Reachability standard form

- <u>*Property*</u>. *Each linear system* $\mathcal{S} = \{\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D}\}$ *(continuous or discrete-time) which is* <u>*NOT*</u> *completely reachable can be given the "Reachability standard form", that is it can be transformed to an equivalent system* $\overline{\mathcal{S}} = \{\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}},\ \overline{\mathbf{D}}\}$ *where matrices* $\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$, $\overline{\mathbf{C}} = \mathbf{C}\mathbf{T}$ *and* $\overline{\mathbf{D}}$ *have the following structure:*

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_{1,1} & \overline{\mathbf{A}}_{1,2} \\ 0 & \overline{\mathbf{A}}_{2,2} \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{B}}_1 \\ 0 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2 \end{bmatrix}, \qquad \overline{\mathbf{D}} = \mathbf{D}$$

- *Let* $\rho = \dim(\mathcal{X}^+) < n$. *The transformation matrix* $\mathbf{T}$ *to be used for obtaining the reachability standard form is the following:*

$$\mathbf{T} = [\mathbf{T}_1,\ \mathbf{T}_2], \qquad \mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$$

  where $\mathbf{T}_1$, *with dimensions* $n \times \rho$, *is a base matrix of the reachability subspace* $\mathcal{X}^+$, *and where* $\mathbf{T}_2$, *with dimensions* $n \times (n-\rho)$, *is a free matrix such that* $\mathbf{T}$ *is a full rank transformation matrix.*

- <u>*Property*</u>. *The subsystem of dimension* ρ *characterized by matrices* $\overline{\mathbf{A}}_{1,1}$ *and* $\overline{\mathbf{B}}_1$ *is completely reachable.*

- The subsystem $(\overline{\mathbf{A}}_{1,1},\ \overline{\mathbf{B}}_1,\ \overline{\mathbf{C}}_1)$, called <u>*reachable subsystem*</u>, describes completely the dynamics of the given system when the initial state belongs to the reachable subspace:

$$\text{if} \quad \mathbf{x}_2(0) = 0 \qquad \Rightarrow \qquad \mathbf{x}_2(t) = 0, \qquad \forall t \geq 0$$

- The subsystem $(\overline{\mathbf{A}}_{2,2},\ 0,\ \overline{\mathbf{C}}_2)$, characterized by matrices $\overline{\mathbf{A}}_{2,2}$, $\overline{\mathbf{B}}_2 = 0$ and $\overline{\mathbf{C}}_2$, called <u>*not reachable subsystem*</u>, has a dynamics which depends only on the initial state $\overline{\mathbf{x}}_2(0)$ and is not influenced by the input signal $\mathbf{u}$.

