

Summation Notation and Series

In this section we need to do a brief review of summation notation or sigma notation. We'll start out with two integers, n and m , with $n < m$ and a list of numbers denoted as follows,

$$a_n, a_{n+1}, a_{n+2}, \dots, a_{m-2}, a_{m-1}, a_m$$

We want to add them up, in other words we want,

$$a_n + a_{n+1} + a_{n+2} + \dots + a_{m-2} + a_{m-1} + a_m$$

For large lists this can be a fairly cumbersome notation so we introduce summation notation to denote these kinds of sums. The case above is denoted as follows.

$$\sum_{i=n}^m a_i = a_n + a_{n+1} + a_{n+2} + \dots + a_{m-2} + a_{m-1} + a_m$$

The i is called the index of summation. This notation tells us to add all the a_i s up for all integers starting at n and ending at m .

or instance,

$$\begin{aligned} \sum_{i=0}^4 \frac{i}{i+1} &= \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} = \frac{163}{60} = 2.716\bar{6} \\ \sum_{i=4}^6 2^i x^{2i+1} &= 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16x^9 + 32x^{11} + 64x^{13} \\ \sum_{i=1}^4 f(x_i^*) &= f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) \end{aligned}$$

Properties

Here are a couple of formulas for summation notation.

- $\sum_{i=i_0}^n ca_i = c \sum_{i=i_0}^n a_i$ where c is any number. So, we can factor constants out of a summation.
- $\sum_{i=i_0}^n (a_i \pm b_i) = \sum_{i=i_0}^n a_i \pm \sum_{i=i_0}^n b_i$ So, we can break up a summation across a sum or difference.

Note that we started the series at i_0 to denote the fact that they can start at any value of i that we need them to. Also note that while we can break up sums and differences as we did in **2** above we can't do the same thing for products and quotients. In other words

$$\sum_{i=i_0}^n (a_i b_i) \neq \left(\sum_{i=i_0}^n a_i \right) \left(\sum_{i=i_0}^n b_i \right) \quad \sum_{i=i_0}^n \frac{a_i}{b_i} \neq \frac{\sum_{i=i_0}^n a_i}{\sum_{i=i_0}^n b_i}$$

Formulas

Here are a couple of nice formulas that we will find useful in a couple of sections. Note that these formulas are only true if starting at $i = 1$. You can, of course, derive other formulas from these for different starting points if you need to.

- $\sum_{i=1}^n c = cn$
- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$

Here is a quick example on how to use these properties to quickly evaluate a sum that would not be easy to do by hand.

Example: Using the formulas and properties from above determine the value of the following summation.

$$\sum_{i=1}^{100} (3 - 2i)^2$$

The first thing that we need to do is square out the stuff being summed and then break up the summation using the properties as follows,

$$\begin{aligned} \sum_{i=1}^{100} (3 - 2i)^2 &= \sum_{i=1}^{100} 9 - 12i + 4i^2 \\ &= \sum_{i=1}^{100} 9 - \sum_{i=1}^{100} 12i + \sum_{i=1}^{100} 4i^2 \\ &= \sum_{i=1}^{100} 9 - 12 \sum_{i=1}^{100} i + 4 \sum_{i=1}^{100} i^2 \end{aligned}$$

Now, using the formulas, this is easy to compute,

$$\begin{aligned} \sum_{i=1}^{100} (3 - 2i)^2 &= 9(100) - 12 \left(\frac{100(101)}{2} \right) + 4 \left(\frac{100(101)(201)}{6} \right) \\ &= 1293700 \end{aligned}$$

Doing this by hand would definitely taken some time and theres a good chance that we might have made a minor mistake somewhere along the line

The next idea is used often when taking derivatives of series, it the concept of **shifting the index**. The basic idea behind index shifts is to start a series at a different value for whatever the reason (and yes, there are legitimate reasons for doing that).

Consider the following series,

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n}$$

Suppose that for some reason we wanted to start this series at $n = 0$, but we didnt want to change the value of the series. This means that we cant just change the $n = 2$ to $n = 0$ as this would add in two new terms to the series and thus change its value.

