

## 28 Real Numbers

In the previous section we introduced the set of rational numbers. We have seen that integers and fractions are rational numbers. Now, what about decimal numbers? To answer this question we first mention that a decimal number can be terminating, nonterminating and repeating, nonterminating and nonrepeating.

If the number is terminating then we can use properties of powers of 10 to write the number as a rational number. For example,  $-0.123 = -\frac{123}{1000}$  and  $15.34 = \frac{1534}{100}$ . Thus, every terminating decimal number is a rational number. Now, suppose that the decimal number is nonterminating and repeating. For the sake of argument let's take the number  $12.341341 \dots$  where 341 repeats indefinitely. Then  $12.341341 \dots = 12 + 0.341341 \dots$ . Let  $x = 0.341341 \dots$ . Then  $1000x = 341.341341 \dots = 341 + x$ . Thus,  $999x = 341$  and therefore  $x = \frac{341}{999}$ . It follows that

$$12.341341 \dots = 12 + \frac{341}{999} = \frac{12329}{999}$$

Hence, every nonterminating repeating decimal is a rational number.

Next, what about nonterminating and non repeating decimals. Such numbers are not rationals and the collection of all such numbers is called the set of **irrational numbers**. As an example of irrational numbers, let's consider finding the hypotenuse  $c$  of a right triangle where each leg has length 1. Then by the Pythagorean formula we have

$$c^2 = 1^2 + 1^2 = 2.$$

We will show that  $c$  is irrational. Suppose the contrary. That is, suppose that  $c$  is rational so that it can be written as

$$\frac{a}{b} = c$$

where  $a$  and  $b \neq 0$  are integers. Squaring both sides we find  $\frac{a^2}{b^2} = c^2 = 2$  or  $a^2 = 2 \cdot b^2$ . If  $a$  has an even number of prime factors then  $a^2$  has an even number of prime factors. If  $a$  has an odd number of prime factors then  $a^2$  has an even number of prime factors. So,  $a^2$  and  $b^2$  have both even number of prime factors. But  $2 \cdot b^2$  has an odd number of prime factors. So we have that  $a^2$  has an even number and an odd number of prime factors. This

cannot happen by the Fundamental Theorem of Arithmetic which says that every positive integer has a unique number of prime factors. In conclusion,  $c$  cannot be written in the form  $\frac{a}{b}$  so it is not rational. Hence, its decimal form is nonterminating and nonrepeating.

**Remark 28.1**

It follows from the above discussion that the equation  $c^2 = 2$  has no answers in the set of rationals. This is true for the equation  $c^2 = p$  where  $p$  is a prime number. Since there are infinite numbers of primes then the set of irrational numbers is infinite.

We define the set of **real numbers** to be the union of the set of rationals and the set of irrationals. We denote it by the letter  $\mathbb{R}$ . The relationships among the sets of all numbers discussed in this book is summarized in Figure 28.1

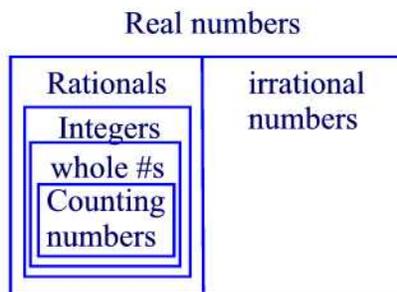


Figure 28.1

Now, in the set of real numbers, the equation  $c^2 = 2$  has a solution denoted by  $\sqrt{2}$ . Thus,  $(\sqrt{2})^2 = 2$ . In general, we say that a positive number  $b$  is the **square root** of a positive number  $a$  if  $b^2 = a$ . We write

$$\sqrt{a} = b.$$

**Representing Irrational Numbers on a Number Line**

Figure 28.2 illustrates how we can accurately plot the length  $\sqrt{2}$  on the number line.

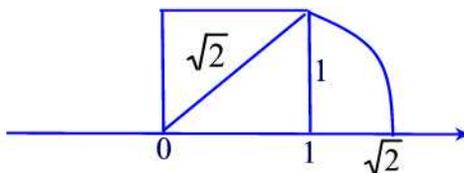


Figure 28.2

## Practice Problems

### Problem 28.1

Given a decimal number, how can you tell whether the number is rational or irrational?

### Problem 28.2

Write each of the following repeating decimal numbers as a fraction.

(a)  $0.\overline{8}$  (b)  $0.\overline{37}$  (c)  $0.\overline{02714}$

### Problem 28.3

Which of the following describe 2.6?

- (a) A whole number
- (b) An integer
- (c) A rational number
- (d) An irrational number
- (e) A real number

### Problem 28.4

Classify the following numbers as rational or irrational.

