The Real Numbers (Theorems)

Original Notes adopted from November 6, 2001 (Week9)

Prove $\sqrt{2} + \sqrt{3}$ is irrational

Suppose $\sqrt{2} + \sqrt{3}$ rational

$\sqrt{2} + \sqrt{3} = m/n$

$\sqrt{2} = m/n - \sqrt{3}$

$2 = m^2/n^2 - 2\, m/n \sqrt{3} + 3$

$2m/n \sqrt{3} = m^2/n^2 + 1$

$\sqrt{3} = n/2m\, (m^2/n^2 + 1)$

But $\sqrt{3}$ irrational. $\therefore$ Contradiction.

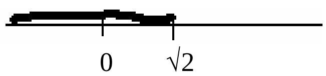
Extra problem : Is $\sqrt{2} + \sqrt{3} + \sqrt{5}$ rational?

Several ways of constructing real numbers from the rationals. We'll use **"Dedekind cuts".**

Q = set of rational numbers

Each real number will be a set of rational numbers.

-It's like the set of rationals less than the number.



Eg. $\sqrt{2} = \{ p \in Q: p < 0 \text{ or } p^2 < 2 \}$

Definition: A real number x is a subset of Q such that:
1) $x \neq \emptyset$, $x \neq Q$ (set of all Q)
2) if $p \in x$ & $s \in Q$ with $s < p$, then $s \in x$.
3) There is no largest number x.

Eg. $3 = \{ p \in Q: p < 3 \}$

To each rational number m/n, associate the set $\{p \in Q: p < m/n \} = m/n$ $x \le y$ if $x \subset y$ $x < y$ if $x \subset y$ & $x \neq y$.

$\sqrt{2} = \{ p \in Q: p < 0 \text{ or } p^2 < 2 \}$

Is $\sqrt{2}$ a real number?

$\sqrt{2} \neq \emptyset$ (eg. $1 \in \sqrt{2}$)

$\sqrt{2} \neq Q$ (eg. $7 \notin \sqrt{2}$)

Show $p \in \sqrt{2}$ & $q < p \Rightarrow q \in \sqrt{2}$ Two Cases:

1) if $p \le 0$
   if $q < p$, then $q < 0$, so $q \in \sqrt{2}$

2) if $p > 0$ & $p \in \sqrt{2}$, & if $q < p$,
   $p^2 < 2$,
   if $q < 0$, $q \in \sqrt{2}$
   if $q > 0$, $0 \le q < p \Rightarrow q^2 < p^2$
   $\therefore q^2 < 2$, so $q \in \sqrt{2}$

Must show no largest element of $\sqrt{2}$.

Suppose $p \in \sqrt{2}$, show there exists $q \in \sqrt{2}$ with $q > p$.

If $p \le 0$, take $q = 1$.

if $p > 0$, $p \in \sqrt{2} \Rightarrow p^2 < 2$

$(p + 1/n)^2 = p^2 + 2/n\,(p) + 1/n^2$. To show some $q \in \sqrt{2}$ that is bigger than p.

Want n such that $p^2 + 2/n\,(p) + 1/n^2 < 2$ or $2/n\,(p) + 1/n^2 < 2 - p^2$

Choose $n \in N$ large enough that

$$2/n\,(p) + 1/n^2 < 2 - p^2$$

Let $q = p + 1/n$. Then $q \in \sqrt{2}$ & $q > p$

Let R denote the set of all real numbers.
Definition: If $x,y \in$ R, we define $x + y = \{ p + q: p \in x, q \in y\}$

$\boxed{\text{Theorem: If } x, y \in R\text{, then } (x + y) \in R.}$
Proof: Must show x + y satisfies the three properties
$x + y \neq 0$, We can choose $p \in x, q \in y$, then $p + q \in x + y$.
To show $x + y \neq Q$, pick some $n \notin x$ & some $m \notin y$ $(n,m \in Q)$
Claim: $n + m \notin x + y$.
Note: If $n \notin x, n \in Q$, then $p \in x \Rightarrow p < n$, for if $p \geq n, p \in x$ would imply $n \in x$ (since x has property 2).
$\therefore p \in x \Rightarrow p < n$
$q \in y \Rightarrow q < m$
$\therefore$ For every $p \in x$ & $q \in y$, $p + q < n + m$
$\therefore n + m \notin x + y$.
Given $p \in x + y$, must show that $q < p, q \in Q \Rightarrow q \in x + y$
$p = r + s$, some $r \in x$, $s + y$.
$q < r + s$ $\qquad \therefore q - r = t$, some $t \in y$
$q - r < s$ $\qquad q = r + t, r \in x, t \in y$ $\therefore q \in x + y$.

