

Summation Notation

Sequences.

A **sequence** is a real-valued function whose domain is all integers greater than or equal to a some initial integer.

Examples.

1. $a_n = n!$, $n \geq 0$: $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, $a_4 = 24$, ...

2. $a_i = 2(i - 2)^3$, $i \geq 1$: $a_1 = -2$, $a_2 = 0$, $a_3 = 2$, $a_4 = 16$, ...

A sequence may be defined **inductively** by defining one or more initial values and defining each subsequent value in terms of previous values.

Examples.

3. $b_1 = 1$; $b_{k+1} = b_k + (k + 1)$, $k \geq 1$:

$$b_1 = 1, b_2 = b_1 + 2 = 1 + 2 = 3, b_3 = b_2 + 3 = 3 + 3 = 6, b_4 = b_3 + 4 = 6 + 4 = 10, \dots$$

The sum of a sequence.

The (finite) **sum** of a sequence is the sum of all the terms of a sequence between two integers in its domain. It is denoted with \sum as follows:

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-2} + a_{n-1} + a_n.$$

If $n = m$, then $\sum_{i=m}^n a_i = a_m = a_n$. By convention, if $n < m$, then $\sum_{i=m}^n a_i = 0$. The variable i is called the **index variable** or the **summation variable**.

Example.

4. For a general sequence b_i , write out the sum $\sum_{i=2}^5 b_i$ using the plus sign.

Solution: $\sum_{i=2}^5 b_i = b_2 + b_3 + b_4 + b_5$. (Note that the i is a dummy variable in the \sum notation, which is to say that i does not appear in the expanded sum.)

5. For a general sequence b_i , write out the sum $\sum_{j=2}^5 b_j$ using the plus sign.

Solution: The answer is the same as the preceding problem. (It does not matter whether the index variable is named i or j or whatever.)

6. Write out the sum $\sum_{i=0}^3 \frac{1}{i+1}$ using the plus sign.

$$\text{Solution: } \sum_{i=0}^3 \frac{1}{i+1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

7. Write $2^3 + 3^3 + 4^3 + 5^3 + 6^3$ using \sum notation.

$$\text{Solution: } 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = \sum_{i=2}^6 i^3.$$

8. Write $2^3 + 3^3 + 4^3 + 5^3 + 6^3$ using \sum notation with the index variable starting at 1.

$$\text{Solution: } 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = \sum_{j=1}^5 (j+1)^3.$$

Rigorous Definition of Sigma.

The \sum notation can be defined inductively as follows. Fix a starting integer m . Let a_i be a sequence defined for $i \geq m$.

$$(1) \text{ Define } \sum_{i=m}^m a_i = a_m.$$

$$(2) \text{ Let } k \text{ be an arbitrary positive integer. Assume } \sum_{i=m}^k a_i \text{ is defined.}$$

$$\text{Define } \sum_{i=m}^{k+1} a_i = \sum_{i=m}^k a_i + a_{k+1}.$$

This defines $\sum_{i=m}^n a_i$ for all $n \geq m$. For convenience, if $n < m$, we define $\sum_{i=m}^n a_i$ to be zero.

Properties.

These are proved by induction except 9(b) which follows directly from the definition of \sum . Assume $n \geq m$.

9. Additivity:

$$(a) \sum_{i=m}^n a_i = a_m + \sum_{i=m+1}^n a_i$$

$$(b) \sum_{i=m}^n a_i = \sum_{i=m}^{n-1} a_i + a_n$$

$$(c) \text{ If } m < k < n, \sum_{i=m}^k a_i + \sum_{i=k+1}^n a_i = \sum_{i=m}^n a_i$$

10. Linearity:

$$(a) \text{ If } c \text{ is a real number, } \sum_{i=m}^n c a_i = c \sum_{i=m}^n a_i$$

$$(b) \sum_{i=m}^n (a_i + b_i) = \sum_{i=m}^n a_i + \sum_{i=m}^n b_i$$

$$(c) \sum_{i=m}^n (a_i - b_i) = \sum_{i=m}^n a_i - \sum_{i=m}^n b_i$$

Examples.