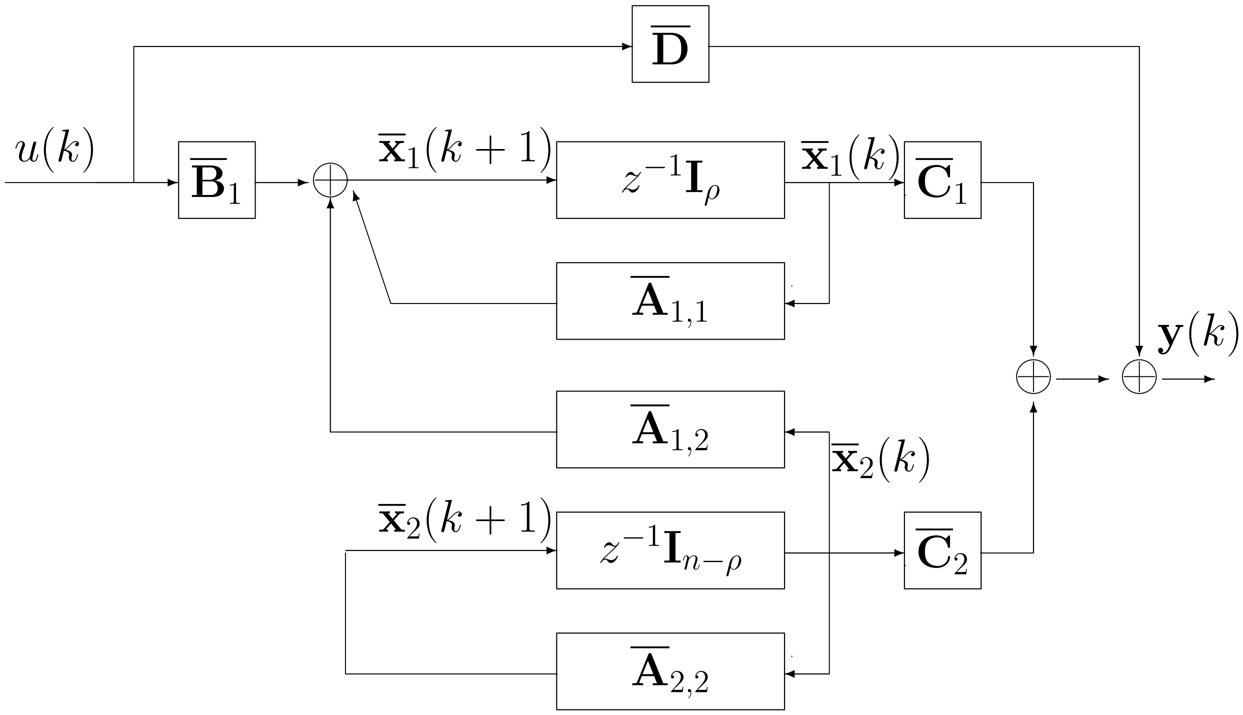
- The eigenvalues of matrix $\mathbf{A}$ are split in two parts: the eigenvalues of the reachable part (the eigenvalues of matrix $\overline{\mathbf{A}}_{1,1}$) and the eigenvalues of the not reachable part (the eigenvalues of matrix $\overline{\mathbf{A}}_{2,2}$).

- Acting on the input u it is <u>not possible</u> to change in any way the dynamics of the <u>not reachable part</u> of the system.

- For discrete systems, the transformed state vector $\overline{\mathbf{x}}(k)$ is divided in two parts: the *reachable* part $\overline{\mathbf{x}}_1$ and the *not reachable* part $\overline{\mathbf{x}}_2$: $\overline{\mathbf{x}} = \begin{bmatrix} \overline{\mathbf{x}}_1 & \overline{\mathbf{x}}_2 \end{bmatrix}^T$ where $\dim(\overline{\mathbf{x}}_1) = \rho$. The system equations can be written as follows:

$$\begin{cases} \overline{\mathbf{x}}_1(k+1) = & \overline{\mathbf{A}}_{1,1}\overline{\mathbf{x}}_1(k) + & \overline{\mathbf{A}}_{1,2}\overline{\mathbf{x}}_2(k) + & \overline{\mathbf{B}}_1\mathbf{u}(k) \\ \overline{\mathbf{x}}_2(k+1) = & & \overline{\mathbf{A}}_{2,2}\overline{\mathbf{x}}_2(k) & \\ \mathbf{y}(k) = & \overline{\mathbf{C}}_1\overline{\mathbf{x}}_1(k) + & \overline{\mathbf{C}}_2\overline{\mathbf{x}}_2(k) + & \overline{\mathbf{D}}\mathbf{u}(k) \end{cases}$$

The corresponding block scheme is:



- A similar decomposition holds also for linear continuous-time systems.

- <u>*Property*</u>*. The transfer matrix $\mathbf{H}(z)$ [or $\mathbf{H}(s)$] of a linear dynamic system, is always equal to the transfer matrix of the* reachable *part of the system.*

