

## 4.1 SIGMA NOTATION AND RIEMANN SUMS

One strategy for calculating the area of a region is to cut the region into simple shapes, calculate the area of each simple shape, and then add these smaller areas together to get the area of the whole region. We will use that approach, but it is useful to have a notation for adding a lot of values together: the sigma ( $\Sigma$ ) notation.

Summation	Sigma notation	A way to read the sigma notation
$1^2 + 2^2 + 3^2 + 4^2 + 5^2$	$\sum_{k=1}^5 k^2$	the sum of $k$ squared for $k$ equals 1 to $k$ equals 5
$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$	$\sum_{k=3}^7 \frac{1}{k}$	the sum of 1 divided by $k$ for $k$ equals 3 to $k$ equals 7
$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$	$\sum_{j=0}^5 2^j$	the sum of 2 to the $j^{\text{th}}$ power for $j$ equals 0 to $j$ equals 5
$a_2 + a_3 + a_4 + a_5 + a_6 + a_7$	$\sum_{i=2}^7 a_i$	the sum of $a$ sub $i$ from $i$ equals 2 to $i$ equals 7

The variable (typically  $i$ ,  $j$ , or  $k$ ) used in the summation is called the **counter** or **index variable**.

The function to the right of the sigma is called the **summand**, and the numbers below and above the sigma are called the **lower and upper limits of the summation**. (Fig. 1)

**Practice 1:** Write the summation denoted by each of the following:

(a)  $\sum_{k=1}^5 k^3$ , (b)  $\sum_{j=2}^7 (-1)^j \frac{1}{j}$ , (c)  $\sum_{m=0}^4 (2m+1)$ .

In practice, the sigma notation is frequently used with the standard function notation:

$$\sum_{k=1}^3 f(k+2) = f(1+2) + f(2+2) + f(3+2) = f(3) + f(4) + f(5) \text{ and}$$

$$\sum_{i=1}^4 f(x_i) = f(x_1) + f(x_2) + f(x_3) + f(x_4)$$

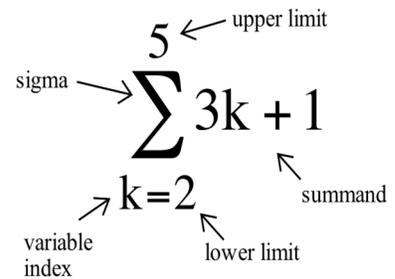


Fig. 1

x	f(x)	g(x)	h(x)
1	2	4	3
2	3	1	3
3	1	-2	3
4	0	3	3
5	3	5	3

Table 1

**Example 1:** Use the values in Table 1 to evaluate  $\sum_{k=2}^5 2 \cdot f(k)$  and  $\sum_{j=3}^5 (5 + f(j-2))$ .

**Solution:**  $\sum_{k=2}^5 2 \cdot f(k) = 2 \cdot f(2) + 2 \cdot f(3) + 2 \cdot f(4) + 2 \cdot f(5) = 2 \cdot (3) + 2 \cdot (1) + 2 \cdot (0) + 2 \cdot (3) = 14$

$$\begin{aligned} \sum_{j=3}^5 (5 + f(j-2)) &= (5 + f(3-2)) + (5 + f(4-2)) + (5 + f(5-2)) = (5 + f(1)) + (5 + f(2)) + (5 + f(3)) \\ &= (5+2) + (5+3) + (5+1) = 21. \end{aligned}$$

**Practice 2:** Use the values of f, g and h in Table 1 to evaluate the following:

(a)  $\sum_{k=2}^5 g(k)$       (b)  $\sum_{j=1}^4 h(j)$       (c)  $\sum_{k=3}^5 (g(k) + f(k-1))$

**Example 2:** For  $f(x) = x^2 + 1$ , evaluate  $\sum_{k=0}^3 f(k)$ .

**Solution:**  $\sum_{k=0}^3 f(k) = f(0) + f(1) + f(2) + f(3)$   
 $= (0^2 + 1) + (1^2 + 1) + (2^2 + 1) + (3^2 + 1) = 1 + 2 + 5 + 10 = 18.$

**Practice 3:** For  $g(x) = 1/x$ , evaluate  $\sum_{k=2}^4 g(k)$  and  $\sum_{k=1}^3 g(k+1)$ .

The summand does not have to contain the index variable explicitly: a sum from  $k=2$  to  $k=4$  of the constant function  $f(k) = 5$  can be written as

$$\sum_{k=2}^4 f(k) \quad \text{or} \quad \sum_{k=2}^4 5 = 5 + 5 + 5 = 3 \cdot 5 = 15. \quad \text{Similarly,} \quad \sum_{k=3}^7 2 = 2 + 2 + 2 + 2 + 2 = 5 \cdot 2 = 10.$$

Since the sigma notation is simply a notation for addition, it has all of the familiar properties of addition.

### Summation Properties

Sum of Constants:  $\sum_{k=1}^n C = C + C + C + \dots + C$  (n terms) =  $n \cdot C$

Addition:  $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$

Subtraction:  $\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$

Constant Multiple:  $\sum_{k=1}^n C \cdot a_k = C \cdot \sum_{k=1}^n a_k$

Problems 16 and 17 illustrate that similar patterns for sums of products and quotients are **not valid**.