$$\textbf{11.} \text{ Show } \sum_{i=0}^n (i^2 - 3i) = \sum_{i=1}^n (i^2 - 3i).$$

$$\text{Solution: By additivity, } \sum_{i=0}^n (i^2 - 3i) = (0^2 - 3 \cdot 0) + \sum_{i=1}^n (i^2 - 3i) = \sum_{i=1}^n (i^2 - 3i).$$

$$\textbf{12.} \text{ Suppose } \sum_{i=1}^n a_i = -1 \text{ and } \sum_{i=1}^n b_i = -4. \text{ Find } \sum_{i=1}^n (3a_i - 2b_i) = \sum_{i=1}^n (i^2 - 3i).$$

$$\text{Solution: By linearity, } \sum_{i=1}^n (3a_i - 2b_i) = \sum_{i=1}^n 3a_i - \sum_{i=1}^n 2b_i = 3 \sum_{i=1}^n a_i - 2 \sum_{i=1}^n b_i = 3(-1) - 2(-4) = 5.$$

Some Common Sums.

These are also proved by induction. Let n be a positive integer.

$$\textbf{13.} \sum_{i=1}^n 1 = n; \sum_{i=1}^n c = cn; \sum_{i=m}^n c = c(n-m+1)$$

$$\textbf{14.} \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\textbf{15.} \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\textbf{16.} \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\textbf{17.} \text{ (Geometric sum.) } \sum_{i=1}^n ar^i = \frac{ar - ar^{n+1}}{1-r} = \frac{\text{first} - \text{ratio} \times \text{last}}{1-\text{ratio}}$$

Examples.

$$\textbf{18.} \text{ Find } \sum_{j=1}^n (3j^2 - 2j + 4).$$

$$\text{Solution: } \sum_{j=1}^n (3j^2 - 2j + 4) = 3 \sum_{j=1}^n j^2 - 2 \sum_{j=1}^n j + \sum_{j=1}^n 4 = 3 \frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} + 4n = n^3 + \frac{1}{2}n^2 + \frac{5}{2}n.$$

19. Find $\sum_{i=0}^{n-1} (2i^3 - 3)$.

Solution: $\sum_{i=0}^{n-1} (2i^3 - 3) = 2 \sum_{i=0}^{n-1} i^3 - \sum_{i=0}^{n-1} 3 = 2 \sum_{i=1}^{n-1} i^3 - 3n = 2 \left[\frac{(n-1)n}{2} \right]^2 - 3n = \frac{1}{2} n^4 - n^3 + \frac{1}{2} n^2 - 3n.$

20. Find $\sum_{i=1}^n 3(2^i)$.

Solution: First term is 6, the last term is $3 \cdot 2^n$ and the ratio between successive terms is 2. Therefore $\sum_{i=1}^n 3(2^i) = \frac{6 - 2 \times (3 \cdot 2^n)}{1 - 2} = 3 \cdot 2^{n+1} - 6$.

21. Find $\sum_{i=0}^{n-1} e^{2i}$.

Solution: First term is $e^0 = 1$, the last term is $e^{2(n-1)}$ and the ratio between successive terms is e^2 . Therefore $\sum_{i=0}^{n-1} e^{2i} = \frac{1 - e^2 \times e^{2(n-1)}}{1 - e^2} = \frac{e^{2n} - 1}{e^2 - 1}$.

22. Let $f(x) = x^2$, $x_i = 2 + \frac{3i}{n}$, and $\Delta x_i = \frac{3}{n}$. Find $\sum_{i=1}^n f(x_i) \Delta x_i$.

Solution: First $f(x_i) = x_i^2 = \left(2 + \frac{3i}{n}\right)^2 = 4 + \frac{12i}{n} + \frac{9i^2}{n^2}$. Therefore

$$\begin{aligned} \sum_{i=1}^n f(x_i) \Delta x_i &= \sum_{i=1}^n \left(4 + \frac{12i}{n} + \frac{9i^2}{n^2}\right) \frac{3}{n} = \sum_{i=1}^n \left(\frac{12}{n} + \frac{36}{n^2} i + \frac{27}{n^3} i^2\right) = \sum_{i=1}^n \frac{12}{n} + \frac{36}{n^2} \sum_{i=1}^n i + \frac{27}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{12}{n} n + \frac{36}{n^2} \frac{n(n+1)}{2} + \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} = 12 + \frac{18(n+1)}{n} + \frac{9(n+1)(2n+1)}{2n^2}. \end{aligned}$$

Exercises.**23.** Write out the first three terms:

(a) $\sum_{i=2}^1 2 \left(\frac{i}{i-1} \right)^2$. (b) $\sum_{i=1}^n \frac{4}{3^{i-2}}$. (c) $\sum_{i=0}^{n-1} (3i^2 - 2)$.

