drops.
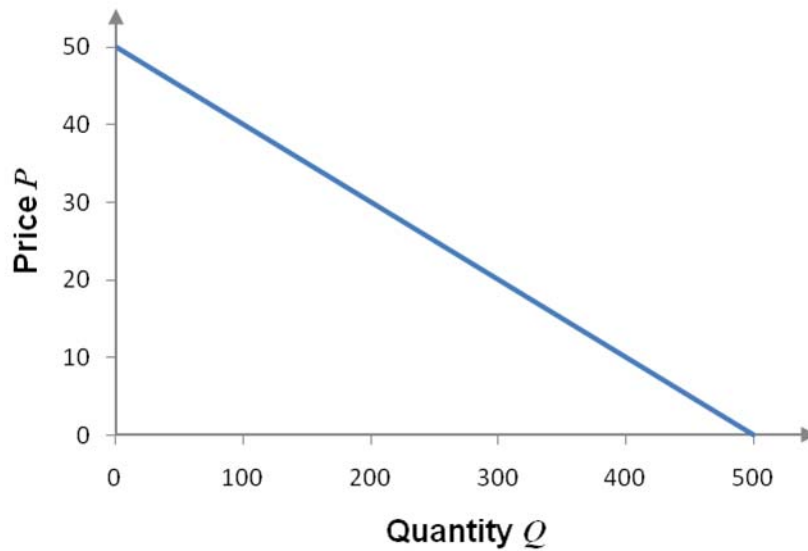
Figure 2 – As the quantity Q increases from 0 to 500 horizontally, the price P drops from 50 to 0. This means that the quantity demanded by consumers increases as the price drops.

This function's graph is a straight line. A function whose graph is a straight line is called a linear function.

Looking at the graph, it is easy to recognize a linear function. We would also like to be able to recognize a linear function from its equation.

Any equation that can be written in the form

$$y = mx + b$$

is a linear function. In this form, we say that y is a linear function of x. The letters m and b are real numbers corresponding to constants and x and y are variables. The graph of this equation is a straight line.

It is easy to read this definition without examining how it applies to the line $P = -\frac{1}{10}Q + 50$. The equations $y = mx + b$ and $P = -\frac{1}{10}Q + 50$ may look different, but are really very similar.

The definition for a linear function contains four letters: y, m, x, and b. Some of these letters are variables and others are constants. To insure that the user of this function knows which letters are variables and which letters are constants, we need to define the variables.

One way of doing this is to write, y is a linear function of x. By writing this phrase, we know that the variables in the equation $y = mx + b$ are x and y. All other letters are constants representing numbers.

By modifying the phrase, we can write other linear functions with different variables. The phrase "P is a linear function of Q" corresponds to the equation

$$P = mQ + b$$

where Q is the independent variable, P is the dependent variable and m and b are constants. An example of a linear function in which P is written as a linear function of Q is

$$P = -\tfrac{1}{10}Q + 50$$

In this case, the value of m is $-\tfrac{1}{10}$ and b is 50.

Example 4   Does An Equation Correspond to a Linear Function?

Decide if the equation $5P + 10Q = 100$ can be written so that P is a linear function of Q.

Solution   To decide if the equation can be written so that P is a linear function of Q, we need to rewrite the equation in the form $P = mQ + b$. By stating "linear function of Q", the example implies that the variable on the right side is Q. Solve the equation for P to put the equation in this form:

$$5P + 10Q = 100$$

$$5P = -10Q + 100 \qquad \text{Subtract } 10Q \text{ from both sides}$$

$$P = -2Q + 20 \qquad \text{Divide each term by 5}$$

The equation can be written in the proper form with $m = -2$ and $b = 20$. ■

Example 5 Does An Equation Correspond to a Linear Function?

Decide if the equation $5P + 10Q = 100$ can be written so that Q is a linear function of P.

Solution Compared to Example 4, this example reverses the role of the variables. "Q is a linear function of P" means that we want to rewrite the equation in the form $Q = mP + b$. To accomplish this task, we must solve the equation for Q:

$$5P + 10Q = 100$$

$$10Q = -5P + 100 \qquad \text{Subtract } 5P \text{ from both sides}$$

$$Q = -\tfrac{1}{2}P + 10 \qquad \text{Divide each term by 10}$$

The equation can be written in the proper form with $m = -\tfrac{1}{2}$ and $b = 10$.

Example 4 and Example 5 suggest that an equation must be solved for the dependent variable to determine if the equation is a linear function. The phrase

Dependent Variable is a linear function of the Independent Variable

allows you to obtain the appropriate form for the linear function. The dependent variable is always written first in this statement and the independent variable is always written after "linear function of". Based on this phrase, we know the form will be

$$\text{Dependent Variable} = m \cdot \text{Independent Variable} + b$$

An equation that can be written in this form, by solving for the dependent variable, is a linear function. If a function cannot be written in this form, it is not a linear function. ■

Example 6   Does An Equation Correspond to a Linear Function?

Can the equation $uv = 5$ be written so that u is a linear function of v?

Solution We need to rewrite the equation in the form $u = mv + b$. To check if this is possible, solve the equation for u.

$$uv = 5$$

$$u = \frac{5}{v} \qquad \text{Divide both sides by v.}$$

In the form for a linear function, a number m must multiply the variable v. However, for this equation a number is *divided* by the variable v. Since it is not possible to write the equation in the proper format, this equation does not define u as a function of v. ■