**Sums of Areas of Rectangles**

In Section 4.2 we will approximate the areas under curves by building rectangles as high as the curve, calculating the area of each rectangle, and then adding the rectangular areas together.

**Example 3:** Evaluate the sum of the rectangular areas in Fig. 2, and write the sum using the sigma notation.

Solution:

$$\begin{aligned} \{ \text{sum of the rectangular areas} \} &= \{ \text{sum of (base) \cdot (height) for each rectangle} \} \\ &= (1) \cdot (1/3) + (1) \cdot (1/4) + (1) \cdot (1/5) = 47/60. \end{aligned}$$

Using the sigma notation,

$$(1) \cdot (1/3) + (1) \cdot (1/4) + (1) \cdot (1/5) = \sum_{k=3}^5 \frac{1}{k} .$$

**Practice 4:** Evaluate the sum of the rectangular areas in Fig. 3, and write the sum using the sigma notation.

The bases of the rectangles do not have to be equal. For the rectangular areas in Fig. 4 ,

rectangle	base	height	area
1	3-1=2	f(2)=4	2·4 = 8
2	4-3=1	f(4)=16	1·16 = 16
3	6-4=2	f(5)=25	2·25 = 50

so the sum of the rectangular areas is  $8 + 16 + 50 = 74$ .

**Example 4:** Write the sum of the areas of the rectangles in Fig. 5 using the sigma notation.

Solution: The area of each rectangle is (base)·(height).

rectangle	base	height	area
1	$x_1 - x_0$	$f(x_1)$	$(x_1 - x_0) \cdot f(x_1)$
2	$x_2 - x_1$	$f(x_2)$	$(x_2 - x_1) \cdot f(x_2)$
3	$x_3 - x_2$	$f(x_3)$	$(x_3 - x_2) \cdot f(x_3)$

The area of the  $k^{\text{th}}$  rectangle is  $(x_k - x_{k-1}) \cdot f(x_k)$  , and the total area of the

rectangles is the sum 
$$\sum_{k=1}^3 (x_k - x_{k-1}) \cdot f(x_k) .$$

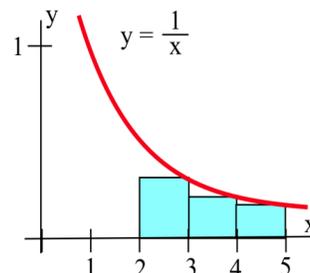


Fig. 2

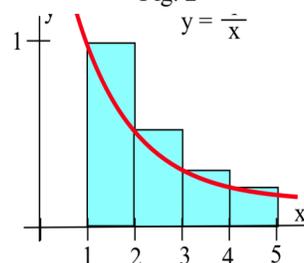


Fig. 3

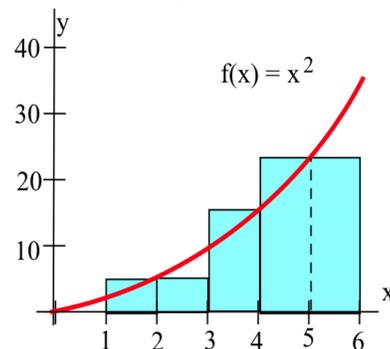


Fig. 4

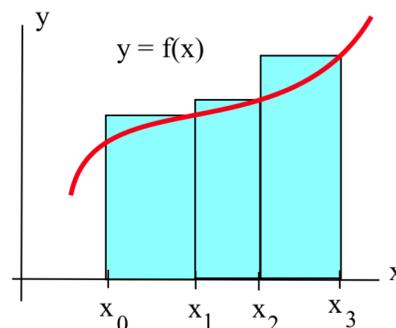


Fig. 5

**Practice 5:** Write the sum of the areas of the shaded rectangles in Fig. 6 using the sigma notation.

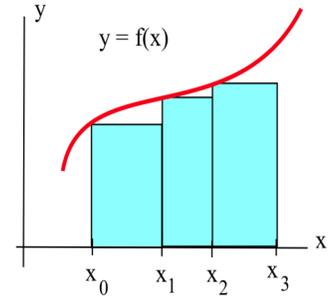


Fig. 6

**Area Under A Curve — Riemann Sums**

Suppose we want to calculate the area between the graph of a positive function  $f$  and the interval  $[a, b]$  on the  $x$ -axis (Fig. 7). The

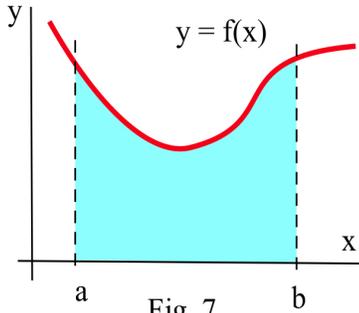


Fig. 7

Riemann Sum method is to build several rectangles with bases on the interval  $[a, b]$  and sides that reach up to the graph of  $f$  (Fig. 8). Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of  $f$  on  $[a, b]$ .

The area of the region formed by the rectangles is an **approximation** of the area we want.