24. Find the following sums:

(a) $\sum_{i=2}^5 \frac{6}{i-1}$. (b) $\sum_{i=1}^4 (i+1)$. (c) $\left(\sum_{i=1}^4 i \right) + 1$. (d) $\sum_{i=1}^{1000} i$. (e) $\sum_{i=1}^n (3 + 2i - i^2)$.
(f) $\sum_{i=1}^{10} 3^i$. (g) $\sum_{i=0}^{n-1} 4^{-i}$. (h) $\sum_{i=1}^n \frac{3^{i-1}}{2^{2i}}$. (i) $\sum_{i=1}^n e^{i/2}$.

25. Suppose $\sum_{i=1}^n a_i = -1$, $\sum_{i=1}^n b_i = 5$, $\sum_{i=1}^n a_i^2 = 5$, $\sum_{i=1}^n a_i b_i = -4$, and $\sum_{i=1}^n b_i^2 = 13$. Find the following.

(a) $\sum_{j=1}^n (b_j - 2a_j)$. (b) $\sum_{j=1}^n (a_j^2 + b_j^2)$. (c) $\left(\sum_{i=1}^n a_i \right)^2 + \left(\sum_{i=1}^n b_i \right)^2$. (d) $\sum_{i=1}^n (a_i + b_i)^2$.
(e) $\left(\sum_{i=1}^n a_i + \sum_{i=1}^n b_i \right)^2$. (f) $\sum_{i=1}^n (2a_i + b_i)^2$.

26. Let $f(x) = x^2 - 2x$, $x_i = -1 + \frac{2i}{n}$, and $\Delta x_i = \frac{2}{n}$. Find $\sum_{i=1}^n f(x_i) \Delta x_i$.**27.** Let $f(x) = 2x - 1$, $x_i = 1 + \frac{4i}{n}$, and $\Delta x_i = \frac{4}{n}$. Find $\sum_{i=0}^{n-1} f(x_i) \Delta x_i$.**28.** Let $f(x) = e^{4x}$, $x_i = \frac{3i}{n}$, and $\Delta x_i = \frac{3}{n}$. Find $\sum_{i=1}^n f(x_i) \Delta x_i$.**29.** Let $f(x) = x^5$, $x_i = 2^{i/n}$, and $\Delta x_i = x_i - x_{i-1}$. Find $\sum_{i=1}^n f(x_i) \Delta x_i$.**Answers.**

23. (a) $4 + \frac{9}{4} + \frac{16}{9}$. (b) $12 + 4 + \frac{4}{3}$. (c) $-2 + 1 + 10$.

24. (a) $25/2$. (b) 14 . (c) 11 . (d) $500, 500$. (e) $\frac{23}{6} n + \frac{1}{2} n^2 - \frac{1}{3} n^3$. (f) $\frac{3}{2} (3^{10} - 1)$.
(g) $\frac{4}{3} (1 - 1/4^n)$. (h) $(2^{2n} - 3^n)/2^{2n}$. (i) $\sqrt{e} (e^{n/2} - 1)/(\sqrt{e} - 1)$.

25. (a) 7 . (b) 18 . (c) 26 . (d) 10 . (e) 16 . (f) 17 . **26.** $\frac{2(n^2 - 6n + 2)}{3n^2}$.

27. $20 - 16/n$. **28.** $\frac{3e^{12/n}(e^{12} - 1)}{n(e^{12/n} - 1)}$. **29.** $\frac{(2^6 - 1)2^{5/n}(2^{1/n} - 1)}{2^{6/n} - 1}$.