Performing an index shift is a fairly simple process to do. Well start by defining a new index, say i , as follows,

$$i = n - 2$$

Now, when $n = 2$, we will get $i = 0$. Notice as well that if $n = \infty$ then $i = \infty - 2 = \infty$, so only the lower limit will change here. Next, we can solve this for n to get,

$$n = i + 2$$

We can now completely rewrite the series in terms of the index i instead of the index n simply by plugging in our equation for n in terms of i .

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \sum_{i=0}^{\infty} \frac{(i+2)+5}{2^{i+2}} = \sum_{i=0}^{\infty} \frac{i+7}{2^{i+2}}$$

To finish the problem out well recall that the letter we used for the index doesnt matter and so well change the final i back into an n to get,

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \sum_{n=0}^{\infty} \frac{n+7}{2^{n+2}}$$

To convince yourselves that these really are the same summation lets write out the first couple of terms for each of them,

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{n+5}{2^n} &= \frac{7}{2^2} + \frac{8}{2^3} + \frac{9}{2^4} + \frac{10}{2^5} + \cdots \\ \sum_{n=0}^{\infty} \frac{n+7}{2^{n+2}} &= \frac{7}{2^2} + \frac{8}{2^3} + \frac{9}{2^4} + \frac{10}{2^5} + \cdots \end{aligned}$$

So, sure enough the two series do have exactly the same terms.

There is actually an easier way to do an index shift. The method given above is the technically correct way of doing an index shift. However, notice in the above example we decreased the initial value of the index by 2 and all the ns in the series terms increased by 2 as well.

This will always work in this manner. If we decrease the initial value of the index by a set amount then all the other ns in the series term will increase by the same amount. Likewise, if we increase the initial value of the index by a set amount, then all the ns in the series term will decrease by the same amount

Lets do an example using this shorthand method for doing index shifts.

Example:

Consider the following series starting at $n = 0$

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Write $\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n = 3$.

Write

$$\sum_{n=1}^{\infty} ar^{n-1}$$

as the series that starts at $n = 0$.

In this case we need to decrease the initial value by 1 and so the ns (okay the single n) in the term must increase by 1 as well.

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^n$$

Problem 1. Perform an index shift so that the series starts at $n = 3$

i. $\sum_{n=1}^{\infty} (n2^n - 3^{1-n})$

iii. $\sum_{n=2}^{\infty} \frac{(-1)^{n-3} (n+2)}{5^{1+2n}}$

ii. $\sum_{n=1}^{\infty} (n2^n - 3^{1-n})$

iv. $\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2 + 1}$

Next lets recall some basic properties of convergent series.

Properties

If $\sum a_n$ and $\sum b_n$ are both convergent series then,

- $\sum ca_n$, where c is any number, is also convergent and

$$\sum ca_n = c \sum a_n$$

- $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$ is also convergent and,

$$\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty} (a_n \pm b_n)$$

The first property is simply telling us that we can always factor a multiplicative constant out of an infinite series and again recall that if we don't put in an initial value of the index that the series can start at any value. Also recall that in these cases we won't put an infinity at the top either.

The second property says that if we add/subtract series all we really need to do is add/subtract the series terms. Note as well that in order to add/subtract series we need to make sure that both have the same initial value of the index and the new series will also start at this value.

Problem 2. Given the above properties determine the following

i. Given that $\sum_{n=0}^{\infty} \frac{1}{n^3 + 1} = 1.686$ determine the value of $\sum_{n=2}^{\infty} \frac{1}{n^3 + 1}$.

ii. Given that $\sum_{n=4}^{\infty} n 4^{-n} = 0.02257$ determine the value of $\sum_{n=1}^{\infty} n 4^{-n}$.

iii. Given that $\sum_{n=3}^{\infty} \frac{n+1}{n^3} = 0.47199$ determine the value of $\sum_{n=5}^{\infty} \frac{n+1}{n^3}$