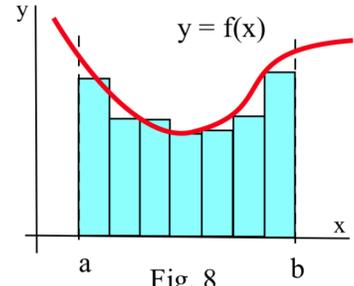


Fig. 8

**Example 5:** Approximate the area in Fig. 9a between the graph of  $f$  and the interval  $[2, 5]$  on the  $x$ -axis by summing the areas of the rectangles in Fig. 9b.

**Solution:** The total area of rectangles is  $(2)(3) + (1)(5) = 11$  square units.

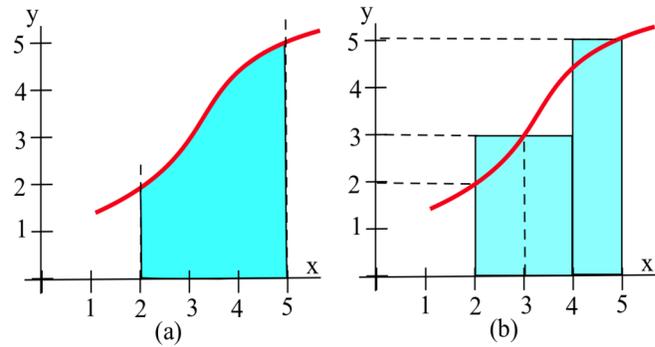


Fig. 9

In order to effectively describe this process, some new vocabulary is helpful: a **partition** of an interval and the **mesh** of the partition.

A **partition**  $P$  of a closed interval  $[a, b]$  into  $n$  subintervals is a set of  $n+1$  points  $\{x_0 = a, x_1, x_2, x_3, \dots, x_{n-1}, x_n = b\}$  in increasing order,  $a = x_0 < x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n = b$ .

(A partition is a collection of points on the axis and it does not depend on the function in any way.)

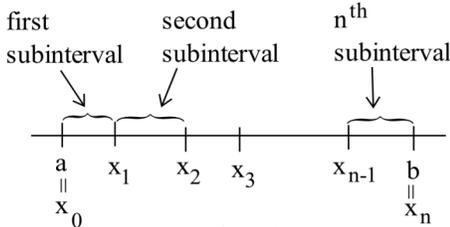


Fig. 10

The points of the partition  $P$  divide the interval into  $n$  subintervals (Fig. 10):  $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots,$  and  $[x_{n-1}, x_n]$  with lengths  $\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \Delta x_3 = x_3 - x_2, \dots,$  and  $\Delta x_n = x_n - x_{n-1}$ . The points  $x_k$  of the partition  $P$  are the locations of the vertical lines for the sides of the rectangles, and the bases of the rectangles have lengths  $\Delta x_k$  for  $k = 1, 2, 3, \dots, n$ .

The **mesh** or **norm** of partition P is the length of the longest of the subintervals  $[x_{k-1}, x_k]$ , or, equivalently, the maximum of  $\Delta x_k$  for  $k = 1, 2, 3, \dots, n$ .

For example, the set  $P = \{2, 3, 4.6, 5.1, 6\}$  is a partition of the interval  $[2,6]$  (Fig. 11) and divides the interval  $[2,6]$  into 4 subintervals with lengths  $\Delta x_1 = 1$ ,  $\Delta x_2 = 1.6$ ,  $\Delta x_3 = .5$  and  $\Delta x_4 = .9$ . The mesh of this partition is 1.6, the maximum of the lengths of the subintervals. (If the mesh of a partition is "small," then the length of each one of the subintervals is the same or smaller.)

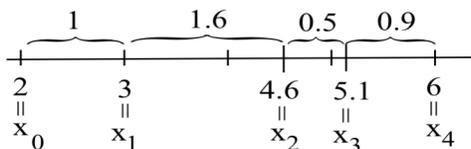


Fig. 11

**Practice 6:**  $P = \{3, 3.8, 4.8, 5.3, 6.5, 7, 8\}$  is a partition of what interval? How many subintervals does it create? What is the mesh of the partition? What are the values of  $x_2$  and  $\Delta x_2$ ?

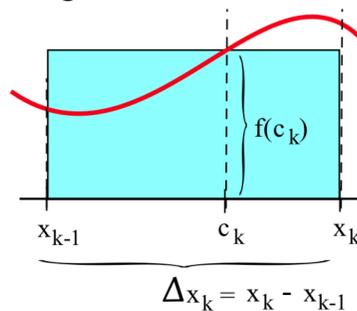


Fig. 12

A function, a partition, and a point in each subinterval determine a Riemann sum.

Suppose  $f$  is a positive function on the interval  $[a,b]$

$P = \{x_0 = a, x_1, x_2, x_3, \dots, x_{n-1}, x_n = b\}$  is a partition of  $[a,b]$

$c_k$  is an  $x$ -value in the  $k^{\text{th}}$  subinterval  $[x_{k-1}, x_k]$ :  $x_{k-1} \leq c_k \leq x_k$ .

Then the area of the  $k^{\text{th}}$  rectangle is  $f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$ . (Fig. 12)

Definition: A summation of the form 
$$\sum_{k=1}^n f(c_k) \cdot \Delta x_k$$
 is called a **Riemann Sum** of  $f$  for the partition  $P$ .