Must show : x + y has no largest element.
Suppose $p + q \in x + y$, with $p \in x, q \in y$ since x & y are real numbers.
There exists $s \in x$ with $s > p$ & a $t \in y$ with $t > q$, then $s + t \in x + y$, & $s + t > p + q$.
$\therefore x + y$ is a real number

Define $\underline{0} = \{ p \in Q: p < 0 \}$ Clearly $\underline{0} \in$ R.
Theorem: $\underline{0} + x = x \quad \forall x \in$ R.

Proof: First, to show $\underline{0} + x \subset x$
$\underline{0} + x = \{ p + q : p < 0 + q \in x \}$
If $p + q \in \underline{0} + x$, $p + q < q, q \in x \Rightarrow p + q \in x$
$\therefore 0 + x \subset x$.

To show: $x \subset \underline{0} + x$
Suppose $r \in x$, then there exists $s \in x$ with $s > r$.
$s + (r - s) = q + p$ with $q \in x, p \in \underline{0}$

$\therefore r = p + q \in \underline{0} + x$.
$x < y$ if $x \subset y$ & $x \neq y$.

Definition: x is positive if $\underline{0} < x$.
For $x, y \geq \underline{0}$, we define $xy = \{ p \in Q: p \leq 0$ or $p = st$ with $s > 0$ & $s \in x, t > 0$ & $t \in y \}$ Must show: xy has the properties of a real number.
------0--------x**----------negatives-------**
For x a real number,
$-x =\{ - p: p \in Q, p \notin x\}$ taking away a largest element if there is one.
$\underline{2} = \{ q : q < 2\}$
$\boxed{\text{Theorem: For every } x, x + (-x) = \underline{0}}$ Define $x * \underline{0} = \underline{0} * x = \underline{0} \; \forall x$
Define $|x| = \{ x$ if $x \geq \underline{0}$
$\{-x$ if $x < \underline{0}$
Define $xy = \{ |x||y|$ if $x \geq \underline{0}$ $y \geq \underline{0}$
$\{$ or if $x \leq 0, y \leq 0$
$\{-(|x||y|)$ if $x \geq \underline{0}$ $y \leq \underline{0}$
$\{$ or $x \leq 0$ & $y > \underline{0}$

$\underline{1} = \{ p \in Q: p < 1\}$
$\boxed{\text{Theorem : } \underline{1} * x = x \; \forall x}$
Definition: If $x > \underline{0}$, define $1/x$ ---0---1/x--------x---------

$1/x = \{ p \in Q : p \leq 0$ or $p = 1/q$ with $q \in Q, q \notin x \}$ take away biggest element.
Definition: If $x \leq \underline{0}$. Define $1/x = - 1/|x|$
Theorem: For all $x \neq \underline{0}$, $x * 1/x = 1$.
Prove that addition and multiplication are associative and commutative.
Prove distributive law: $x (y + z) = xy + xz$
For any set S of real numbers, $b \in R$ is an **upperbound** for S if $x \leq b, \forall x \in S$.
If each member of S is a subset of Q, $b \subset Q$, $x \subset b \; \forall x \in S$, so b contains the union of all sets in S.
Completeness Property of R: Every subset of R other than $\varnothing$ that has an upper bound has a lub.
Proof: Suppose $S \neq \varnothing$, b is an upperbound for S. Let c be the union of all the sets of rational numbers that are in S. $c \subset Q$. Claim: c is the least upperbound of S.
Must Show:
1) c is a real number
2) c is an upperbound of S
3) If b any upperbound of S, $b \geq c$.
1) $S \neq \varnothing \Rightarrow c \neq \varnothing$.
b an upperbound for $S \Rightarrow$ every member of s is contained b.
Pick $r \in Q$ such that $r \notin b$ ($b \in R$)
Then $r \notin c$, for if $r \in c$, $r \in x$ some $x \in S$.
(Then every rational less than r would be in x) (b wouldn't contain x) $\therefore$ $c \neq Q$.

If $p \in c$ & $q < p$, $p \in x$ some $x \in R$, $q \subset p \Rightarrow q \in x$ ($x \in R$) $\therefore$ $q \in c$.
Any largest element of c would be a largest element of some $x \in S$. $\therefore$ $c \in R$.
2) If $x \in S$, $x \subset c$, so $x \leq c$.
3) If b any upperbound of S, $b \supset x, \forall x \in S$
$\Rightarrow b \supset$ union of x's = c.