(a)  $\sqrt{11}$  (b)  $\frac{3}{7}$  (c)  $\pi$  (d)  $\sqrt{16}$

### Problem 28.5

Classify the following numbers as rational or irrational.

(a)  $0.34938661\cdots$  (b)  $0.\overline{26}$  (c)  $0.565665666\cdots$

### Problem 28.6

Match each word in column A with a word in column B.

A	B
Terminating	Rational
Repeating	Irrational
Infinite nonrepeating	

### Problem 28.7

- (a) How many whole numbers are there between  $-3$  and  $3$  (not including  $3$  and  $-3$ )?
- (b) How many integers are there between  $-3$  and  $3$ ?
- (c) How many real numbers are there between  $-3$  and  $3$ ?

**Problem 28.8**

Prove that  $\sqrt{3}$  is irrational.

**Problem 28.9**

Show that  $1 + \sqrt{3}$  is irrational.

**Problem 28.10**

Find an irrational number between  $0.\overline{37}$  and  $0.\overline{38}$

**Problem 28.11**

Write an irrational number whose digits are twos and threes.

**Problem 28.12**

Classify each of the following statement as true or false. If false, give a counter example.

- (a) The sum of any rational number and any irrational number is a rational number.
- (b) The sum of any two irrational numbers is an irrational number.
- (c) The product of any two irrational numbers is an irrational number.
- (d) The difference of any two irrational numbers is an irrational number.

**Arithmetic of Real Numbers**

Addition, subtraction, multiplication, division, and "less than" are defined on the set of real numbers in such a way that all the properties on the rationals still hold. These properties are summarized next.

**Closure:** For any real numbers  $a$  and  $b$ ,  $a + b$  and  $ab$  are unique real numbers.

**Commutative:** For any real numbers  $a$  and  $b$ ,  $a + b = b + a$  and  $ab = ba$ .

**Associative:** For any real numbers,  $a$ ,  $b$ , and  $c$  we have  $a + (b + c) = (a + b) + c$  and  $a(bc) = (ab)c$ .

**Identity element:** For any real numbers  $a$  we have  $a + 0 = a$  and  $a \cdot 1 = a$ .

**Inverse Elements:** For any real numbers  $a$  and  $b \neq 0$  we have  $a + (-a) = 0$  and  $b \cdot \frac{1}{b} = 1$ .

**Distributive:** For any real numbers  $a$ ,  $b$ , and  $c$  we have  $a(b + c) = ab + ac$ .

**Transitivity:** If  $a < b$  and  $b < c$  then  $a < c$ .

**Addition Property:** If  $a < b$  then  $a + c < b + c$ .

**Multiplication Property:** If  $a < b$  then  $ac < bc$  if  $c > 0$  and  $ac > bc$  if

$c < 0$ .

**Density:** If  $a < b$  then  $a < \frac{a+b}{2} < b$ .

## Practice Problems

### Problem 28.13

Construct the lengths  $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \dots$  as follows.

- First construct a right triangle with both legs of length 1. What is the length of the hypotenuse?
- This hypotenuse is the leg of the next right triangle. The other leg has length 1. What is the length of the hypotenuse of this triangle?
- Continue drawing right triangles, using the hypotenuse of the preceding triangle as a leg of the next triangle until you have constructed one with length  $\sqrt{6}$ .

### Problem 28.14

Arrange the following real numbers in increasing order.

$0.56, 0.\overline{56}, 0.5\overline{66}, 0.56565556\dots, 0.\overline{566}$

### Problem 28.15

Which property of real numbers justify the following statement

$$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$$

### Problem 28.16

Find  $x$  :  $x + 2\sqrt{2} = 5\sqrt{2}$ .

### Problem 28.17

Solve the following equation:  $2x - 3\sqrt{6} = 3x + \sqrt{6}$

### Problem 28.18

Solve the inequality:  $\frac{3}{2}x - 2 < \frac{5}{6}x + \frac{1}{3}$

### Problem 28.19

Determine for what real number  $x$ , if any, each of the following is true:

- (a)  $\sqrt{x} = 8$  (b)  $\sqrt{x} = -8$  (c)  $\sqrt{-x} = 8$  (d)  $\sqrt{-x} = -8$

## Radical and Rational Exponents

By rational exponents we mean exponents of the form

$$a^{\frac{m}{n}}$$

where  $a$  is any real number and  $m$  and  $n$  are integers. First we consider the simple case  $a^{\frac{1}{n}}$  (where  $n$  is a positive integer) which is defined as follows

$$a^{\frac{1}{n}} = b \text{ is equivalent to } b^n = a.$$

We call  $b$  the  **$n$ th root** of  $a$  and we write  $b = \sqrt[n]{a}$ . We call  $a$  the **radicand** and  $n$  the **index**. The symbol  $\sqrt[n]{\phantom{a}}$  is called a **radical**. It follows that

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Note that the above definition requires  $a \geq 0$  for  $n$  even since  $b^n$  is always greater than or equal to 0.