This Riemann sum is the total of the areas of the rectangular regions and is an approximation of the area between the graph of  $f$  and the  $x$ -axis.

**Example 6:** Find the Riemann sum for  $f(x) = 1/x$  and the partition  $\{1, 4, 5\}$  using the values  $c_1 = 2$  and  $c_2 = 5$ . (Fig. 13)

Solution: The 2 subintervals are  $[1,4]$  and  $[4,5]$  so  $\Delta x_1 = 3$  and  $\Delta x_2 = 1$ .

Then the Riemann sum for this partition is

$$\begin{aligned} \sum_{k=1}^2 f(c_k) \cdot \Delta x_k &= \sum_{k=1}^2 f(c_k) \cdot \Delta x_k = f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 = f(2) \cdot (3) + f(5) \cdot (1) \\ &= \frac{1}{2} (3) + \frac{1}{5} (1) = 1.7 . \end{aligned}$$

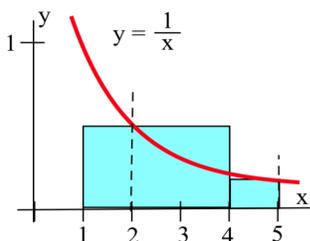


Fig. 13

**Practice 7:** Calculate the Riemann sum for  $f(x) = 1/x$  on the partition  $\{1, 4, 5\}$  using the values  $c_1 = 3, c_2 = 4$ .

**Practice 8:** What is the smallest value a Riemann sum for  $f(x) = 1/x$  and the partition  $\{1, 4, 5\}$  can have? (You will need to select values for  $c_1$  and  $c_2$ .) What is the largest value a Riemann sum can have for this function and partition?

Table 2 shows the results of a computer program that calculated Riemann sums for the function  $f(x) = 1/x$  with different numbers of subintervals and different ways of selecting the points  $c_i$  in each subinterval.

When the mesh of the partition is small (and the number of subintervals large), all of the ways of selecting the  $c_i$  lead to approximately the same number for the Riemann sums. For this decreasing function, using the left endpoint of the subinterval always resulted in a sum that was larger than the area. Choosing the right end point gave a value smaller than the area. Why?

Table 2: Riemann sums for  $f(x) = 1/x$  on the interval  $[1,5]$

number of subintervals	mesh	Values of the Riemann sum for different choices of $c_k$		
		$c_k = \text{left edge} = x_{k-1}$	$c_k = \text{"random" point in } [x_{k-1}, x_k]$	$c_k = \text{right edge} = x_k$
4	1	2.083333	1.473523	1.283333
8	.5	1.828968	1.633204	1.428968
16	.25	1.714406	1.577806	1.514406
40	.1	1.650237	1.606364	1.570237
400	.01	1.613446	1.609221	1.605446
4000	.001	1.609838	1.609436	1.609038

As the mesh gets smaller, all of the Riemann Sums seem to be approaching the same value, approximately 1.609. ( $\ln 5 = 1.609437912$ )

**Example 7:** Find the Riemann sum for the function  $f(x) = \sin(x)$  on the interval  $[0, \pi]$  using the partition  $\{0, \pi/4, \pi/2, \pi\}$  with  $c_1 = \pi/4, c_2 = \pi/2, c_3 = 3\pi/4$ .

Solution: The 3 subintervals (Fig. 14) are  $[0, \pi/4], [\pi/4, \pi/2],$  and  $[\pi/2, \pi]$  so  $\Delta x_1 = \pi/4, \Delta x_2 = \pi/4$  and  $\Delta x_3 = \pi/2$ . The Riemann sum for this partition is

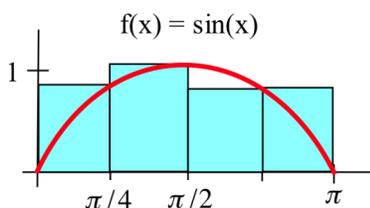


Fig. 14

$$\begin{aligned} \sum_{k=1}^3 f(c_k) \cdot \Delta x_k &= \sin(\pi/4) \cdot (\pi/4) + \sin(\pi/2) \cdot (\pi/4) + \sin(3\pi/4) \cdot (\pi/2) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} + 1 \cdot \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{2} \approx 2.45148. \end{aligned}$$

**Practice 9:** Find the Riemann sum for the function and partition in the previous example, but use  $c_1 = 0, c_2 = \pi/2, c_3 = \pi/2$ .

**Two Special Riemann Sums: Lower & Upper Sums**

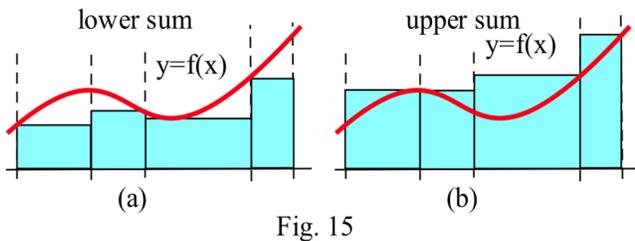
Two particular Riemann sums are of special interest because they represent the extreme possibilities for Riemann sums for a given partition.