<u>Proof</u>. The transfer matrix $\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \overline{\mathbf{C}}(z\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}$ is:

$$\begin{aligned} \mathbf{H}(z) &= \begin{bmatrix} \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} z\mathbf{I} - \overline{\mathbf{A}}_{1,1} & -\overline{\mathbf{A}}_{1,2} \\ 0 & z\mathbf{I} - \overline{\mathbf{A}}_{2,2} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{B}}_1 \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} (z\mathbf{I} - \overline{\mathbf{A}}_{1,1})^{-1} & ****\\ 0 & (z\mathbf{I} - \overline{\mathbf{A}}_{2,2})^{-1} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{B}}_1 \\ 0 \end{bmatrix} = \\ &= \overline{\mathbf{C}}_1(z\mathbf{I} - \overline{\mathbf{A}}_{1,1})^{-1}\overline{\mathbf{B}}_1 \end{aligned}$$

Example. Let us consider the following discrete system:

$$\left\{\begin{array}{rcl} \mathbf{x}(k+1) &=& \overbrace{\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 1&0&0&1 \end{bmatrix}}^{\mathbf{A}} \mathbf{x}(k) + \overbrace{\begin{bmatrix} 0&0\\ 0&1\\ 0&0\\ 1&0 \end{bmatrix}}^{\mathbf{B}} u(k) \\ y(k) &=& \underbrace{\begin{bmatrix} 1&1&1&1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x}(k) \end{array}\right.$$

The reachable matrix of the system is:

$$\mathcal{R}^+ = \left[\begin{array}{cc|cc|cc|cc} 0&0&0&0&\vdots&0&\vdots&\vdots\\ 0&1&0&0&\vdots&0&\vdots&\vdots\\ 0&0&0&1&\vdots&0&\vdots&\vdots\\ 1&0&1&0&\vdots&0&\vdots&\vdots \end{array}\right], \qquad \mathcal{X}^+ = \mathrm{Im}\mathcal{R}^+ = \mathrm{Im}\begin{bmatrix} 0&0&0\\ 0&1&0\\ 0&0&1\\ 1&0&0 \end{bmatrix}$$

The system is <u>not</u> completely reachable and therefore <u>it is possible</u> to give the system a reachability standard form. Using the following matrix:

$$\mathbf{T} = \left[\begin{array}{cc} \mathbf{T}_1 & \mathbf{T}_2 \end{array}\right] = \left[\begin{array}{ccc|c} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{array}\right], \qquad \mathbf{T}^{-1} = \mathbf{T}$$

the transformed system has the following form:

$$\left\{\begin{array}{rcl} \bar{\mathbf{x}}(k+1) &=& \left[\begin{array}{ccc|c} 1&0&0&1\\ 0&0&1&0\\ 0&0&0&0\\ \hline \mathbf{0}&\mathbf{0}&\mathbf{0}&0 \end{array}\right] \bar{\mathbf{x}}(k) + \left[\begin{array}{cc} 1&0\\ 0&0\\ 0&1\\ \hline \mathbf{0}&\mathbf{0} \end{array}\right] u(k) \\ y(k) &=& \left[\begin{array}{ccc|c} 1&1&1&1 \end{array}\right] \bar{\mathbf{x}}(k) \end{array}\right.$$

The zeros shown in bold are the structural zeros of the reachability standard form. The not reachable part is characterized by a zero eigenvalue. The transfer matrix $\mathbf{H}(z)$ of the system is equal to the transfer matrix of the reachable part of the system:

$$\begin{aligned} \mathbf{H}(z) &= \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \mathbf{C}_1(z\mathbf{I}-\mathbf{A}_{11})^{-1}\mathbf{B}_1 \\ &= \mathbf{C}_1 \begin{bmatrix} z-1&0&0\\ 0&z&-1\\ 0&0&z \end{bmatrix}^{-1} \mathbf{B}_1 = \begin{bmatrix} 1&1&1 \end{bmatrix} \begin{bmatrix} (z-1)^{-1}&0&0\\ 0&z^{-1}&z^{-2}\\ 0&0&z^{-1} \end{bmatrix} \mathbf{B}_1 \\ &= \begin{bmatrix} (z-1)^{-1} & z^{-1} & (z^{-1}+z^{-2}) \end{bmatrix} \begin{bmatrix} 1&0\\ 0&0\\ 0&1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{z-1} & \dfrac{z+1}{z^2} \end{bmatrix} \end{aligned}$$