**Example 28.1**

Compute each of the following.

(a)  $25^{\frac{1}{2}}$  (b)  $(-8)^{\frac{1}{3}}$  (c)  $(-16)^{\frac{1}{4}}$  (d)  $-16^{\frac{1}{4}}$

**Solution.**

(a) Since  $5^2 = 25$  then  $25^{\frac{1}{2}} = 5$ .

(b) Since  $(-2)^3 = -8$  then  $(-8)^{\frac{1}{3}} = -2$ .

(c) Since the radicand is negative and the index is even then  $(-16)^{\frac{1}{4}}$  is undefined.

(d) Since  $2^4 = 16$  then  $-16^{\frac{1}{4}} = -2$  ■

As the last two parts of the previous example have once again shown, we really need to be careful with parentheses. In this case parenthesis makes the difference between being able to get an answer or not.

For a negative exponent we define

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}}.$$

Now, for a general rational exponent we define

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

That is,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

Now, if  $b = (a^{\frac{1}{n}})^m$  then  $b^n = ((a^{\frac{1}{n}})^m)^n = (a^{\frac{1}{n}})^{mn} = ((a^{\frac{1}{n}})^n)^m = (a^{\frac{n}{n}})^m = a^m$ . This shows that  $b = (a^m)^{\frac{1}{n}}$ . Hence, we have established that

$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$$

or

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

**Example 28.2**

Rewrite each expression using rational exponents.

(a)  $\sqrt[4]{5xy}$    (b)  $\sqrt[3]{4a^2b^5}$

**Solution.**

(a)  $\sqrt[4]{5xy} = (5xy)^{\frac{1}{4}}$

(b)  $\sqrt[3]{4a^2b^5} = (4a^2b^5)^{\frac{1}{3}}$  ■

**Example 28.3**

Express the following values without exponents.

(a)  $9^{\frac{3}{2}}$    (b)  $125^{-\frac{4}{3}}$ .

**Solution.**

(a)  $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27.$

(b)  $125^{-\frac{4}{3}} = (125^{\frac{1}{3}})^{-4} = 5^{-4} = \frac{1}{625}.$  ■

The properties of integer exponents also hold for rational exponents. These properties are equivalent to the corresponding properties of radicals if the expressions involving radicals are meaningful.

**Theorem 28.1**

Let  $a$  and  $b$  real numbers and  $n$  a nonzero integer. Then

$$\begin{aligned} (ab)^{\frac{1}{n}} &= a^{\frac{1}{n}}b^{\frac{1}{n}} & \sqrt[n]{ab} &= \sqrt[n]{a}\sqrt[n]{b} \\ \left(\frac{a}{b}\right)^{\frac{1}{n}} &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} & \sqrt[n]{\frac{a}{b}} &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \end{aligned}$$

**Practice Problems****Problem 28.20**

Express the following values without exponents.

(a)  $25^{\frac{1}{2}}$    (b)  $32^{\frac{1}{5}}$    (c)  $9^{\frac{5}{2}}$    (d)  $(-27)^{\frac{4}{3}}$

**Problem 28.21**

Write the following radicals in simplest form if they are real numbers.

(a)  $\sqrt[3]{-27}$    (b)  $\sqrt[4]{-16}$    (c)  $\sqrt[5]{32}$

**Problem 28.22**

A student uses the formula  $\sqrt{a}\sqrt{b} = \sqrt{ab}$  to show that  $-1 = 1$  as follows:

$$-1 = (\sqrt{-1})^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

What's wrong with this argument?

**Problem 28.23**

Give an example where the following statement is true:  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

**Problem 28.24**

Give an example where the following statement is false:  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ .

**Problem 28.25**

Find an example where the following statement is false:  $\sqrt{a^2 + b^2} = a + b$ .

**Problem 28.26**

Express the following values without using exponents:

(a)  $(3^{10})^{\frac{3}{5}}$  (b)  $81^{-\frac{5}{4}}$

**Problem 28.27**

If possible, find the square root of each of the following without using a calculator.

(a) 225 (b) 169 (c) -81 (d) 625

**Problem 28.28**

Write each of the following in the form  $a\sqrt{b}$  where  $a$  and  $b$  are integers and  $b$  has the least value possible.

(a)  $\sqrt{180}$  (b)  $\sqrt{363}$  (c)  $\sqrt{252}$