Definition: Suppose  $f$  is a positive function on  $[a,b]$ , and  $P$  is a partition of  $[a,b]$ .  
 Let  $\mathbf{m}_k$  be the  $x$ -value in the  $k$ th subinterval so that  $f(\mathbf{m}_k)$  is the minimum value of  $f$  in that interval, and let  $\mathbf{M}_k$  be the  $x$ -value in the  $k$ th subinterval so that  $f(\mathbf{M}_k)$  is the maximum value of  $f$  in that interval.

LS<sub>P</sub>:  $\sum_{k=1}^n f(\mathbf{m}_k) \cdot \Delta x_k$  is the **lower sum** of  $f$  for the partition  $P$ .

US<sub>P</sub>:  $\sum_{k=1}^n f(\mathbf{M}_k) \cdot \Delta x_k$  is the **upper sum** of  $f$  for the partition  $P$ .

Geometrically, the lower sum comes from building rectangles under the graph of  $f$  (Fig. 15a), and the lower sum (every lower sum) is less than or equal to the exact area  $A$ :  $LS_P \leq A$  for every partition  $P$ . The upper sum comes from building rectangles over the graph of  $f$  (Fig. 15b), and the upper sum (every upper sum) is greater than or equal to the exact area  $A$ :



$US_P \geq A$  for every partition  $P$ . The lower and upper sums provide bounds on the size of the exact area:

$$LS_P \leq A \leq US_P.$$

For any  $c_k$  value in the  $k$ th subinterval,

$f(\mathbf{m}_k) \leq f(c_k) \leq f(\mathbf{M}_k)$ ; so, for any choice of the  $c_k$  values, the Riemann sum  $RS_P = \sum_{k=1}^n f(c_k) \cdot \Delta x_k$  satisfies

$$\sum_{k=1}^n f(\mathbf{m}_k) \cdot \Delta x_k \leq \sum_{k=1}^n f(c_k) \cdot \Delta x_k \leq \sum_{k=1}^n f(\mathbf{M}_k) \cdot \Delta x_k \text{ or, equivalently, } LS_P \leq RS_P \leq US_P.$$

The lower and upper sums provide bounds on the size of all Riemann sums.

The exact area  $A$  and every Riemann sum  $RS_P$  for partition  $P$  both lie between the lower sum and the upper sum for  $P$  (Fig. 16). Therefore, if the lower and upper sums are close together then the area and any Riemann sum for  $P$  must also be close together. If we know that the upper and lower sums for a partition  $P$  are within 0.001 units of each other, then we can be sure that every Riemann sum for partition  $P$  is within 0.001 units of the exact area.

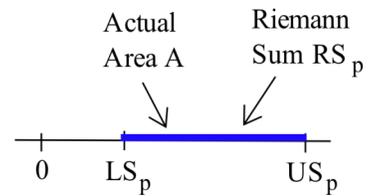


Fig. 16

$A$  and  $RS_P$  are both between  $LS_P$  and  $US_P$

Unfortunately, finding minimums and maximums can be a time-consuming business, and it is usually not practical to determine lower and upper sums for "wiggly" functions. If  $f$  is monotonic, however, then it is easy to find the values for  $m_k$  and  $M_k$ , and sometimes we can explicitly calculate the limits of the lower and upper sums.

For a monotonic bounded function we can guarantee that a Riemann sum is within a certain distance of the exact value of the area it is approximating.

Theorem: If  $f$  is a positive, **monotonically** increasing, bounded function on  $[a,b]$ ,  
 then for any partition  $P$  and any Riemann sum for  $P$ ,

$$\left\{ \begin{array}{l} \text{distance between the} \\ \text{Riemann sum and} \\ \text{the exact area} \end{array} \right\} \leq \left\{ \begin{array}{l} \text{distance between the} \\ \text{upper sum and} \\ \text{the lower sum} \end{array} \right\} \leq \{f(b) - f(a)\} \cdot (\text{mesh of } P).$$

Proof: The Riemann sum and the exact area are both between the upper and lower sums so the distance between the Riemann sum and the exact area is less than or equal to the distance between the upper and lower sums. Since  $f$  is monotonically increasing, the areas representing the difference of the

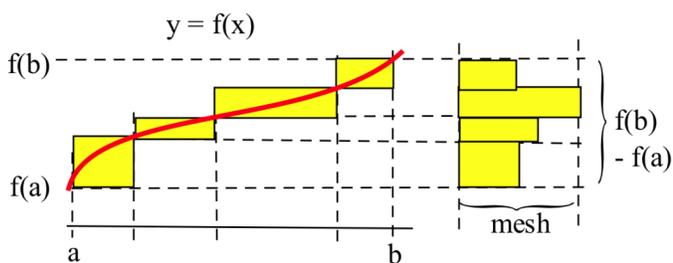


Fig. 17

upper and lower sums can be slid into a rectangle (Fig. 17) whose height equals  $f(b) - f(a)$  and whose base equals the mesh of  $P$ . Then the total difference of the upper and lower sums is less than or equal to the area of the rectangle,  $\{f(b) - f(a)\} \cdot (\text{mesh of } P)$ .

