NOTES ON FUNCTIONS

These notes will cover some terminology regarding functions not included in Solow's book. You should read Appendix A.2 in the book before reading these notes.

Definition 1. We say that two functions f and g are equal if they have the same domain and codomain, and $f(a) = g(a)$ for all a in the domain.

Note that we require the functions to have the same domain and codomain for them to be equal. For example, the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{Z}$ given by

$$f(x) = x^2 \text{ and } g(x) = x^2$$

are defined by the same formula, but they are not equal since they have different domains and codomains. The function $h : \mathbb{R} \to \{x \in R \mid x \geq 0\}$ given by $h(x) = x^2$ again is defined by the same formula as f, and now has the same domain as f, but since they have different codomains, they are not equal.

Definition 2. The *identity function* on a set A, denoted by id_A, is the function from A to itself such that $\mathrm{id}_A(a) = a$ for all $a \in A$.

Definition 3. If $f : A \to B$ and $g : B \to C$ are functions, we define their *composition*, denoted by $g \circ f$, to be the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$.

Definition 4. A function $f : A \to B$ is called *bijective* if it is both injective and surjective.

Again, we have to be careful about the domain and codomain on which a function is defined. Consider a function given by the formula $f(x) = x^2$. It makes no sense to say this is injective, surjective, nor bijective without specifying what domain and codomain we are consideirng. For example, as a function from $\mathbb{R}$ to $\mathbb{R}$, f is neither injective nor surjective; as a function from $\mathbb{R}$ to $\{x \in \mathbb{R} \mid x \geq 0\}$, it is surjective but not injective; and as a function from $\{x \in \mathbb{R} \mid x \geq 0\}$ to itself, it is bijective.

Definition 5. Let $f : A \to B$ be a function. We say that f is *invertible* if there is a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case we call g the inverse of f and denote it by f^{-1}.

This notion also depends on the domain and codomain; the function $h(x) = x^2$ is invertible as a function from the set of positive real numbers to itself (its inverse in this case is the square root function), but it is not invertible as a function from $\mathbb{R}$ to $\mathbb{R}$. The following theorem shows why:

Theorem 1. *A function is invertible if and only if it is bijective.*

Proof. Suppose that the function $f : A \to B$ is invertible and let f^{-1} be its inverse. First we show that f is injective. To this end, suppose $a, b \in A$ are such that $f(a) = f(b)$. Then we can apply f^{-1} to both sides to get

$$(f^{-1} \circ f)(a) = (f^{-1} \circ f)(b).$$

By the definition of inverses, both compositions above are the identity on A, so the above equality becomes $a = b$, showing that f is injective. To show that f is surjective, let $b \in B$ and let $a = f^{-1}(b)$. Then

$$f(a) = f(f^{-1}(b)) = (f \circ f^{-1})(b) = \mathrm{id}_B(b) = b$$

as required. We conclude that f is bijective.

Conversely, suppose that f is bijective. We must construct an inverse. Let $b \in B$. Since f is surjective, there exists $a_b \in A$ so that $f(a_b) = b$, and since f is injective, there is only one such a_b. We define a function $g : B \to A$ by setting $g(b) = a_b$. It is then easy to check (which you should!) that g satisfies the defining properties of the inverse of f, so f is invertible. $\square$

Next we look at certain sets which are associated to functions.

Definition 6. Let $f : A \to B$ be a function. The *image* of a subset $U \subseteq A$ under f, denoted by $f(U)$, is the set of all elements b of B for which there exists $a \in U$ so that $f(a) = b$; in set notation this means

$$f(A) = \{b \in B \mid \exists a \in U \text{ such that } f(a) = b\} = \{f(a) \in B \mid a \in U\}.$$

In particular, the image of A is called the image (or *range*) of f; in addition to the notation $f(A)$, the image of f is also denoted by $\operatorname{im} f$.

If $C \subseteq B$, the *preimage* (or *inverse image*) of C, denoted by $f^{-1}(C)$, is the set of all $a \in A$ so that $f(a) \in C$; in set notation this means

$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}.$$

For $b \in B$, the notation $f^{-1}(b)$ is commonly used for $f^{-1}(\{b\})$.

So, the image of a subset U of A under a function $f : A \to B$ is the set of all things in B which you can possibly get by applying f to elements of U, and the preimage of a set C is the set of all things in A which are sent into C by f. The preimage of a single element b of C, i.e. the set of all things in A that map to b, is commonly called the *fiber* of f above b; we won't be using this terminology in this class, and indeed you probably won't see it again unless you take a geometry or topology course later on. If you really do want to know why we use the word "fiber", I'd be happy to tell you in office hours!

Exercises

1. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if the composition $g \circ f$ is injective, then f is injective. Prove that if the composition $g \circ f$ is surjective, then g is surjective.

2. Let $f : A \to B$ be a function, and let $U \subseteq A$. Prove that $U \subseteq f^{-1}(f(U))$. Are these sets necessarily equal? Why or why not? If not, what are some conditions on f under which they will be equal?

3. Let $g : C \to D$ be a function, and let $V \subseteq D$. Prove that $f(f^{-1}(V)) \subseteq V$. Are these sets necessarily equal? Why or why not? If not, what are some conditions on g under which they will be equal?

4. Let $f : A \to B$ be a function, and let $C, D \subseteq A$. Must it be true that $f(C \cap D) = f(C) \cap f(D)$? Must it be true that $f(C \cup D) = f(C) \cup f(D)$? For each of these, either prove the given equality or give an example in which it fails.

5. There is a one-to-one correspondence between functions from A to B and relations Γ between A and B satisfying the property that if $(a, b) \in \Gamma$ and $(a, c) \in \Gamma$, then $b = c$. Explicitly, a function $f : A \to B$ corresponds to the relation

$$\gamma = \{(a, f(a)) \in A \times B\},$$

called the *graph* of the function, and conversely a relation

$$\gamma = \{(a, b) \in A \times B\}$$

satisfying the above mentioned property corresponds to the function f defined by $f(a) = b$. Indeed, this is essentially the definition of a function given in the book.

Under this correspondence between functions and relations, if the graph of a function is actually an equivalence relation, what can be said about the function in question?