### PROBLEMS

In problems 1 – 6, rewrite the sigma notation as a summation and perform the indicated addition.

1.  $\sum_{k=2}^4 k^2$

2.  $\sum_{j=1}^5 (1 + j)$

3.  $\sum_{n=1}^3 (1 + n)^2$

4.  $\sum_{k=0}^5 \sin(\pi k)$

5.  $\sum_{j=0}^5 \cos(\pi j)$

6.  $\sum_{k=1}^3 1/k$

In problems 7 – 12, rewrite each summation using the sigma notation. Do not evaluate the sums.

7.  $3 + 4 + 5 + \dots + 93 + 94$

8.  $4 + 6 + 8 + \dots + 24$

9.  $9 + 16 + 25 + 36 + \dots + 144$

10.  $\frac{3}{4} + \frac{3}{9} + \frac{3}{16} + \dots + \frac{3}{100}$

11.  $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \dots + 7 \cdot 2^7$

12.  $3 + 6 + 9 + \dots + 30$

In problems 13 – 15 , use the values of  $a_k$  and  $b_k$  in Table 3 and verify that the value in part (a) does equal the value in part (b).

Table 3	$k$	$a_k$	$b_k$
	1	1	2
	2	2	2
	3	3	2

$$13. \text{ (a) } \sum_{k=1}^3 (a_k + b_k) \quad \text{(b) } \sum_{k=1}^3 a_k + \sum_{k=1}^3 b_k \quad 14. \text{ (a) } \sum_{k=1}^3 (a_k - b_k) \quad \text{(b) } \sum_{k=1}^3 a_k - \sum_{k=1}^3 b_k$$

$$15. \text{ (a) } \sum_{k=1}^3 5a_k \quad \text{(b) } 5 \cdot \sum_{k=1}^3 a_k$$

For problems 16 – 18 , use the values of  $a_k$  and  $b_k$  in Table 3 and verify that the value in part (a) does **not** equal the value in part (b).

$$16. \text{ (a) } \sum_{k=1}^3 a_k \cdot b_k \quad \text{(b) } \sum_{k=1}^3 a_k \cdot \sum_{k=1}^3 b_k \quad 17. \text{ (a) } \sum_{k=1}^3 (a_k^2) \quad \text{(b) } \left( \sum_{k=1}^3 a_k \right)^2$$

$$18. \text{ (a) } \sum_{k=1}^3 a_k/b_k \quad \text{(b) } \left( \sum_{k=1}^3 a_k \right) / \left( \sum_{k=1}^3 b_k \right)$$

For problems 19 – 30 ,  $f(x) = x^2$  ,  $g(x) = 3x$  , and  $h(x) = 2/x$  . Evaluate each sum.

$$19. \sum_{k=0}^3 f(k)$$

$$20. \sum_{k=0}^3 f(2k)$$

$$21. \sum_{j=0}^3 2f(j)$$

$$22. \sum_{i=0}^3 f(1+i)$$

$$23. \sum_{m=1}^3 g(m)$$

$$24. \sum_{k=1}^3 g(f(k))$$

$$25. \sum_{j=1}^3 g^2(j)$$

$$26. \sum_{k=1}^3 k \cdot g(k)$$

$$27. \sum_{k=2}^4 h(k)$$

$$28. \sum_{i=1}^4 h(3i)$$

$$29. \sum_{n=1}^3 f(n) \cdot h(n)$$

$$30. \sum_{k=1}^7 g(k) \cdot h(k)$$

For problems 31 – 36 , write out each summation and simplify the result. These are examples of "telescoping sums".

$$31. \sum_{k=1}^7 ((k)^2 - (k-1)^2)$$

$$32. \sum_{k=1}^6 ((k)^3 - (k-1)^3)$$

$$33. \sum_{k=1}^5 \left( \frac{1}{k} - \frac{1}{k+1} \right)$$

$$34. \sum_{k=0}^4 ((k+1)^3 - (k)^3)$$

$$35. \sum_{k=0}^8 (\sqrt{k+1} - \sqrt{k})$$

$$36. \sum_{k=1}^5 (x_k - x_{k-1})$$

For problems 37–43, (i) list the subintervals determined by the partition  $P$ , (ii) find the values of  $\Delta x_i$ ,

(iii) find the mesh of  $P$ , and (iv) calculate  $\sum_{i=1}^n \Delta x_i$ .

37.  $P = \{ 2, 3, 4.5, 6, 7 \}$

38.  $P = \{ 3, 3.6, 4, 4.2, 5, 5.5, 6 \}$

39.  $P = \{ -3, -1, 0, 1.5, 2 \}$

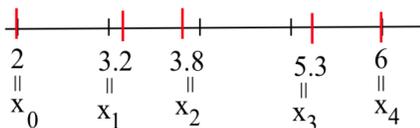
40.  $P$  is given in Fig. 18.41.  $P$  is given in Fig. 19.42.  $P$  is given in Fig. 20.

Fig. 18

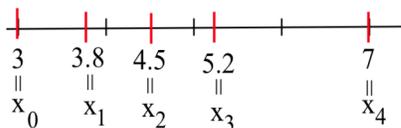


Fig. 19

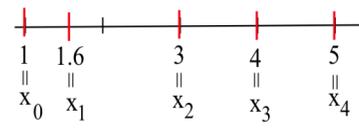


Fig. 20

43. For  $\Delta x_i = x_i - x_{i-1}$ , verify that  $\sum_{i=1}^n \Delta x_i = \text{length of the interval } [a, b]$ .

For problems 44–48, (i) sketch the graph of  $f$  on the given interval, (ii) draw vertical lines at each point of the given partition, (iii) evaluate each  $f(c_i)$  and sketch the corresponding rectangle, and (iv)

calculate and add together the areas of the rectangles.

44.  $f(x) = x + 1$ ,  $P = \{ 1, 2, 3, 4 \}$  (a)  $c_1 = 1, c_2 = 3$ , and  $c_3 = 3$ . (b)  $c_1 = 2, c_2 = 2$ , and  $c_3 = 4$ .

45.  $f(x) = 4 - x^2$ ,  $P = \{ 0, 1, 1.5, 2 \}$  (a)  $c_1 = 0, c_2 = 1$ , and  $c_3 = 2$ . (b)  $c_1 = 1, c_2 = 1.5$ , and  $c_3 = 1.5$ .

46.  $f(x) = \sqrt{x}$ ,  $P = \{ 0, 2, 5, 10 \}$  (a)  $c_1 = 1, c_2 = 4$ , and  $c_3 = 9$ . (b)  $c_1 = 0, c_2 = 3$ , and  $c_3 = 7$ .

47.  $f(x) = \sin(x)$ ,  $P = \{ 0, \pi/4, \pi/2, \pi \}$  (a)  $c_1 = 0, c_2 = \pi/4$ , and  $c_3 = \pi/2$ . (b)  $c_1 = \pi/4, c_2 = \pi/2$ , and  $c_3 = \pi$ .

48.  $f(x) = 2^x$ ,  $P = \{ 0, 1, 3 \}$  (a)  $c_1 = 0, c_2 = 2$ . (b)  $c_1 = 1, c_2 = 3$ .

For problems 49–52, sketch the function and find the smallest possible value and the largest possible value for a Riemann sum of the given function and partition.

49.  $f(x) = 1 + x^2$  (a)  $P = \{ 1, 2, 4, 5 \}$  (b)  $P = \{ 1, 2, 3, 4, 5 \}$  (c)  $P = \{ 1, 1.5, 2, 3, 4, 5 \}$

50.  $f(x) = 7 - 2x$  (a)  $P = \{ 0, 2, 3 \}$  (b)  $P = \{ 0, 1, 2, 3 \}$  (c)  $P = \{ 0, .5, 1, 1.5, 2, 3 \}$

51.  $f(x) = \sin(x)$  (a)  $P = \{ 0, \pi/2, \pi \}$  (b)  $P = \{ 0, \pi/4, \pi/2, \pi \}$  (c)  $P = \{ 0, \pi/4, 3\pi/4, \pi \}$

52.  $f(x) = x^2 - 2x + 3$  (a)  $P = \{ 0, 2, 3 \}$  (b)  $P = \{ 0, 1, 2, 3 \}$  (c)  $P = \{ 0, .5, 1, 2, 2.5, 3 \}$

### Upper and Lower Sum Problems

53. Suppose  $LS_P = 7.362$  and  $US_P = 7.402$  for a positive function  $f$  and a partition  $P$  of the interval

$[1, 5]$ . We can be certain that every Riemann sum for the partition  $P$  is within what distance of the exact value of the area between the graph of  $f$  and the interval  $[1, 5]$ ? (b) What if  $LS_P = 7.372$  and  $US_P = 7.390$ ?

54. Suppose we divide the interval  $[1, 4]$  into 100 equally wide subintervals and calculate a Riemann sum for  $f(x) = 1 + x^2$  by randomly selecting a point  $c_i$  in each subinterval. We can be certain that the value of the Riemann sum is within what distance of the exact value of the area between the graph of  $f$  and the interval  $[1, 4]$ ? (b) What if we take 200 equally long subintervals?

55. If we divide the interval  $[2, 4]$  into 50 equally wide subintervals and calculate a Riemann sum for  $f(x) = 1 + x^3$  by "randomly" selecting a point  $c_i$  in each subinterval (any point  $c_i$  in the subinterval), then we can be certain that the Riemann sum is within what distance of the exact value of the area between  $f$  and the interval  $[2, 4]$ ?

56. If  $f$  is monotonic decreasing on  $[a, b]$  and we divide the interval  $[a, b]$  into  $n$  equally wide subintervals (Fig. 21), then we can be certain that the Riemann sum is within what distance of the exact value of the area between  $f$  and the interval  $[a, b]$ .

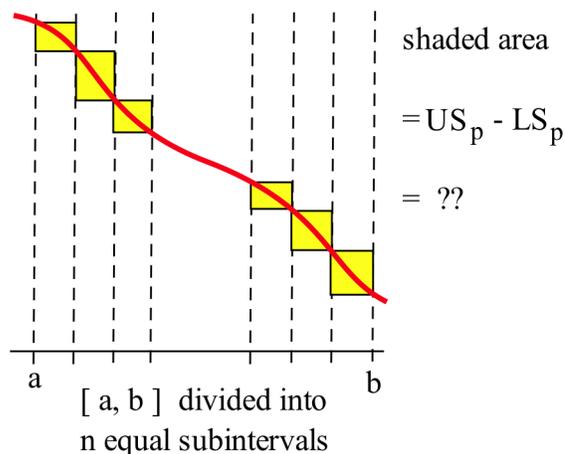


Fig. 21

**Summing Powers of Consecutive Integers**

Explicit formulas for some commonly encountered summations are known and are useful for explicitly evaluating some Riemann sums and their limits. The formulas below are included here for your reference. They will not be used or needed in the following sections.

The summation formula for the first  $n$  positive integers is relatively well-known, has several easy but clever proofs, and even has an interesting story.

$$1 + 2 + 3 + \dots + (n-1) + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Proof: Let  $S$  represent the sum  $1 + 2 + 3 + \dots + (n-2) + (n-1) + n$ . Rearranging the summands, we also know  $S = n + (n-1) + (n-2) + \dots + 3 + 2 + 1$ . Adding these 2 representations of  $S$  together,

$$\begin{array}{r} S = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n \\ + S = n + (n-1) + (n-2) + \dots + 3 + 2 + 1 \\ \hline 2S = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1) = n(n+1) \end{array}$$

so  $S = \frac{n(n+1)}{2}$ , the desired formula.

It is said that Gauss discovered this formula for himself at the age of 5. His teacher, planning on keeping the class busy for a while, asked the students to add the integers from 1 to 100. Gauss thought a few minutes, wrote his answer on his slate, and turned it in. According to the story, he then sat smugly while his classmates struggled with the problem.

57. Find the sum of the first 100 positive integers in 2 ways: (1) using Gauss' formula, and (2) using Gauss' method.
58. Find the sum of the first 10 odd integers.  
(Each odd integer can be written in the form  $2k - 1$  for  $k = 1, 2, 3, \dots$ )
59. Find the sum of the integers from 10 to 20.

Formulas for other integer powers of the first  $n$  integers have been discovered:

$$\sum_{k=1}^n k = \frac{1}{2} n^2 + \frac{1}{2} n = \frac{n(n+1)}{2} \qquad \sum_{k=1}^n k^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{2}{12} n = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{3}{12} n^2 + 0 \cdot n = \left( \frac{n(n+1)}{2} \right)^2$$

$$\sum_{k=1}^n k^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{4}{12} n^3 + 0 \cdot n^2 - \frac{1}{30} n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

In problems 60 – 62, use the properties of summation and the formulas for powers to evaluate each sum.

60.  $\sum_{k=1}^{10} (3 + 2k + k^2)$

61.  $\sum_{k=1}^{10} k \cdot (k^2 + 1)$

62.  $\sum_{k=1}^{10} k^2 \cdot (k - 3)$

### Section 4.1

### PRACTICE Answers

**Practice 1:**

(a)  $\sum_{k=1}^5 k^3 = 1 + 8 + 27 + 64 + 125.$

(b)  $\sum_{j=2}^7 (-1)^j \frac{1}{j} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}$

(c)  $\sum_{m=0}^4 (2m+1) = 1 + 3 + 5 + 7 + 9.$

**Practice 2:** (a)  $\sum_{k=2}^5 g(k) = g(2) + g(3) + g(4) + g(5) = 1 + (-2) + 3 + 5 = 7$

(b)  $\sum_{j=1}^4 h(j) = h(1) + h(2) + h(3) + h(4) = 3 + 3 + 3 + 3 = 12$

(c)  $\sum_{k=3}^5 (g(k) + f(k-1)) = (g(3) + f(2)) + (g(4) + f(3)) + (g(5) + f(4)) = (-2+3)+(3+1)+(5+0) = 10$

**Practice 3:** For  $g(x) = 1/x$ ,  $\sum_{k=2}^4 g(k) = g(2) + g(3) + g(4) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$  and

$$\sum_{k=1}^3 g(k+1) = g(2) + g(3) + g(4) = \frac{13}{12} .$$

**Practice 4:** Rectangular areas  $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = \sum_{j=1}^4 \frac{1}{j} .$

**Practice 5:**  $f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$

$$= \sum_{j=1}^3 f(x_{j-1})(x_j - x_{j-1}) \text{ or } \sum_{k=0}^2 f(x_k)(x_{k+1} - x_k) .$$

**Practice 6:** Interval is  $[3, 8]$ . 6 subintervals. mesh = length of longest subinterval = 1.2 .  
 $x_2 = 4.8$  and  $\Delta x_2 = x_2 - x_1 = 4.8 - 3.8 = 1 .$

**Practice 7:**  $RS = (3)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{4}\right) = 1.25$

**Practice 8:** smallest  $RS = (3)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{5}\right) = 0.95$       largest  $RS = (3)\left(1\right) + (1)\left(\frac{1}{4}\right) = 3.25$

**Practice 9:**  $RS = (0)\left(\frac{\pi}{4}\right) + (1)\left(\frac{\pi}{4}\right) + (1)\left(\frac{\pi}{2}\right) \approx 2.356 .$